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DARBOUX INTEGRABILITY AND RATIONAL REVERSIBILITY IN CUBIC SYSTEMS WITH TWO INVARIANT STRAIGHT LINES

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Abstract. We find conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system with two distinct invariant straight lines. The presence of a center at $O(0,0)$ is proved by using the method of Darboux integrability and the rational reversibility.

1. Introduction and statement of results

A cubic system with a singular point with pure imaginary eigenvalues ($\lambda_{1,2} = \pm i$, $i^2 = -1$) by a nondegenerate transformation of variables and time rescaling can be brought to the form

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 = P(x,y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) = Q(x,y), \end{aligned} \tag{1.1}$$

where the variables x, y and coefficients $a, b, \dots, s$ in (1.1) are assumed to be real. Then the origin $O(0,0)$ is a singular point of a center or a focus type for (1.1). The problem arises of distinguishing between a center and a focus, i.e. of finding the coefficient conditions under which $O(0,0)$ is, for example, a center. These conditions are called the conditions for a center existence or the center conditions and the problem - the problem of the center.

The derivation of necessary conditions for a center existence often involves extensive use of computer algebra (see, for example, [12], [13]), in many cases making very heavy demands on the available algorithms and hardware. The necessary conditions are shown to be sufficient by a variety of methods. A number of techniques, of progressively wider application, have been developed.

A theorem of Poincaré in [15] says that a singular point $O(0,0)$ is a center for (1.1) if and only if the system has a nonconstant analytic first integral F in the neighborhood of $O(0,0)$. It is known [1] that the origin is a center for system (1.1) if and only if the system has in the neighborhood of $O(0,0)$ an analytic integrating

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factor of the form

$$\mu(x,y) = 1 + \sum_{k=1}^{\infty} \mu_k(x,y),$$

where μ_k are homogeneous polynomials of degree k.

There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1.1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$: $dF/dt = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j$. Quantities $L_j,\ j=\overline{1,\infty}$ are polynomials with respect to the coefficients of system (1.1) called to be the Lyapunov quantities [14]. The origin $O(0,0)$ is a center for (1.1) if and only if $L_j = 0,\ j = \overline{1,\infty}$. The set of these conditions, which are polynomial equations in the coefficients of the system (1.1), is denumerable [9] and hence by Hilbert's basis theorem, it is sufficient that a finite number of them be satisfied.

A singular point $O(0,0)$ is a center for (1.1) if the equations of (1.1) are invariant under reflection in a line through the origin and reversion of time, called time-reversible systems. The classical condition is that the system is invariant under one or other of the transformations $(x,y,t) \to (-x,y,-t)$ or $(x,y,t) \to (x,-y,-t)$. The first corresponds to reflection in the y-axis and the second to reflection in the x-axis.

Żołądek [23] mentioned three general mechanisms for producing centers: searching for (1) a Darboux first integral or (2) a Darboux–Schwarz–Christoffel first integral or by (3) generating centers by rational reversibility, and he claimed that these are sufficient for producing all cases of real polynomial differential systems with centers. This conjecture is still open, even for cubic systems (1.1).

The time-reversibility in two-dimensional autonomous systems was studied in [16] and the relation between time-reversibility and the center-focus problem was discussed in [21].

The problem of the center was solved for quadratic systems and for cubic symmetric systems. The problem of finding a finite number of necessary and sufficient conditions for the center in the cubic case (for cubic system (1.1)) is still open. It was possible to find the conditions for the center only in some particular cases (see, for example, [2, 3, 4, 5, 6, 7, 8, 11, 12, 17, 18, 19, 20, 22]).

The problem of the center for cubic differential systems (1.1) with invariant straight lines (real or complex) was considered in [3], [4], [5], [6], [11], [19], [20]. In these papers, the problem of the center was completely solved for cubic systems with at least three invariant straight lines. The main results of these works is that every center in the cubic system (1.1) with at least three invariant straight lines comes from a Darboux integrability or from a rational reversibility.

The goal of this paper is to obtain center conditions for a cubic differential system (1.1) with two distinct invariant straight lines by using the method of Darboux integrability and rational reversibility. Our main result is the following one.

Theorem 1.1. *The origin is a center for a cubic differential system* (1.1)*, with at least two invariant straight lines, if one of the conditions* (i)–(xiv), (1)– (26) *hold.*

The paper is organized as follows. In Section 2 we summarize the results obtained for cubic differential systems with at least three invariant straight lines and centers. In Section 3 we find four series of conditions for the existence of two distinct invariant straight lines. In Section 4 we study the Darboux integrability in cubic systems with two distinct invariant straight lines and obtain the center conditions (i)–(xiv). In Section 5 we describe the algorithm to transform a cubic system (1.1) to one which is symmetric in a line by means of a rational transformation. In Section 6 for cubic system (1.1) with at least two invariant straight lines we obtain conditions (1)–(26) for the system to be rationally reversible. In the last section, we prove the main theorem.

2. Cubic systems with at least three invariant straight lines

We shall study the problem of the center for cubic system (1.1) assuming that (1.1) has invariant straight lines.

Definition 2.1. An algebraic invariant curve (or an algebraic particular integral) of (1.1) is the solution set in $\mathbb{C}^2$ of an equation $f(x,y)=0$, where f is a polynomial in x,y with complex coefficients such that

$$\frac{df}{dt}=\dot{f}=\frac{\partial f}{\partial x}P+\frac{\partial f}{\partial y}Q=fK,$$

for some polynomial in x,y, $K=K(x,y)$ with complex coefficients, called the cofactor of the invariant algebraic curve $f(x,y)=0$.

By the above definition, a straight line

$$L\equiv C+Ax+By=0,\quad A,B,C\in\mathbb{C},\quad (A,B)\neq(0,0), \tag{2.1}$$

is an invariant straight line for (1.1) if and only if there exists a polynomial $K(x,y)$ such that the following identity holds

$$A\cdot P(x,y)+B\cdot Q(x,y)\equiv(C+Ax+By)\cdot K(x,y). \tag{2.2}$$

If the cubic system (1.1) has complex invariant straight lines then obviously they occur in complex conjugated pairs

$$L\equiv C+Ax+By=0\quad\text{and}\quad \overline{L}\equiv\overline{C}+\overline{A}x+\overline{B}y=0.$$

According to [3] the cubic system (1.1) cannot have more than four nonhomogeneous invariant straight lines, i.e. invariant straight lines of the form

$$1+Ax+By=0,\quad (A,B)\neq(0,0). \tag{2.3}$$

As homogeneous straight lines $Ax+By=0$, this system can have only the lines $x\pm iy=0, i^2=-1$.

From (2.2) it results that (2.3) is an invariant straight line of (1.1) if and only if A and B are the solutions of the system

$$\begin{gathered}
F_1(A,B)\equiv AB^2-fAB+bB^2+rA-lB=0,\\
F_2(A,B)\equiv A^2B+aA^2-gAB-kA+sB=0,\\
F_3(A,B)\equiv B^3-2A^2B+fA^2+(c-b)AB-dB^2-pA+nB=0,\\
F_4(A,B)\equiv A^3-2AB^2-cA^2+(d-a)AB+gB^2+mA-qB=0.
\end{gathered} \tag{2.4}$$

The cofactor of (2.3) is

$$\begin{aligned}
K(x,y)={}&-Bx+Ay+(aA-gB+AB)x^2+(cA-dB+B^2-A^2)xy\\
&+(fA-bB-AB)y^2.
\end{aligned}$$

The problem of the center for cubic systems with at least three invariant straight lines was completely solved. The main results of these works are summarized in the following three theorems.

Theorem 2.2 ([3, 4]). *Let the cubic differential system have four invariant straight lines (real, real and complex, complex). Then any singular point with pure imaginary eigenvalues of this system is a center if and only if the first two Liapunov quantities vanish ($L_1 = L_2 = 0$).*

Theorem 2.3 ([5, 6, 19, 20]). *Let the cubic differential system have exactly three invariant straight lines (real, real and complex). Then any singular point with pure imaginary eigenvalues of this system is a center if and only if the first seven Liapunov quantities vanish ($L_j = 0$, $j = 1, \dots, 7$).*

Theorem 2.4. *Every center in the cubic differential system* (1.1) *with:*

1. *four invariant straight lines comes from a Darboux first integral or a Darboux integrating factor;*
2. *three invariant straight lines comes from a Darboux integrating factor or a rational reversibility.*

3. Cubic systems with two invariant straight lines

Let the cubic system (1.1) have two distinct invariant straight lines L_1 and L_2 real or complex. If $L_1,\ L_2$ are complex and $L_2 \neq \overline{L_1}$, then the straight lines $\overline{L_1}, \overline{L_2}$ conjugate with $L_1,\ L_2$ will be also invariant for (1.1) (the coefficients in (1.1) are real). In this case the system (1.1) has four distinct invariant straight lines and the problem of the center is solved by Theorem 2.2. If L_1 is complex and L_2 is real, then the problem of the center is solved by Theorem 2.3.

In this section, we shall consider cubic systems (1.1) with two distinct invariant straight lines, where L_1, L_2 are real or L_1, L_2 are complex ($L_2 = \overline{L_1}$). It is easy to see that for the relative positions of two distinct invariant straight lines three cases can occur:

1. two parallel invariant straight lines;
2. two homogeneous invariant straight lines;
3. two nonhomogeneous and nonparallel invariant straight lines.

3.1. **Two parallel invariant straight lines.**

Let the cubic system (1.1) have two parallel invariant straight lines L_1, L_2, then by a rotation of axes we can make them parallel to the axis of ordinates (Oy). Note that by a rotation of axes of coordinates the linear part of (1.1) preserves the form.

Assume L_1 and L_2 are complex, then $L_2 = \overline{L_1}$. From $L_1 || \overline{L_1}$, it follows that L_1 looks as $1 + A(x + By) = 0$, where A is a complex number and B is real. In this case, via a rotation of axes about the origin, it is also possible to make the straight lines L_1 and L_2 to be parallel to the axis Oy.

In order that the cubic system (1.1) had two invariant straight lines L_1, L_2 parallel to the axis Oy, it is necessary and sufficient that the following coefficient conditions to be satisfied

$$a = f = k = p = r = 0, \quad m(c^2 - 4m) \neq 0. \tag{3.1}$$

In this case the invariant straight lines L_1 and L_2 are

$$L_{1,2} \equiv 1 + \frac{c \pm \sqrt{c^2 - 4m}}{2} x = 0. \tag{3.2}$$

3.2. **Two homogeneous invariant straight lines.** For homogeneous invariant straight lines, it is easily verified that the cubic system (1.1) can have only the lines $x \pm iy = 0$, $i^2 = -1$. These lines are invariant if and only if the following conditions hold

$$g = b + c, \quad f = a + d, \quad q = p + l - k, \quad s = m + n - r. \tag{3.3}$$

3.3. **Two nonhomogeneous and nonparallel invariant straight lines.** Let the cubic system (1.1) have two nonhomogeneous and nonparallel invariant straight lines L_1, L_2 intersecting at a point (x_0, y_0). The intersection point (x_0, y_0) is a singular point for (1.1) and has real coordinates. By rotating the system of coordinates $(x \to x\cos\varphi - y\sin\varphi, y \to x\sin\varphi + y\cos\varphi)$ and rescaling the axes of coordinates $(x \to \alpha x,\, y \to \alpha y)$, we obtain $L_1 \cap L_2 = (0, 1)$. In this case the invariant straight lines can be written as

$$L_j \equiv 1 + A_j x - y = 0, \quad A_j \in \mathbb{C}, \quad j = 1, 2;\ A_1 - A_2 \neq 0. \tag{3.4}$$

As the point $(0, 1)$ is a singular point for (1.1), then $P(0, 1) = Q(0, 1) = 0$. These equalities yield $r = -f - 1, l = -b$. Substituting $B = -1, r = -f - 1$ and $l = -b$ in (2.4) we find that the straight lines (3.4) are invariant for (1.1) if and only if A_1 and A_2 are the solutions of the system

$$\begin{gathered}
F_2(A_1) \equiv (a-1)A_1^2 + (g-k)A_1 - s = 0, \\
F_3(A_1) \equiv (f+2)A_1^2 + (b-c-p)A_1 - d - n - 1 = 0, \\
F_4(A_1) \equiv A_1^3 - cA_1^2 + (a-d+m-2)A_1 + g + q = 0, \\
F_2(A_2) \equiv (a-1)A_2^2 + (g-k)A_2 - s = 0, \\
F_3(A_2) \equiv (f+2)A_2^2 + (b-c-p)A_2 - d - n - 1 = 0, \\
F_4(A_2) \equiv A_2^3 - cA_2^2 + (a-d+m-2)A_2 + g + q = 0.
\end{gathered} \tag{3.5}$$

It is easy to see from (3.5) that the system (1.1) can have two distinct invariant straight lines of the form (3.4) if and only if the following coefficient conditions are satisfied

$$\begin{gathered}
k = (a-1)(A_1 + A_2) + g, \quad l = -b, \quad r = -f - 1, \\
m = -A_1^2 - A_1A_2 - A_2^2 + c(A_1 + A_2) - a + d + 2, \\
n = -(f+2)A_1A_2 - (d+1), \quad s = (1-a)A_1A_2, \\
p = (f+2)(A_1 + A_2) + b - c, \quad q = (A_1 + A_2 - c)A_1A_2 - g.
\end{gathered} \tag{3.6}$$

Theorem 3.1. *The cubic differential system* (1.1) *has at least two distinct invariant straight lines if and only if one of the sets of conditions* (3.1), (3.3) *and* (3.6) *holds.*

4. Darboux integrability in cubic systems with two invariant straight lines

Let the cubic system (1.1) have sufficiently many invariant algebraic curves $f_j(x, y) = 0$, $j = 1, \dots, q$ with cofactors $K_j(x, y)$. Then in most cases a first integral (an integrating factor) can be constructed in the Darboux form [10]

$$f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_q^{\alpha_q} \tag{4.1}$$

and we say that the cubic system (1.1) is Darboux integrable. The function (4.1), with $\alpha_i \in \mathbb{C}$ not all zero, is a first integral (an integrating factor) for (1.1) if and only if

$$\sum_{i=1}^{q} \alpha_i K_i \equiv 0 \quad \Big(\sum_{i=1}^{q} \alpha_i K_i \equiv \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\Big).$$

The method of Darboux turns out to be very useful and elegant one to prove integrability for some classes of systems depending on parameters.

In this section we shall find center conditions for cubic system (1.1) with two invariant straight lines by constructing an integrating factor of the Darboux form

$$\mu = L_1^{\alpha_1} L_2^{\alpha_2}, \tag{4.2}$$

where $L_j = 0$, $j = 1, 2$ are invariant straight lines for (1.1) with cofactor $K_j(x, y)$ and $\alpha_j \in \mathbb{C}$. The cubic system (1.1) will have an integrating factor of the form (4.2) if and only if the numbers α_j satisfy the following identity

$$\alpha_1 K_1(x, y) + \alpha_2 K_2(x, y) \equiv \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}. \tag{4.3}$$

4.1. **Centers of system** (1.1) **with two parallel invariant straight lines and Darboux integrability.**

Lemma 4.1. *The following set of conditions is sufficient condition for the origin to be a center for system* (1.1)*:*

(i) $a = d = f = k = l = p = q = r = 0$.

Proof. Let the conditions (3.1) hold, then the cubic system (1.1) has two invariant straight lines of the form (3.2) with cofactors $K_{1,2}(x, y) = [y(c + 2mx \pm \sqrt{c^2 - 4m})]/2$. Taking into account the cofactors, the identity (4.3) yields $d = q = l = 0$ and

$$\alpha_{1,2} = [(n - m)\sqrt{c^2 - 4m} \pm (2bm - cn)]/(m\sqrt{c^2 - 4m}),$$

we obtain the center conditions (i). □

4.2. **Centers of system** (1.1) **with two homogeneous invariant straight lines and Darboux integrability.**

Lemma 4.2. *The following three sets of conditions are sufficient conditions for the origin to be a center for system* (1.1)*:*

(ii) $c = -2b$, $d = -2a$, $f = -a$, $g = -b$, $n = 2r - m$, $p = -l$, $q = -k$, $s = r$;

(iii) $a = d = f = 0$, $g = b+c$, $k = l$, $m = (2br+cn-cr)/(2b)$, $p = q = [l(b+c)]/b$, $s = (2bn + cn - cr)/(2b)$;

(iv) $c = (bd)/a$, $f = a + d$, $g = [b(a + d)]/a$, $p = q = [l(a + d)]/a$, $k = l$, $m = (2ar + dn - dr)/(2a)$, $s = (2an + dn - dr)/(2a)$.

Proof. Assume the conditions (3.3) are satisfied, then the cubic system (1.1) has two homogeneous invariant straight lines $x \pm iy = 0$ with cofactors

$$\begin{aligned} K_1(x, y) = {} & -i + (a - ib - ic)x - (b + ia + id)y + (k - im - in + ir)x^2 \\ & + (r - n - il - ip)xy - (l + ir)y^2, \end{aligned}$$

$$K_2 = \overline{K_1}.$$

In this case the system (1.1) will have an integrating factor of the form (4.2) if and only if the identity (4.3) holds. Substituting in this identity the expressions of the cofactors and identifying the coefficients of x^0, x, y, x^2, xy and y^2, we obtain that $\alpha_2 = \alpha_1$ and α_1 obey the following system of algebraic equations:

$$\begin{gathered}(r-n)\alpha_1 + m - n = 0, \quad 2a(\alpha_1+1) - d = 0, \quad 2b(\alpha_1+1) - c = 0,\\ 2k(\alpha_1+2) - l - p = 0, \quad (\alpha_1+2)(k-l) = 0.\end{gathered} \tag{4.4}$$

Let $\alpha_1 = -2$, then from (4.4) we obtain the conditions (ii). Assume $\alpha_1 \neq -2$, then $k = l$. If $a = 0$, then $b \neq 0$, $\alpha_1 = (c-2b)/(2b)$ and from (4.4) we get the conditions (iii). If $a \neq 0$, then $\alpha_1 = (d-2a)/(2a)$ and (4.4) implies the conditions (iv).

In each of the cases (ii)–(iv), the system (1.1) has an integrating factor of the form (4.2) and therefore the origin is a center for (1.1). □

4.3. **Centers of system (1.1) with two nonhomogeneous and nonparallel invariant straight lines and Darboux integrability.** Let the coefficient conditions (3.6) hold. Denote $\lambda = a - 1$, $\gamma = f + 2$ and consider the following two cases:

4.3.1. $\lambda = 0$. In this case $a = 1$ and (3.6) yields the following conditions

$$\begin{gathered}a = 1, \quad k = g, \quad l = -b, \quad q = [(d+n+1)(c\gamma + b - c - p) - g\gamma^2]/\gamma^2,\\ m = [(\gamma(d+1) + c^2)(\gamma-1) - (b-p)(c(\gamma-2) + b - p) - n\gamma]/\gamma^2,\\ r = 1-\gamma, \quad s = 0, \quad \gamma[(b-c-p)^2 + 4\gamma(d+n+1)] \neq 0\end{gathered} \tag{4.5}$$

for the existence of two distinct invariant straight lines of the form (3.4) where A_j, $j = 1, 2$ are the solutions of the equation

$$\gamma A^2 + (b-c-p)A - d - n - 1 = 0. \tag{4.6}$$

Lemma 4.3. *The following five sets of conditions are sufficient conditions for the origin to be a center for system* (1.1)*:*

- (v) $a = \gamma = 1$, $d = -2$, $k = -q = g$, $p = -l = b$, $m = -n$, $r = s = 0$;
- (vi) $a = n = 1$, $b = l = s = 0$, $d = -2$, $k = -q = g$, $p = c(\gamma-1)$, $f = \gamma - 2$, $m = -1$, $r = 1-\gamma$;
- (vii) $a = n = 1$, $d = -2$, $f = \gamma - 2$, $k = -q = g$, $l = -b$, $r = 1-\gamma$, $s = 0$, $c = [2b(\gamma-2)]/\gamma$, $m = -(4b^2\gamma - 4b^2 + \gamma^2)/\gamma^2$, $p = b(4-3\gamma)/\gamma$;
- (viii) $a = 1$, $d = -1$, $f = (-3)/2$, $k = g = q = s = 0$, $l = -b$, $m = -2n$, $p = (2b-c)/2$, $r = 1/2$;
- (ix) $a = 1$, $k = g$, $l = -b$, $q = [(d+n+1)(c\gamma+b-c-p) - g\gamma^2]/\gamma^2$, $m = [(\gamma(d+1)+c^2)(\gamma-1) - (b-p)(c(\gamma-2)+b-p) - n\gamma]/\gamma^2$, $f = \gamma-2$, $r = 1-\gamma$, $s = 0$, $p = b(1-d) + (c-2b)\gamma - c$, $g = [b((d+\gamma)^2 - \gamma^2 + (n+1)(d+2\gamma))]/[(d+2)\gamma^2]$, $n = [b(d+2\gamma)(c\gamma - 2bd - 2b\gamma) + d\gamma(d+1)(\gamma-1)]/[\gamma(d+2\gamma)]$.

Proof. Indeed, if the conditions (4.5) hold, then the cubic system (1.1) has two invariant straight lines of the form $L_{1,2} \equiv 1 + A_{1,2}x - y = 0$ with cofactors

$$K_{1,2}(x,y) = x + A_{1,2}y + gx^2 + (1 + d - A_{1,2}^2 + cA_{1,2})xy + ((\gamma-1)A_{1,2} + b)y^2,$$

where A_1, A_2 are the roots of the equation (4.6):

$$A_{1,2} = (p - b + c \pm \sqrt{(b-c-p)^2 + 4\gamma(d+n+1)}\,)/(2\gamma).$$

In this case system (1.1) will have an integrating factor of the form (4.2) if and only if the identity (4.3) holds. Substituting in (4.3) the expressions of the cofactors and identifying the coefficients of x, y, x^2, xy and y^2, we obtain that

$$\alpha_1 = d - 2 - \alpha_2, \quad \alpha_2 = [(d-2)A_1 - 2b + c]/(A_1 - A_2)$$

and

$$\begin{gathered} p = b(1-d) + (c-2b)\gamma - c,\ g(d+2)\gamma^2 - b(d+n+1)(d+2\gamma) = 0, \\ (d^2 - 4b^2 + 2bc + d - 2n)\gamma^2 + d\gamma(bc - 6b^2 - d - n - 1) - 2b^2d^2 = 0. \end{gathered} \tag{4.7}$$

Let $d = -2$. If $\gamma = 1$, then from (4.7) we obtain the conditions (v); if $\gamma \neq 1$ and $b = 0$ – the conditions (vi); if $b(\gamma - 1) \neq 0$ and $n = 1$ – the conditions (vii).

Assume $d \neq -2$. If $d + 2\gamma = 0$, then we get the conditions (viii) and if $d + 2\gamma \neq 0$ – the conditions (ix). In each of the cases (v)–(ix), the system (1.1) has an integrating factor of the form (4.2) and therefore the origin is a center for (1.1). $\square$

4.3.2. $\lambda \neq 0$. In this case (3.6) yields the following conditions

$$\begin{gathered} p = [(b-c)\lambda + (k-g)\gamma]/\lambda, \quad q = [\lambda(cs - g\lambda) + s(g-k)]/\lambda^2, \\ l = -b, \quad m = [(d - \lambda + 1)\lambda^2 + \lambda(c(k-g) - s) - (k-g)^2]/\lambda^2, \\ r = 1 - \gamma, \quad n = [s\gamma - (1+d)\lambda]/\lambda, \quad (g-k)^2 + 4s\lambda \neq 0 \end{gathered} \tag{4.8}$$

for the existence of two distinct invariant straight lines of the form (3.4) where $A_j,\ j = 1, 2$ are the solutions of the equation

$$\lambda A^2 + (g-k)A - s = 0. \tag{4.9}$$

Lemma 4.4. *The following five sets of conditions are sufficient conditions for the origin to be a center for system* (1.1)*:*

- (x) $a = \lambda + 1,\ b = -(2c\lambda + g)/(2\lambda),\ d = 2\lambda - 1,\ f = (-3)/2,\ k = c\lambda + g,\ l = (2c\lambda + g)/(2\lambda),\ m = (\lambda^2 - s)/\lambda,\ q = -g,\ n = (s - 4\lambda^2)/(2\lambda),\ p = -(3c\lambda + g)/(2\lambda),\ r = 1/2$;
- (xi) $a = \lambda + 1,\ d = -2,\ f = \lambda - 1,\ k = c\lambda + g,\ l = -b = c - g,\ q = -g,\ m = -\lambda - 1 - s\lambda^{-1},\ n = s + 1 + s\lambda^{-1},\ p = c(\lambda - 1) + g,\ r = -\lambda$;
- (xii) $a = \lambda + 1,\ b = l = 0,\ c = [g(2\lambda - d - 2)]/(2\lambda),\ k = [g(2\lambda - d)]/2,\ f = \gamma - 2,\ p = [(b-c)\lambda + (k-g)\gamma]/\lambda,\ n = [s\gamma - (1+d)\lambda]/\lambda,\ m = [(d - \lambda + 1)\lambda^2 + \lambda(c(k-g) - s) - (k-g)^2]/\lambda^2,\ r = 1 - \gamma,\ q = -g,\ s = [\lambda(d^2 - 2d\lambda + 3d - 4\lambda + 2)]/(d + 2\gamma - 2\lambda)$;
- (xiii) $a = \lambda + 1,\ f = -2,\ d = 2\lambda,\ n = -(2\lambda + 1),\ b = [(c\lambda + g - k)(c\lambda + 2(g-k)) - 2\lambda^2(\lambda + 1)]/[2\lambda(c\lambda + g - k)],\ p = (b - c),\ q = [\lambda(cs - g\lambda) + s(g-k)]/\lambda^2,\ r = 1,\ l = -b,\ m = [\lambda^2(\lambda + 1) + \lambda(c(k-g) - s) - (k-g)^2]/\lambda^2,\ s = -\lambda^2(2\lambda^2(\lambda + 1) + (k+g)(k - g - c\lambda))/(c\lambda + g - k)^2$;
- (xiv) $a = \lambda + 1,\ f = \gamma - 2,\ r = 1 - \gamma,\ l = -b,\ b = [\gamma(c\lambda + g - k)]/[\lambda(d + 2(\gamma - \lambda))],\ n = [s\gamma - (1+d)\lambda]/\lambda,\ p = [(b-c)\lambda + (k-g)\gamma]/\lambda,\ q = [\lambda(cs - g\lambda) + s(g-k)]/\lambda^2,\ m = [(d - \lambda + 1)\lambda^2 + \lambda(c(k-g) - s) - (k-g)^2]/\lambda^2,\ s = [\lambda^2((2b - c - 2g)\lambda + 3k - g + dg)]/(c\lambda + g - k),\ 2(d+2)\lambda^3 + ((c-b)^2 - b^2 - d^2 - 3d - 2s - 2)\lambda^2 + ((3c - 2b)(g-k) + (d + 2\gamma)s)\lambda + 2(g-k)^2 = 0$.

Proof. Indeed, if the conditions (4.8) hold, then the cubic system (1.1) has two invariant straight lines of the form $L_{1,2} \equiv 1 + A_{1,2}x - y = 0$ with cofactors

$K_{1,2}(x,y) = x + A_{1,2}y + (g + \lambda A_{1,2})x^2 + (1 + d + cA_{1,2} - A_{1,2}^2)xy + (b + (\gamma - 1)A_{1,2})y^2$, where A_1, A_2 are the roots of the equation (4.9):

$$A_{1,2} = (k - g \pm \sqrt{(g-k)^2 + 4\lambda s})/(2\lambda).$$

In this case the system (1.1) will have an integrating factor of the form (4.2) if and only if the identity (4.3) holds. Substituting in this identity the expressions of the cofactors and identifying the coefficients of x, y, x^2, xy and y^2, we obtain that

$$\alpha_1 = d - 2(\lambda + 1) - \alpha_2,\ \alpha_2 = [c - 2b + (d - 2 - 2\lambda)A_1]/(A_1 - A_2)$$

and

$$\begin{gathered} b = [(c + 2g)\lambda^3 + (g - dg - 3k)\lambda^2 + (c\lambda + g - k)s]/(2\lambda^3), \\ 2b\lambda^2 + ((c - 2b)\gamma - bd)\lambda + (g - k)\gamma = 0, \\ 2(d + 2)\lambda^3 + ((c - b)^2 - b^2 - d^2 - 3d - 2s - 2)\lambda^2 \\ + ((3c - 2b)(g - k) + (d + 2\gamma)s)\lambda + 2(g - k)^2 = 0. \end{gathered} \tag{4.10}$$

Let $k = c\lambda + g$. If $d = 2(\lambda - \gamma)$, then from (4.10) we obtain the conditions (x) and (xi); if $d \neq 2(\lambda - \gamma)$, then (4.10) implies the conditions (xii).

Let $k \neq c\lambda + g$. If $d = 2(\lambda - \gamma)$, then from (4.10) we get the conditions (xiii) and if $d \neq 2(\lambda - \gamma)$, then (4.10) yields the conditions (xiv).

In each of the cases (x)–(xiv), the system (1.1) has an integrating factor of the form (4.2) and therefore the origin is a center for (1.1). □

Taking into account Lemmas 4.1–4.4, for cubic differential system (1.1) with two distinct invariant straight lines (real or complex conjugated), it was proved the following theorem.

Theorem 4.5. *The differential system* (1.1) *with two distinct invariant straight lines has a Darboux integrating factor of the form* (4.2) *if and only if one of the sets of conditions* (i)–(xiv) *is satisfied.*

5. Rational transformation in cubic systems

It is well known from Poincaré [15] that if a differential system with a singular point $O(0,0)$ of a center or a focus type is invariant by the reflection with respect, for example, to the axis $X = 0$ and reversion of time then $O(0,0)$ is a center for (1.1) ($X = 0$ is called the axis of symmetry). It is clear that (1.1) has a center at $O(0,0)$ if there exists a diffeomorphism

$$\Phi : U \to V, \quad \Phi = \{X = \varphi(x,y),\ Y = \psi(x,y)\}, \quad \Phi(0,0) = (0,0), \tag{5.1}$$

which brings system (1.1) to a system with the axis of symmetry. In particular, if $\varphi(x,y)$ and $\psi(x,y)$ are rational functions in (5.1), then we say that (1.1) is rationally reversible ([24]).

In [13] is described an algorithm based on application of Gröebner bases in the search for a bilinear transformation, which is invertible in a neighbourhood of the origin and transform a given system to one which is symmetric in a line. This algorithm is applied to find center conditions for some cubic systems.

In this section we shall consider a general mechanism to produce center by rational reversibility. We seek a transformation of the form

$$x = \frac{a_1X + b_1Y}{a_3X + b_3Y - 1}, \quad y = \frac{a_2X + b_2Y}{a_3X + b_3Y - 1} \tag{5.2}$$

with $a_1b_2 - b_1a_2 \neq 0$ and $a_j, b_j \in \mathbb{R}$, $j = 1,2,3$. The condition $a_1b_2 - b_1a_2 \neq 0$ guarantees that (5.2) is invertible in a neighborhood of $O(0,0)$ and the singular point is mapped to $X = Y = 0$. Applying the transformation (5.2) to (1.1) we obtain a system of the form

$$\dot{X} = \frac{P(X,Y)}{R(X,Y)}, \quad \dot{Y} = \frac{Q(X,Y)}{R(X,Y)},$$

whose orbits in some neighborhood of $O(0,0)$ are the same as those of the system

$$\dot{X} = \sum_{i+j=0}^{4} U_{ij}X^iY^j \equiv P(X,Y), \quad \dot{Y} = \sum_{i+j=0}^{4} V_{ij}X^iY^j \equiv Q(X,Y), \tag{5.3}$$

where U_{ij}, V_{ij} are polynomials in the coefficients of the original system and the parameters $a_1, a_2, a_3, b_1, b_2, b_3$ of the transformation.

The requirement is to show that $a_1, a_2, a_3, b_1, b_2, b_3$ can be chosen so that the system (5.3) is symmetric in the Y-axis; i.e. the transformation (5.2) brings in some neighborhood of $O(0,0)$ the system (1.1) to one equivalent with a polynomial system

$$\frac{dX}{dt} = Y + M(X^2, Y), \quad \frac{dY}{dt} = -X(1 + N(X^2, Y)). \tag{5.4}$$

The obtained system has an axis of symmetry $X = 0$ and therefore $O(0,0)$ is a center for (1.1). The system (5.4) is equivalent to the system (5.3) if the following conditions are satisfied:

$$U_{31} \equiv V_{22} = 0, \quad U_{13} \equiv V_{04} = 0, \quad U_{10} \equiv V_{01} = 0, \quad V_{40} = 0, \quad U_{00} = 0, \quad V_{00} = 0$$

and

$$\begin{gathered}
V_{04} \equiv a_3[sb_1^4 + ((k+q)b_1^2 + (m+n)b_1b_2 + (l+p)b_2^2)b_1b_2 + rb_2^4] = 0, \\
\begin{aligned}
V_{22} \equiv a_3[&ma_1^2b_2^2 + ca_1b_2^2a_3 + (2p - 3k - q)a_1a_2b_2^2 + da_2b_1^2a_3 \\
&+ (3l + p - 2q)a_2^2b_1b_2 + (3(r+s) - 2(m+n))a_2^2b_2^2 + na_2^2b_1^2 \\
&+ (2b + c - 2g)a_2b_1b_2a_3 + (2f - 2a - d)a_2b_2^2a_3 + a_3^2] = 0,
\end{aligned} \\
\begin{aligned}
U_{30} \equiv\ &2aa_1^2b_2a_3 + [(m-s)a_1 + (p-q)a_2 + 2(c-g)a_3]a_1a_2b_2 \\
&+ ka_1^3b_2 + a_2^3(lb_1 - nb_2 + rb_2) + 2a_2^2a_3(bb_1 - db_2 + fb_2) = 0,
\end{aligned} \\
\begin{aligned}
U_{12} \equiv\ &(qa_2 + 2ga_3)b_1^3 + [2(a+d)a_3 + (m + 2n - 3s)a_2]b_1^2b_2 + \big[(3l - 3k \\
&+ 2p - 2q)a_2 + 2(b+c)a_3\big]b_1b_2^2 + [pa_1 - (2m + n - 3r)a_2 + 2fa_3]b_2^3 = 0,
\end{aligned} \\
\begin{aligned}
V_{03} \equiv\ &(ka_2 - ga_3)b_1^3 + [(m-s)a_2 - (a+d)a_3]b_1^2b_2 \\
&+ [(p-q)a_2 - (b+c)a_3]b_1b_2^2 + (la_1 - (n-r)a_2 - fa_3)b_2^3 = 0,
\end{aligned} \\
\begin{aligned}
V_{21} \equiv\ &qa_1^3b_2 + (m + 2n - 3s)a_1^2a_2b_2 + (d-a)a_1^2a_3b_2 \\
&+ (3l - 3k + 2p - 2q)a_1a_2^2b_2 + [pb_1 + (3r - 2m - n)b_2]a_2^3 \\
&+ (2b - g)a_1a_2a_3b_2 + [(f - 2a)b_2 - (b-c)b_1]a_2^2a_3 = 0,
\end{aligned} \\
V_{02} \equiv aa_2b_1^2 + (c-g)b_1b_2a_2 + (ba_1 - da_2 + fa_2)b_2^2 - a_3 = 0, \\
V_{20} \equiv ga_1^3 + (a+d)a_1^2a_2 + (b+c)a_1a_2^2 + fa_2^3 + 2a_3 = 0, \\
U_{11} \equiv [db_1 + (2b + c - 2g)b_2]a_2b_1 + [ca_1 + (2f - 2a - d)a_2]b_2^2 + 3a_3 = 0, \\
U_{01} \equiv b_1^2 + b_2^2 - 1 = 0, \quad U_{10} \equiv a_1b_1 + a_2b_2 = 0, \quad V_{10} \equiv a_1^2 + a_2^2 - 1 = 0.
\end{gathered} \tag{5.5}$$

Next we shall study the compatibility of (5.5) assuming that the cubic system (1.1) has two distinct invariant straight lines (real or complex conjugated). If (5.5) is compatible, then the cubic system (1.1) with two distinct invariant straight lines is rationally reversible and a singular point $O(0,0)$ is a center.

6. Rationally reversible cubic systems with at least two invariant straight lines

In this section we shall find conditions on the coefficients, for cubic system (1.1) with two distinct invariant straight lines, which allow us to transform the system to the system (5.4), symmetric in a line, by means of the rational transformations (5.2).

It is easy to verify that the equations $U_{01} = 0$, $V_{10} = 0$ of (5.5) admit the following parametrization

$$a_1 = (2u)/(u^2+1), \quad a_2 = (u^2-1)/(u^2+1),$$
$$b_1 = (2v)/(v^2+1), \quad b_2 = (v^2-1)/(v^2+1),$$

where u and v are some real parameters. In this case $U_{10} \equiv j_1 j_2 = 0$, where $j_1 = uv + u - v + 1$, $j_2 = uv - u + v + 1$.

Next assume $j_1 = 0$, then $v = (1+u)/(1-u)$ and $U_{10} \equiv 0$. The case $j_2 = 0$ is equivalent with $j_1 = 0$ if we take into consideration that $j_2(u,v) = j_1(-u,-v)$.

6.1. **Centers of system** (1.1) **with two parallel invariant straight lines and reversibility.**

Consider the system of algebraic equations (5.5) and let the conditions (3.1) hold.

6.1.1. $a_3 = 0$. In this case $V_{04} \equiv 0$ and $V_{22} \equiv 0$. If $u = 0$; $u = -1$ or $u(u+1) \neq 0$, then from the equations of (5.5) we obtain, respectively, the following three sets of conditions for the existence of a center:

(1) $a = d = f = k = l = p = q = r = 0$;
(2) $a = b = c = f = g = k = l = p = q = r = 0$;
(3) $a = f = k = p = r = 0$, $l = [4mu(u^6 - 7u^4 + 7u^2 - 1)]/(u^2+1)^4$, $b = [c(6u^2 - u^4 - 1)]/(u^2+1)^2$, $s = [m(u^4 - 6u^2 + 1)^2]/(u^2+1)^4$, $g = -b$, $q = -3l$, $d = [2cu(10u^2 - 3u^4 - 3)]/[(u^2+1)^2(u^2-1)]$, $n = [-2m(u^8 - 20u^6 + 54u^4 - 20u^2 + 1)]/(u^2+1)^4$.

6.1.2. $a_3 \neq 0$. In this case from the equation $V_{02} = 0$ of (5.5) we have

$$a_3 = [2u((g-c)u^4 - 2du^3 + 2(2b+c-g)u^2 + 2du + g - c)]/(u^2+1)^3.$$

If $u = 0$, then (5.5) yields the center conditions which are contained in (1).

If $u = -1$, then from the equations of (5.5) we obtain the following conditions for the existence of a center

(4) $a = f = k = l = p = r = 0$, $c = -3b$, $g = -2b$, $m = 2b^2$, $q = -bd$.

Assume $u(u+1) \neq 0$, then from the equations $\{U_{11} = 0,\ V_{04} = 0,\ U_{12} = 0,\ V_{03} = 0,\ V_{21} = 0,\ U_{30} = 0\}$ of (5.5) we express, g, l, s, m, q, n, respectively and

$$V_{22} \equiv V_{20} \equiv (3b+c)(3u^2-1)(u^2-3)u - d(u^4 - 10u^2 + 1)(u^2-1) = 0.$$

If $(3u^2-1)(u^2-3) = 0$, then we obtain the following two sets of conditions for the existence of a center:

(5) $a = d = f = k = p = r = 0$, $g = (c - 7b)/5$, $n = [3c(2b - c)]/20$, $l = [\sqrt{3}(8b^2 + 2bc - 3c^2)]/100$, $m = [2(2c^2 - 2b^2 - 3bc)]/25$, $q = [3\sqrt{3}(3c^2 - 8b^2 - 2bc)]/100$, $s = [3(16b^2 - 6bc - c^2)]/100$;

(6) $a = d = f = k = p = r = 0$, $g = (c - 7b)/5$, $n = [3c(2b - c)]/20$, $l = [\sqrt{3}(-8b^2 - 2bc + 3c^2)]/100$, $m = [2(2c^2 - 2b^2 - 3bc)]/25$, $q = [3\sqrt{3}(-3c^2 + 8b^2 + 2bc)]/100$, $s = [3(16b^2 - 6bc - c^2)]/100$.

If $(3u^2 - 1)(u^2 - 3) \neq 0$, then we get the following conditions for the existence of a center

(7) $a = f = k = p = r = 0$, $m = [h(du^2 - 4bu - 8cu - d)]/(100u^2)$, $g = [4(b + 2c)(u^5 + u) + d(1 - 19u^2 + 19u^4 - u^6) - 8(4b + 3c)u^3]/[10u(u^2 - 1)^2]$, $l = [h(u^2 - 1)(d(u^6 - 19u^4 + 19u^2 - 1) - (19b + 3c)(u^5 + u) + 6(7b - c)u^3)]/[25u(u^2 + 1)^4]$, $h = d + 4bu - 2cu - du^2$,

$$\begin{aligned} n &= [h(d(1 - 9u^2 + 230u^4 - 230u^6 + 9u^8 - u^{10}) + 3(3b + c)(u + u^9) \\ &\quad + 16(c - 12b)(u^7 + u^3) + 2(279b + 13c)u^5)]/[50u^2(u^2 + 1)^4], \\ q &= [h(d(1 - 21u^2 + 458u^4 - 458u^6 + 21u^8 - u^{10}) + 12(2b - c)(u + u^9) \\ &\quad + 40(c - 9b)(u^3 + u^7) + 8(144b + 13c)u^5)]/[50u(u^2 - 1)(u^2 + 1)^4], \\ s &= [2hu(d(4u^7 - 76u^5 + 76u^3 - 4u) + 5(b - c)(u^8 + 1) \\ &\quad + 8(c - 7b)(u^6 + u^2) + 2(99b + 13c)u^4)]/[25(u^2 - 1)^2(u^2 + 1)^4], \\ c &= [d(u^4 - 10u^2 + 1)(u^2 - 1)]/[(3u^2 - 1)(u^2 - 3)u] - 3b. \end{aligned}$$

Remark 6.1. In each of the cases (3) and (7) the system (1.1) has four invariant straight lines. Thus, in conditions (3) besides the invariant straight lines (3.2), the system (1.1) has two more invariant straight lines $L_{3,4} = (c \pm \sqrt{c^2 - 4m})[(u^4 - 6u^2 + 1)x + 4u(1 - u^2)y] + 2(u^2 + 1)^2$; in conditions (7): $L_3 = [4bu(3u^2 - 1)(u^2 - 3) - 2d(u^2 + 2u - 1)(u^2 - 2u - 1)(u^2 - 1)](u^2 y + 2ux - y) - (3u^2 - 1)(u^2 + 1)^2(u^2 - 3)$, $L_4 = (3u^2 - 1)(u^2 - 3)[4bu(u^2 - 1)(u^2 x - 2uy - x) - 2u(u^2 + 1)^2] - d(u^2 - 1)(u^8 x - 12u^6 x + 32u^5 y + 38u^4 x - 32u^3 y - 12u^2 x + x)$.

6.2. **Centers of system** (1.1) **with two homogeneous invariant straight lines and reversibility.** Consider the system of algebraic equations (5.5) and let the conditions (3.3) hold.

6.2.1. $a_3 = 0$. In this case $V_{04} \equiv V_{22} \equiv 0$ and we have the following possibilities:

If $u = -1$ or $u(u+1) \neq 0$, then from the equations of (5.5) we obtain respectively the following two sets of conditions for the existence of a center:

(8) $b = c = g = k = l = p = q = 0$, $f = a + d$, $r = m + n - s$;

(9) $b = [a(1 - u^2)]/(2u)$, $c = [d(1 - u^2)]/(2u)$, $g = [(a + d)(1 - u^2)]/(2u)$, $n = [(q - 3k)(u^4 - 6u^2 + 1) + 4m(u^3 - u)]/[4u(u^2 - 1)]$, $l = k$, $f = a + d$, $r = [(q - k)(u^4 - 6u^2 + 1) + 4m(u^3 - u)]/[4u(u^2 - 1)]$, $s = [k(6u^2 - u^4 - 1) + 2mu(u^2 - 1)]/[2u(u^2 - 1)]$, $p = q$.

If $u = 0$, then (5.5) yields the symmetric set of conditions to (8).

6.2.2. $a_3 \neq 0$. In this case from the equation $V_{02} = 0$ of (5.5) we find

$$a_3 = [a(u^2 - 1) + 2bu]/(u^2 + 1).$$

If $u = -1$, then (5.5) yields the following conditions for the existence of a center

(10) $c=-3b$, $f=a+d$, $g=-2b$, $k=-2ab$, $l=b(a+d)$, $m=2b^2$, $p=-2b(a+d)$, $q=b(a-d)$, $r=0$, $s=2b^2+n$.

In the case $u=0$, we get the symmetric to (10) set of conditions for the existence of a center.

Let $u(u+1)\neq 0$, then from the equations of (5.5) we obtain the following conditions for the existence of a center

(11) $c=[(3a+d)(1-u^2)-6bu]/(2u)$, $g=[(3a+d)(1-u^2)-4bu]/(2u)$,

$$
\begin{aligned}
l &= [a(3a+d)(u^2-1)+2(3ab+bd+k)u]/(2u),\\
m &= [r(u^2+1)^4+2(au^2-a+2bu)((5a+2d)(u^6-1)\\
&\quad +(11a-2d)(u^2-u^4)+b(10u^5-12u^3+10u))]/(u^2+1)^4,\\
s &= [n(u^2+1)^4+2(au^2-a+2bu)((5a+2d)(u^6-1)\\
&\quad +(11a-2d)(u^2-u^4)+b(10u^5-12u^3+10u))]/(u^2+1)^4,\\
f &= a+d, \quad q=[2pu+(3a+d)(au^2-a+2bu)]/(2u),\\
r &= [2(5ab+bd+k)(u^{11}-u)+2(4b^2-9a^2-3ad)(u^{10}+u^2)\\
&\quad +a(3a+d)(u^{12}+1)+2(3k-5bd-33ab)(u^9-u^3)\\
&\quad +(61a^2-ad-64b^2)(u^8+u^4)+4(45ab-3bd+k)(u^7-u^5)\\
&\quad +4(28b^2-23a^2+3ad)u^6]/[4u^2(u^2+1)^4],\\
n &= [2(k-10ab-2bd)(u^9+u)+8(10ab+k)(u^7+u^3)\\
&\quad +2(14a^2+ad-12b^2)(u^8-u^2)+4(10b^2+ad-8a^2)(u^6-u^4)\\
&\quad +4(3k-14ab+2bd)u^5+2a(2a+d)(1-u^{10})]/[(u^2+1)^4(u^2-1)],\\
p &= [(12ab+2bd+k)(u^9+u)+(12b^2-5ad-19a^2)(u^8-u^2)\\
&\quad +4(k-16ab-2bd)(u^7+u^3)+2(21a^2-3ad-26b^2)(u^6-u^4)\\
&\quad +a(3a+d)(u^{10}-1)+2(52ab-10bd+3k)u^5]/[u(u^2+1)^4].
\end{aligned}
$$

Remark 6.2. In each of the cases (10) and (11) the system (1.1) has three invariant straight lines. Thus, in conditions (10) besides the invariant straight lines $x\pm iy=0$, the system (1.1) has one more invariant straight line $L_3=1-2bx$; in conditions (11): $L_3=(au^2-a+2bu)[4ux+2(u^2-1)y]-(u^2+1)^2=0$.

6.3. **Centers of system (1.1) with two nonhomogeneous and nonparallel invariant straight lines and reversibility.** Consider the system of algebraic equations (5.5) and let the conditions (3.6) hold.

6.3.1. $a_3=0$. In this case $V_{04}\equiv V_{22}\equiv 0$ and we have the following possibilities:

If $u=0$ or $u=-1$, then from the equations of (5.5) we obtain, respectively, the following three sets of conditions for the existence of a center:

(12) $a=b=d=f=k=l=p=q=0$, $c=2g$, $m=g^2+1$, $n=1$, $r=s=-1$, $A_1=g-A_2$, $A_2^2-gA_2-1=0$;

(13) $b=c=g=k=l=p=q=0$, $r=-(f+1)$, $s=(a-1)(d-a-m+2)$, $n=(f+2)(d-a-m+2)-d-1$, $A_1=-A_2$, $A_2^2+a+m-d-2=0$;

(14) $b=c=g=l=p=q=s=0$, $f=-2$, $n=-(d+1)$, $r=1$, $k=(a-1)A_2$, $A_1=0$, $A_2^2+a-d+m-2=0$.

If $u(u+1) \neq 0$, then from the equations $\{V_{02}=0, V_{20}=0, U_{11}=0\}$ of (5.5) we express, b, g and d, respectively. Then $V_{03} \equiv f_1 f_2 f_3 = 0$, where

$$f_1 = 2uA_1 + 1 - u^2,\ f_2 = 2uA_2 + 1 - u^2,$$
$$f_3 = a(u^4+1) + 2(A_1 + A_2 - c)(u^3 - u) + 2(2f - a + 4)u^2.$$

Let $f_1 = 0$, then $A_1 = (u^2-1))/(2u)$ and $V_{03} \equiv V_{21} \equiv 0$. Express A_2 from $U_{30} = 0$ and denote $z = u^4 - 6u^2 + 1$, then $U_{12} \equiv h_1 c + h_2 = 0$, where $h_1 = 2(fz + (u^2+1)^2 - 8au^2)(u^2-1)u$ and $h_2 = f(u^2+1)^4 - 32a^2u^4 + f^2(u^2-1)^2 z - 4a(f-2)(u^2+1)^2u^2$.

If $h_1 = 0$, then $U_{12} = 0$ yields $f = -1$ and $a = 1$. In this case we get the following conditions for the existence of a center

(15) $a = 1$, $b = [u^6 - 15u^4 + 15u^2 - 1 - 2cuz]/[2u(u^2+1)^2]$, $f = -1$, $d = [4(u^2-1)z - 2cu(3z + 8u^2)]/[(u^2+1)^2(u^2-1)]$, $g = k = l = -b$, $p = q = b$, $m = (bz)/[2u(1-u^2)]$, $r = s = 0$, $n = -m$, $z = u^4 - 6u^2 + 1$, $A_1 = (u^2-1)/(2u)$, $A_2 = (2cu - u^2 + 1)/(2u)$.

If $h_1 \neq 0$, then express c form $U_{12} = 0$ and obtain the following conditions for the existence of a center

(16) $g = [(a(1-u^2) + 2cu)z + 8f(1-u^2)u^2]/[2u(1+u^2)^2]$, $s = (1-a)A_1A_2$, $d = [2(a-f)(u^2-1)z - 2cu(3z+8u^2)]/[(u^2-1)(u^2+1)^2]$,

$$c = -[f(u^2+1)^4 - 32a^2u^4 + f^2(u^2-1)^2 z - 4a(f-2)(u^2+1)^2u^2]$$
$$\div [2(fz + (u^2+1)^2 - 8au^2)(u^2-1)u],$$

$b = -g + (a+f)(1-u^2)/(2u)$, $k = (a-1)(A_1+A_2) + g$, $q = (A_1 + A_2 - c)A_1A_2 - g$, $r = -f-1$, $n = A_1A_2(-f-2) - (d+1)$, $p = (f+2)(A_1+A_2) + b - c$, $l = -b$, $m = -A_1^2 - A_1A_2 - A_2^2 + c(A_1+A_2) - a + d + 2$, $z = u^4 - 6u^2 + 1$, $A_1 = (u^2-1)/(2u)$, $A_2 = 2(a-1)u/(u^2-1) + (2cu + fu^2 - f)/(2u)$.

The case $f_2 = 0$ can be reduced to $f_1 = 0$ if we replace A_2 by A_1.

Assume $f_1 f_2 \neq 0$ and $f_3 = 0$. We express A_1 from $f_3 = 0$, a from $V_{21} = 0$ and calculate the resultant of the equations $\{U_{30} = 0, U_{12} = 0\}$ by A_2. We find that $\mathrm{Res}(U_{30}, U_{12}, A_2) \equiv 64 g_1^2 g_2^2 (u^2+1)^{12}(u^2-1)^4u^2$, where $g_1 = 2uz(u^2-1)c + f(u^8+1) - 8(f-1)(u^6+u^2) + 2(23f+24)u^4$, $g_2 = (u^2+1)^2 f + 8u^2$.

If $g_1 = 0$ or $g_2 = 0$, then we get the following center conditions, respectively:

(17) $a = (8u^2 - fz)/[2(u^2-1)^2]$, $n = 2pu/(u^2-1) + (z - 16u^2)/z$,

$$c = [8(f-1)(u^6+u^2) - 2(23f+24)u^4 - f(u^8+1)]/[2zu(u^2-1)],$$
$$d = [4(3f+8)(u^6+u^2) - 8(5f+12)u^4]/[z(u^2-1)^2], \quad l = -b,$$
$$b = [2(f+2)u]/(u^2-1), \quad g = [f(u^4+1) + 2(5f+12)u^2]/[4u(1-u^2)],$$
$$k = [32(f+1)u^2(u^4+1) + f^2(u^2+1)^4]/[8u(u^2-1)^3], \quad r = -(f+1),$$
$$m = [qz(u^2-1) - 2u(5u^4 - 14u^2 + 5)]/[2zu], \quad z = u^4 - 6u^2 + 1,$$
$$p = [f^2(u^2+1)^4 + 48u^2(u^2-1)^2 + 8fu^2(7u^4 - 10u^2 + 7)]/[4uz(1-u^2)],$$
$$s = [(f+2)((f-2)(u^8 + 6u^4 + 1) + 4(f+6)(u^6+u^2))]/[4(u^2-1)^4],$$
$$q = [f^2(u^2+1)^4(u^4 - 14u^2 + 1) + 32(f+3)(u^{10} + 10u^6 + u^2)$$
$$- 192(3f+5)(u^8+u^4)]/[8zu(u^2-1)^3],$$
$$A_1 = [4z(1-u^2)uA_2 - 4(f+10)(u^6+u^2) - 6(f-8)u^4 - f(u^8+1)]/[4zu(u^2-1)],$$

$$4zu(u^2-1)^2A_2^2+(u^2-1)(8(5u^4-6u^2+5)u^2+f(u^2+1)^4)A_2$$
$$+2u((f-2)(u^8+6u^4+1)+4(f+6)(u^6+u^2))=0;$$

(18) $a=(8u^2)/(u^2+1)^2$, $f=-a$, $l=g=-b$, $q=3b$, $p=-3k$, $b=[4u(u^2-1)(u^4-14u^2+1)-cz(u^2+1)^2]/(u^2+1)^4$, $d=[32z(u^4-u^2)-2cu(3u^4-10u^2+3)(u^2+1)^2]/[(u^2+1)^4(u^2-1)]$, $k=[4u(u^2-1)]/(u^2+1)^2$, $s=(-bz)/[4u(u^2-1)]$, $r=-z/(u^2+1)^2$, $m=[4u(u^2-1)(u^4-22u^2+1)+c(u^2+1)^4]/[4u(u^2-1)(u^2+1)^2]$,
$$n=[c(u^8-20u^6+54u^4-20u^2+1)(u^2+1)^2-2u(u^8-68u^6+246u^4$$
$$-68u^2+1)(u^2-1)]/[2u(u^2+1)^4(1-u^2)],\ z=u^4-6u^2+1,$$
$A_1=c-A_2-8(u^3-u)/(u^2+1)^2$, $4u(u^2-1)[(u^2+1)^2(A_2^2-cA_2)+8u(u^2-1)A_2-u^4+14u^2-1]+cz(u^2+1)^2=0$.

6.3.2. $a_3\neq 0$. In this case the equation $V_{02}=0$ of (5.5) yields

$$a_3=[a(u^6-1)+2(g-c)(u^5+u)+(3a+4d-4f)(u^2-u^4)+4(2b+c-g)u^3]/(u^2+1)^3.$$

If $u=0$ or $u=-1$, then from the equations of (5.5) we obtain, respectively, the following three sets of conditions for the existence of a center:

(19) $b=l=s=0$, $a=r=1$, $d=-3$, $n=-f=2$, $k=g$, $p=-c$, $q=-2g$, $A_2^3-cA_2^2+(m+2)A_2-g=0$, $A_1^2+(A_2-c)A_1+A_2^2-cA_2+m+2=0$;

(20) $r=s=0$, $a=1/2$, $c=b+2g$, $d=(-3)/2$, $f=-1$, $k=g/2$, $l=-b$, $m=g(b+g)$, $n=1/2$, $p=-g$, $q=-g$, $A_1=0$, $A_2=g$;

(21) $c=-3b$, $f=-1$, $g=-2b$, $k=-2ab$, $l=-b$, $m=2b^2$, $n=1-a$, $p=2b$, $q=b(a-d)$, $s=3a-a^2+ad-d-2$, $r=0$, $A_1=-A_2-2b$, $A_2^2+2bA_2+a-d-2$.

If $u(u+1)\neq 0$, then we express d from $V_{20}=0$ and replace in $V_{04}=0$. Factoring we obtain $V_{04}\equiv f_1f_2f_3=0$, where

$$f_1=A_1(u^2-1)+2u,\ f_2=A_2(u^2-1)+2u,$$
$$f_3=(a-1)(u^4+1)+2(A_1+A_2-c)(u^3-u)+2(3-a+2f)u^2.$$

Let $f_1=0$, then $A_1=(2u)/(1-u^2)$ and we find $U_{12}\equiv g_1g_2=0$, where

$$g_1=(2a+2f+1)(u^4+1)+4(b+g)(u^3-u)-2(2a+2f-1)u^2,$$
$$g_2=(2a+f)(u^4+1)+2(b-c+g)(u^3-u)+2(f-2a)u^2.$$

Assume $g_1=0$ and express g from $g_1=0$, then $U_{12}\equiv U_{30}\equiv 0$. Replacing g in $V_{03}=0$ and factoring we obtain $V_{03}\equiv h_1h_2=0$, where

$$h_1=4u(1-u^2)A_2+u^4-6u^2+1,$$
$$h_2=(2a-1)(u^4+1)+4(A_2-c)(u^3-u)+2(3-2a+4f)u^2.$$

If $h_1=0$, then $A_2=(u^4-6u^2+1)/[4u(u^2-1)]$. We express a from $V_{22}\equiv V_{21}\equiv U_{11}=0$ and obtain the following set of conditions for the existence of a center

(22)
$$a=[(f+1)(8u^6-u^8+8u^2-1)+2(b+c)(u-u^7)+2(b-7c)(u^3-u^5)$$
$$+2(9-7f)u^4]/[8u^2(u^2-1)^2],$$
$$k=(a-1)(A_1+A_2)+g,\ l=-b,$$
$$d=[(f+1)(-u^8-1)+2(b+c)(u-u^7)+2(4f-1)(u^6+u^2)$$

$$+ 2(7b + 3c)(u^5 - u^3) - 2(7f + 1)u^4]/[4u^2(u^2 - 1)^2],$$
$$r = -(f + 1),$$
$$g = [(f + 1)(u^8 + 1) + 2(b + c)(u^7 - 7u^5 + 7u^3 - u) - 4(4f + 3)(u^6 + u^2) + 2(15f - 13)u^4]/[16u^3(u^2 - 1)],$$

$n = -(f + 2)A_1A_2 - d - 1$, $m = c(A_1 + A_2) - A_1^2 - A_1A_2 - A_2^2 - a + d + 2$, $s = (1 - a)A_1A_2$, $q = A_1A_2(A_1 + A_2 - c) - g$, $p = (f + 2)(A_1 + A_2) + b - c$, $A_1 = (2u)/(1 - u^2)$, $A_2 = (u^4 - 6u^2 + 1)/[4u(u^2 - 1)]$.

If $h_1 \neq 0$, $h_2 = 0$, then express A_2 from $h_2 = 0$ and a from $U_{11} \equiv V_{22} = 0$. We obtain the following two sets of conditions for the existence of a center

(23) $a = [2cu(u^2 - 1) - 4fu^2]/(u^2 - 1)^2$, $k = (a - 1)(A_1 + A_2) + g$, $l = -b$, $b = [(f+1)(10u^2-u^4-1)+2c(u-u^3)]/[2(u^3-u)]$, $r = -(f+1)$, $d = [6(4f+5)u^2-(4f+9)(u^4+1)+4c(u-u^3)]/[2(u^2-1)^2]$, $g = [u^4-2(4f+11)u^2+1]/[4(u^3-u)]$, $n = -(f+2)A_1A_2-d-1$, $m = c(A_1+A_2)-A_1^2-A_1A_2-A_2^2-a+d+2$, $s = (1 - a)A_1A_2$, $q = A_1A_2(A_1 + A_2 - c) - g$, $p = (f + 2)(A_1 + A_2) + b - c$, $A_1 = (2u)/(1 - u^2)$, $A_2 = (u^4 - 6u^2 + 1)/[4u(u^2 - 1)]$;

(24) $a = [2(3f + 4)u^2 - f(u^4 + 1)]/[2(u^2 - 1)^2]$, $r = -(f + 1)$,
$$b = [2c(u - 7u^3 + 7u^5 - u^7) - (f + 1)(u^8 + 1) + 4(3f + 2)(u^6 + u^2) - 2(19f + 7)u^4]/[2u(u^2 - 1)(u^2 + 1)^2],$$
$$k = (a - 1)(A_1 + A_2) + g,$$
$$d = [28(2f + 1)(u^6 + u^2) - 3(2f + 3)(u^8 + 1) + 4c(3u - 13u^3 + 13u^5 - 3u^7) - 6(22f + 9)u^4]/[2(u^4 - 1)^2],$$
$$s = (1 - a)A_1A_2,$$
$$g = [(f + 1)(u^8 - 28u^6 - 28u^2 + 1) + 4c(u^7 - 7u^5 + 7u^3 - u) + 2(35f + 3)u^4]/[4u(u^2 + 1)^2(u^2 - 1)],$$
$$n = -(f + 2)A_1A_2 - d - 1, \quad m = c(A_1 + A_2) - A_1^2 - A_1A_2 - A_2^2 - a + d + 2,$$
$$l = -b, \quad q = A_1A_2(A_1 + A_2 - c) - g, \quad p = (f + 2)(A_1 + A_2) + b - c,$$
$$A_1 = (2u)/(1 - u^2), \quad A_2 = [(f + 1)(u^4 - 14u^2 + 1)]/[4u(u^2 - 1)] + c.$$

Assume now $g_1 \neq 0$ and $g_2 = 0$. We express g from $g_2 = 0$, A_2 from $V_{03} = 0$, c from $U_{30} = 0$ and a from $V_{22} = 0$. Then $U_{11} \equiv g_1 \neq 0$.

The case $f_2 = 0$ can be reduced to $f_1 = 0$ if we replace A_2 by A_1.

Assume $f_1f_2 \neq 0$ and $f_3 = 0$. We reduce the equations of (5.5) by c from $f_3 = 0$. Factoring we obtain that $V_{03} \equiv e_1e_2 = 0$, where

$$e_1 = (a + f + 1)(u^2 - 1) + 2(b + g - A_1)u,$$
$$e_2 = (a + f + 1)(u^2 - 1) + 2(b + g - A_2)u.$$

Let $e_1 = 0$, then $A_1 = [(a + f + 1)(u^2 - 1) + 2(b + g)u]/(2u)$. From the equations $V_{21} \equiv V_{22} = 0$, $U_{12} = 0$ and $f_3 = 0$ of (5.5) we express b, g and A_2, respectively.

If $a = 1$, then $\{U_{11} = 0, U_{30} = 0\}$ yields $f = -2$ and we obtain the following conditions for the existence of a center

(25) $a = r = 1$, $f = -2$, $k = g$, $l = -b$, $q = A_1A_2(A_1 + A_2 - c) - g$, $b = [z(z - 4u^2-2u(u^2-1)c)]/[2u(u^2-1)(u^2+1)^2]$, $p = b-c$, $d = [2u(3A_2u^4-10A_2u^2+3A_2+8u^3-8u)]/[(u^2+1)^2(1-u^2)]$, $g = [(A_2(u^2-1)+2u)z]/[(u^2+1)^2(u^2-1)]$,

$$z = u^4 - 6u^2 + 1,\ m = c(A_1 + A_2) - A_1^2 - A_1A_2 - A_2^2 + d + 1,\ n = -d - 1,$$
$$s = 0,\ A_1 = b + g,\ A_2 = [4c(u^3 - u) - u^4 + 14u^2 - 1]/[4(u^2 - 1)u].$$

If $a \neq 1$, then express c from $U_{11} = 0$ and $U_{30} \equiv (2a+f)(a-1)(u^8+1) - 4(u^6 + u^2)(2a^2 + 4af + f^2 - 2f - 1) + 2u^4(6a^2 + 15af + 2a + 12f^2 + 9f + 4) = 0$. This equation admits the following parametrization

$$a = [(u^4 - 6u^2 + 1)(w^2 + 16u^4) + w(u^8 - 8u^6 + 46u^4 - 8u^2 + 1)]/[w(u^4 - 1)^2],$$
$$f = [(u^4 - 6u^2 + 1)(4u^2 - 2w) - 2w^2]/[w(u^2 + 1)^2], w \neq 0.$$

In this case we obtain the following conditions for the existence of a center

$$\begin{aligned}
&(26)\quad a = [(u^4-6u^2+1)(w^2+16u^4)+w(u^8-8u^6+46u^4-8u^2+1)]/[w(u^4-1)^2],\\
&b = [2u(w - 4u^2)(u^4 + 10u^2 - 2w + 1)]/[(u^2 + 1)^2(u^2 - 1)w],\\
&c = [2u^{10}(u^4 + 6u^2 - 7w - 113) + u^8(w^2 + 88w + 552) - 2u^2(4w^2 + 7w - 1)\\
&\quad - 2u^6(4w^2 + 154w + 113) + 2u^4(23w^2 + 44w + 6) + w^2]\\
&\quad \div [uw(u^2 - 1)(u^4 - 6u^2 + 1)(u^2 + 1)^2],\\
&f = [(u^4 - 6u^2 + 1)(4u^2 - 2w) - 2w^2]/[w(u^2 + 1)^2],\\
&g = [(u^4(12u^2 - w + 56) + u^2(12 - 10w) - w)(4u^2 - w)]/[2(u^2 + 1)^2(u^2 - 1)uw],\\
&d = [(2a + f)(u^6 - 1) + 2(b - c + 2g)(u^5 + u) + (2a - 5f)(u^2 - u^4)\\
&\quad + 4(3b + c)u^3]/[4u^2(u^2 - 1)],\\
&q = (A_1 + A_2 - c)A_1A_2 - g, \quad r = -(f + 1),\\
&m = -A_1^2 - A_1A_2 - A_2^2 + c(A_1 + A_2) - a + d + 2, \quad s = (1 - a)A_1A_2,\\
&l = -b,\ n = -(f + 2)A_1A_2 - (d + 1), \quad p = (f + 2)(A_1 + A_2) + b - c,\\
&A_1 = [2u((u^2 + 1)^2 - w)]/[(u^2 - 1)w], \quad k = (a - 1)(A_1 + A_2) + g,\\
&A_2 = [2cwu(u^2 - 1) - 4u^2(u^2 + 1)^2 - 8fwu^2 - w^2]/[2wu(u^2 - 1)].
\end{aligned}$$

The case $e_2 = 0$ can be reduced to $e_1 = 0$ if we replace A_2 by A_1.

Remark 6.3. In each of the cases (14), (20), (21), (24) the system (1.1) has three invariant straight lines. Thus, in conditions (14) besides the invariant straight lines (3.4), the system (1.1) has one more invariant straight line $L_3 = 1 + (d + 1)y$; in conditions (20): $L_3 = 1 + gx$; in conditions (21): $L_3 = 1 - 2bx$; in conditions (24):

$$\begin{aligned}
L_3 &= (u^8 + 1)(f + 1)x + 4(u^7 - u)(cx - fy - y + 1) - 4(u^6 + u^2)(4cy + 5fx + 3x)\\
&\quad + 4(u^3 - u^5)(7cx - 15fy - 7y - 1) + 2u^4(16cy + 43fx + 19x).
\end{aligned}$$

Remark 6.4. In each of the cases (12), (15), (16), (17), (18), (19), (25), (26) the system (1.1) has four invariant straight lines. Thus, in conditions (12) besides the invariant straight lines (3.4), the system (1.1) has two more invariant straight lines $L_{3,4} = 2 + (g \pm \sqrt{g^2 + 4})x + 2y = 0$; in conditions (15): $L_3 = (u^2 - 1)(bx - 1) - 2buy$, $L_4 = (u^6 - 1)x - 2(u^5 + u)(cx + 3y + 1) + (u^4 - u^2)(8cy - 15x) + 4u^3(3cx + 5y - 1)$; in conditions (16): $L_3 = [2(a - 1)ux - (fy + y + 1)(u^2 - 1)](u^2 + 1)^2 - (u^2 - 1)(fz - 8au^2)$,

$$\begin{aligned}
L_4 &= (u^{10} - 1)(f + 1)^2x - 2(u^9 + u)(f + 1)(fy + y + 1) + (u^2 - u^8)[4a(3f + 2)\\
&\quad + (3f + 1)^2]x + 8(u^7 + u^3)(3afy + 2ay + 2a + 2f^2y + fy + 2f)\\
&\quad + 2(u^6 - u^4)(16a^2 + 26af - 4a + 11f^2 - 4f - 1)x + 4u^5(1 - 16a^2y\\
&\quad - 20afy + 8ay - 8a - 7f^2y + 6fy - 7f + y);
\end{aligned}$$

in conditions (17): $L_{3,4} = fx(u^8+1)-4(f+2)(u^7+3u^6x+3u^2x-u)-4(f+10)(u^5-u^3)-2(13f+24)u^4x \pm ((u^2+1)^2x+4u(u^2-1))\sqrt{A}-8(f+1)u(u^2+1)^2(u^2-1)y$, $A = f^2(u^2+1)^4+48fu^2(u^2+1)^2+64(u^4+3u^2+1)u^2$; in conditions (18): $L_{3,4} = (u^3-u)[(cx-2)(u^8+1)+4(c-2x)(u^2+1)(u^5-u)+4(cx-6)(u^6+u^2)+6(cx+14)u^4] \pm ((u^2+1)^2x+4u(u^2-1))\sqrt{A}-(2u(u^2-1)(u^2+1)^4)y$, $A = u(u^2-1)[c(u^2+1)(u^4-1)-4uz](cu(u^2+1)^2-u^6-9u^4+9u^2+1)$; in conditions (19): $L_3 = 1-2y$, $L_4 = 1+(c-a_1-a_2)x-y$; in conditions (25): $L_3 = 2ux+(u^2-1)(y-1)$,

$$\begin{aligned} L_4 &= (u^8+1)x+4(cx+y+1)(u-u^7)+4(4cy-7x)(u^6+u^2) \\ &\quad +4(7cx+23y-1)(u^5-u^3)+2(67x-16cy)u^4; \end{aligned}$$

in conditions (26): $L_3 = 2(2ux+u^2y-y)(4u^2-w)+w(u^2-1)$, $L_4 = (8u^2-w)[(4u^2-w)(u^4-6u^2+1)x+2u(u^2-1)(u^4-6u^2+2w+1)y]+2uw(u^2-1)(u^4-6u^2+1)$.

In this way we have proved the following theorem.

Theorem 6.5. *The cubic differential system* (1.1) *with at least two invariant straight lines is rationally reversible if and only if one of the sets of conditions* (1)– (26) *is satisfied.*

The proof of the main result, Theorem 1.1, follows directly from Theorems 4.5 and 6.5.

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