Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 228, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO PARABOLIC PROBLEMS WITH NONLINEAR NONLOCAL TERMS

MIGUEL LOAYZA

Abstract. We study the existence and asymptotic behavior of self-similar solutions to the parabolic problem

$$u_t - \Delta u = \int_0^t k(t,s)|u|^{p-1}u(s)ds \quad \text{on } (0,\infty)\times\mathbb{R}^N,$$

with $p>1$ and $u(0,\cdot)\in C_0(\mathbb{R}^N)$.

1. Introduction

In this work we study the existence and asymptotic behavior of global solutions of the semilinear parabolic problem

$$\begin{gathered} u_t - \Delta u = \int_0^t k(t,s)|u|^{p-1}u(s)ds \quad \text{in } (0,\infty)\times\mathbb{R}^N, \\ u(0,x) = \psi(x) \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{1.1}$$

where $p>1$ and $k:\mathcal{R}\to\mathbb{R}$ satisfies

- (K1) k is a continuous function on the region $\mathcal{R}=\{(t,s)\in\mathbb{R}^2; 0<s<t\}$,
- (K2) $k(\lambda t,\lambda s)=\lambda^{-\gamma}k(t,s)$ for all $(t,s)\in\mathcal{R}$, $\lambda>0$ and some $\gamma\in\mathbb{R}$,
- (K3) $k(1,\cdot)\in L^1(0,1)$,
- (K4) $\limsup_{\eta\to 0^+}\eta^l|k(1,\eta)|<\infty$ for some $l\in\mathbb{R}$.

Problem (1.1) models diffusion phenomena with memory effects and has been considered by several authors for some values of the function k (see [1, 4, 6, 7, 10, 12] and the references therein). When $k(t,s)=(t-s)^{-\gamma}$, $\gamma\in[0,1)$ and $\psi\in C_0(\mathbb{R}^N)$, it was shown in [4] that if

$$p>p_*=\max\{1/\gamma, 1+(4-2\gamma)/[(N-2+2\gamma)^+]\}\in(0,\infty],$$

then the solution of (1.1) is global, for $\|\psi\|_{r^*}$ small enough, where $r^*=N(p-1)/[2(2-\gamma)]$. The value p_* is the Fujita critical exponent and is not given by a scaling argument. Similar results were obtained in [6] replacing the operator $-\Delta$ by the operator $(-\Delta)^{\beta/2}$ with $0<\beta\le 2$. When the function k is nonnegative and satisfies conditions (K1)–(K4), with $\gamma<2$ and $l<1$, it was shown in [10] that if

$$p(2-\gamma)/(p-1)<N/2+a \text{ and } p(1-\gamma)<(p-1)a,$$

2000 *Mathematics Subject Classification.* 35K15, 35B40, 35E15.
Key words and phrases. Nonlocal parabolic equation; global solution; self-similar solution.

Submitted August 9, 2012. Published October 16, 2013.

where $a = \min\{1-l, 2-\gamma\}$, then (1.1) has a global solution if $\|\psi\|_{r^*}$ is sufficiently small.

It is clear that if u is a global solution of problem (1.1) then for every $\lambda > 0$, the function $u_\lambda(t,x) = \lambda^\alpha u(\lambda^2 t, \lambda x)$ satisfies

$$\begin{gathered} u_t - \Delta u = \lambda^{2[\alpha(1-p)+2-\gamma]} \int_0^t k(t,s)|u|^{p-1}u(s)ds \quad \text{in } (0,\infty)\times\mathbb{R}^N, \\ u(0,x) = \lambda^{2\alpha}\psi(\lambda x) \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{1.2}$$

In particular, if $\alpha = (2-\gamma)/(p-1)$, then u_λ is also a solution of problem (1.1). A solution satisfying $u = u_\lambda$ for all $\lambda > 0$ is called a self-similar solution of problem (1.1). Note that, in this case, $\psi(x) = \lambda^{2\alpha}\psi(\lambda x)$; that is, the function ψ is a homogeneous function of degree -2α.

Our objective is to determine the asymptotic behavior of global solutions of (1.1) in terms of the self-similar solution w corresponding to the cases (see Theorem 1.5 for details):

(i) $\alpha(p-1) = 2-\gamma$.

$$\begin{gathered} w_t - \Delta w = \int_0^t k(t,s)|w|^{p-1}w(s)ds \quad \text{in } (0,\infty)\times\mathbb{R}^N, \\ w(0,x) = |x|^{-2\alpha} \quad \text{in } \mathbb{R}^N, \end{gathered}$$

(ii) $\alpha(p-1) > 2-\gamma$.

$$\begin{gathered} w_t - \Delta w = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \\ w(0,x) = |x|^{-2\alpha} \quad \text{in } \mathbb{R}^N. \end{gathered}$$

For $\alpha(p-1) < 2-\gamma$, we show that there is no nonnegative global solution of (1.1), if $w(0,x) \sim |x|^{-2\alpha}$ for $|x|$ large enough (see Theorem 1.7 for details).

To show the existence of global solutions to (1.1) we use a contraction mapping argument on the associated integral equation

$$u(t) = e^{t\Delta}\psi + \int_0^t e^{(t-s)\Delta} \int_0^s k(s,\sigma)|u|^{p-1}u(\sigma)d\sigma ds, \tag{1.3}$$

where $(e^{t\Delta})_{t\geq 0}$ is the heat semigroup. Precisely, this contraction mapping argument is done on a given Banach space equipped with a norm chosen so that we obtain directly the global character of the solution. Our approach works for unbounded and sign changing initial data. On the other hand, the self-similar solutions constructed in this work may be not radially symmetric. In fact,we adapt a method introduced by Fujita and Kato [8, 9] and used later in [2, 3, 5, 13, 14].

Since the homogeneous function $\psi = |\cdot|^{-2\alpha}$, does not belong to any $L^p(\mathbb{R}^N)$ space, we consider initial data so that $\sup_{t>0} t^{\alpha - N/(2r_1)}\|e^{t\Delta}\psi\|_{r_1} < \infty$, for some $r_1 \geq 1$. Hence, it is necessary to consider that $\alpha < N/2$ since this condition ensures that ψ belongs to $L^1_{\rm loc}(\mathbb{R}^N)$.

The following result determines the asymptotic behavior for the heat semigroup on homogeneous functions, see [5, page 118] and [13, Proposition 2.3].

Proposition 1.1. *Let $r_1 > N/(2\alpha) > 1$, $\beta_1 = \alpha - N/(2r_1)$ and let φ_h be a tempered distribution homogeneous of degree -2α such that $\varphi_h(x) = \mu(x)|x|^{-2\alpha}$, where $\mu \in L^{r_1}(S^{N-1})$ is a function homogeneous of degree 0. Assume that η is a cut-off function, that is, identically 1 near the origin and of compact support. Then*

(i) $\sup_{t>0} t^{\beta_1}\|e^{t\Delta}\varphi_h\|_{r_1} < \infty$;
(ii) $\sup_{t>0} t^{\beta_1+\delta}\|e^{t\Delta}(\eta\varphi_h)\|_{r_1} < \infty$ *for* $0 < \delta < N/2 - \alpha$;
(iii) $\sup_{t>0} t^{\beta_1}\|e^{t\Delta}(1-\eta)\varphi_h\|_{r_1} < \infty$.

Our first result is technical. It will be used to formulate the global existence and asymptotic behavior results.

Proposition 1.2. *Let* $l < 1, \gamma < 2$ *and set* $a = \min\{1-l, 2-\gamma\}$. *Assume that* $\alpha \in (0, N/2)$ *satisfies*

$$2-\gamma+\alpha < \frac{N}{2} + a, \tag{1.4}$$

$$(2-\gamma+\alpha)\frac{1-\gamma}{2-\gamma} < a. \tag{1.5}$$

Then, there exits $r_1 \geq 1$ *satisfying*
(i) $r_1 > \frac{N}{2\alpha}(2-\gamma), r_1 > \frac{2-\gamma}{\alpha} + 1$ *and* $r_1 > \frac{N}{2\alpha}$.
(ii) $(2-\gamma+\alpha)(1-\frac{N}{2r_1\alpha}) < a$.

We now give the following existence result for problem (1.1) shows the existence of global solutions and its continuous dependence.

Theorem 1.3. *Let* $p > 1$ *and* k *satisfying conditions* $K1) - K4)$ *with* $\gamma < 2$ *and* $l < 1$. *Assume*

$$p > 1 + 2(2-\gamma)/N \tag{1.6}$$

and $\alpha \in (0, N/2)$ *satisfying* (1.4), (1.5) *and*

$$\frac{2-\gamma}{p-1} \leq \alpha < \frac{N}{2}. \tag{1.7}$$

Fix $\tilde{\alpha} > 0$ *such that*

$$\tilde{\alpha} \leq \frac{2-\gamma}{p-1}. \tag{1.8}$$

Let $r_1 > 1$ *be given by Proposition 1.2, and let* $r_2 > 1$ *be defined by* $r_2 = \alpha r_1/\tilde{\alpha}$. *For every* $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ *define* $\mathcal{N}$ *by*

$$\mathcal{N}(\varphi) = \sup_{t>0}\{t^{\beta_1}\|e^{t\Delta}\varphi\|_{r_1}, t^{\beta_2}\|e^{t\Delta}\varphi\|_{r_2}\}, \tag{1.9}$$

where $\beta_1 = \alpha - N/(2r_1)$ *and* $\beta_2 = \tilde{\alpha} - N/(2r_2)$.

Let $M > 0$ *be such that* $C = C(M) < 1$, *where* C *is a positive constant given by* (2.10). *Choose* $R > 0$ *such that* $R + CM \leq M$. *If* φ *is a tempered distribution such that*

$$\mathcal{N}(\varphi) \leq R, \tag{1.10}$$

then there exits a unique global solution u *of* (1.1) *satisfying*

$$\sup_{t>0}\{t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|u(t)\|_{r_2}\} \leq M.$$

In addition, if φ, ψ *satisfy* (1.10) *and if* u_φ *and* u_ψ *respectively are the solutions of* (1.3) *with initial data* φ, ψ, *then*

$$\sup_{t>0}[t^{\beta_1}\|u_\varphi(t) - u_\psi(t)\|_{r_1}, t^{\beta_2}\|u_\varphi(t) - u_\psi(t)\|_{r_2}] \leq (1-C)^{-1}\mathcal{N}(\varphi - \psi). \tag{1.11}$$

Moreover, if φ, ψ *are such that*

$$\mathcal{N}_\delta(\varphi - \psi) = \sup_{t>0}\{t^{\beta_1+\delta}\|e^{t\Delta}(\varphi-\psi)\|_{r_1}, t^{\beta_2+\delta}\|e^{t\Delta}(\varphi-\psi)\|_{r_2}\} < \infty, \tag{1.12}$$

for some $\delta \in (0,\delta_0)$*, where* $\delta_0 = 1 - l - (2-\gamma+\alpha)[1 - N/(2r_1\alpha)] > 0$*. Then*

$$\sup_{t>0}\{t^{\beta_1+\delta}\|u_\varphi - u_\psi\|_{r_1}, t^{\beta_2+\delta}\|u_\varphi - u_\psi\|_{r_2}\} \le (1-C_\delta)^{-1}\mathcal{N}_\delta(\varphi-\psi), \tag{1.13}$$

where C_δ *is given by* (2.16) *below and the constant* $M > 0$ *is chosen small enough so that* $C_\delta < 1$*.*

Remark 1.4. Suppose that $\alpha(p-1) = 2-\gamma$ in Theorem 1.3.

(i) From (1.7) and (1.8), we see that it is possible to choose $\tilde{\alpha} = \alpha$. It follows that $r_1 = r_2$, $\beta_1 = \beta_2$. Therefore, Theorem 1.3 holds replacing the norm $\mathcal{N}$ of (1.9) by $\mathcal{N}_s(\varphi) := \sup_{t>0}\{t^{\beta_1}\|e^{t\Delta}\varphi\|_{r_1}\}$.

(ii) Assume that $k(t,s) = (t-s)^{-\gamma}$ with $\gamma \in (0,1)$. Then k satisfies $K1) - K4)$ with $l = 0$, and therefore $a = \min\{1-l, 2-\gamma\} = 1$. From conditions (1.4)-(1.7) we have that $p(N-2+2\gamma) > N+2$, $p\gamma > 1$ and $p > 1 + 2(2-\gamma)/N$ respectively. Since $p > 1 + (4-2\gamma)/[(N-2+2\gamma)^+] > 1 + 2(2-\gamma)/N$, we conclude that $p > p^* = \max\{1/\gamma, 1 + (4-2\gamma)/[(N-2+2\gamma)^+]\}$ which coincides with the condition encountered in [4].

(iii) Conditions (1.4)-(1.6) become $2(2-\gamma)p < (N+2a)(p-1)$, $p(1-\gamma) < a(p-1)$ and $p > 1 + 2(2-\gamma)/N$ respectively. The last inequality is obtained from the first one, since $2(2-\gamma)p < (N+2a)(p-1) \le [N + 2(1-l)](p-1)$ and $\gamma < 2$. Indeed, $p > 1 + 2(2-\gamma)/[N-2+2(\gamma-l)^+] > 1 + 2(2-\gamma)/N$. These conditions were used in [10] to show global existence of (1.1).

We now state the following asymptotic behavior result for some global solution of problem (1.1) with small initial data with respect to the norm $\mathcal{N}$ given by (1.9).

Theorem 1.5 (Asymptotically self-similar solutions)**.** *Let* $p > 1$ *satisfying* (1.6) *and* k *be a function satisfying conditions* $K1) - K4)$ *with* $\gamma < 2$ *and* $l < 1$*. Let* $\alpha \in (0, N/2)$ *be satisfying* (1.4)*,* (1.5) *and* (1.7)*,* $\tilde{\alpha} > 0$ *satisfying* (1.8)*,* r_1 *given by Proposition 2 and* $r_2 = \alpha r_1/\tilde{\alpha}$*. Set* $\varphi_h(x) = \mu(x)|x|^{-2\alpha}$*, where* μ *is homogeneous of degree* 0 *and* $\mu \in L^{r_1}(S^{N-1})$*.*

Suppose that $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ *satisfies* (1.10)*,* u *is the corresponding solution of* (1.1) *given by Theorem 1.3, and*

$$\sup_{t>0} t^{\beta_1+\delta}\|e^{t\Delta}(\varphi - \varphi_h)\|_{r_1} < \infty \tag{1.14}$$

for some $\delta \in (0,\delta_0)$*, where* $\delta_0 = 1 - l - p(2-\gamma)/(p-1) + Np/(2r_1)$ *when* $\alpha(p-1) = 2-\gamma$ *and given by Lemma 3.1 when* $\alpha(p-1) > 2-\gamma$*. We have the following:*

- (i) *If* $\alpha(p-1) > 2-\gamma$*, then* $\sup_{t>0} t^{\beta_1+\delta}\|u(t) - e^{t\Delta}\varphi_h\|_{r_1} \le C_\delta$*, for some constant* $C_\delta > 0$*.*
- (ii) *If* $\alpha(p-1) = 2-\gamma$ *and* w *is the solution of* (1.1) *given by Theorem 1.3 with initial data* φ_h*(we multiplied* φ_h *by a small constant so that* (1.10) *is satisfied), then* w *is self-similar and* $\sup_{t>0} t^{\beta_1+\delta}\|u(t) - w(t)\|_{r_1} \le C_\delta$ *for some constant* $C_\delta > 0$*.*

Remark 1.6. The class of functions φ satisfying the condition (1.14) is nonempty. Indeed, from Proposition 1.1(2), condition (1.14) is satisfied for $\varphi = (1-\eta)\varphi_h$.

In the following result, we analyze the non existence of global solutions of problem (1.1), under the assumption

(K5) There exist $T>0$ and a nonnegative, non-increasing continuous function $\phi\in C([0,\infty))$ with integrable derivative such that $\phi(0)=1$ and $\phi(t)=0$ for $t\ge T$ satisfying $k(\cdot,t)\phi(\cdot)\in L^1(t,T)$ for $t>0$ and
$$\int_0^T \phi(t)^{p'}\Big(\int_t^T k(s,t)\phi(s)ds\Big)^{-p'/p}dt<\infty, \tag{1.15}$$
where p' is the conjugate of p.

Theorem 1.7. *Let $p>1$ and let k be a nonnegative function satisfying conditions* (K1)–(K3), (K5). *If $\psi\in C_0(\mathbb{R}^N),\psi\ge 0$ satisfies $\liminf_{|x|\to\infty}|x|^{2(2-\gamma)/(p-1)}\psi(x)=\infty$ and u is a corresponding nonnegative solution of problem* (1.1)*, then u is not a global solution.*

Remark 1.8. Regarding Theorem 1.7 we have the following statements:

(i) Under conditions (K1)–(K3), existence of local solutions for (1.1) in the class $C([0,T),C_0(\mathbb{R}^N))$ and initial data $\psi\in C_0(\mathbb{R}^N)$, were studied in [10]. In particular, we know that if k and ψ are nonnegative, then the solution of (1.1) is nonnegative.

(ii) Let $k(t,s)=(t-s)^{-\gamma_1}s^{-\gamma_2}$ for $0<s<t$ and $\gamma_i\in[0,1), i=1,2$. Clearly, k satisfies $K1)-K3)$. We show that k satisfies (K5) with $\phi(t)=[(1-t)^+]^q$, $t\ge 0$, $T=1$ and $q>1/(p-1)$. Indeed, since $\phi\le 1$, we have for $t>0$
$$\int_t^1 k(s,t)\phi(s)ds=t^{-\gamma_2}\int_t^1(s-t)^{-\gamma_1}\phi(s)ds\le\frac{t^{-\gamma_2}}{1-\gamma_1}(1-t)^{1-\gamma_1}<\infty.$$
On the other hand,
$$\int_t^1 k(s,t)\phi(s)ds=t^{-\gamma_2}\int_t^1(s-t)^{-\gamma_1}\phi(s)ds\ge t^{-\gamma_2}\int_t^1\phi(s)ds=\frac{t^{-\gamma_2}}{1+q}(1-t)^{1+q}.$$
Therefore,
$$\int_0^1\phi(t)^{p'}\Big(\int_t^1 k(s,t)\phi(s)ds\Big)^{-p'/p}dt=(1+q)^{p'/p}\int_0^1(1-t)^{p'(q-\frac{1+q}{p})}t^{\frac{\gamma_2 p'}{p}}dt,$$
which is finite, since
$$1+p'(q-\frac{1+q}{p})=\frac{p'}{p}[p-2+q(p-1)]>\frac{p'}{p}(p-1)>0.$$

2. Existence of global solutions

Proof of Proposition 1.2. Let $A=\frac{2\alpha}{N}(1-\frac{a}{2-\gamma+\alpha})$. Since $a>0$ we conclude that $A<2\alpha/N<1$. From (1.5) and (1.4) we have $A<2\alpha/[N(2-\gamma)]$ and $A<\alpha/(2-\gamma+\alpha)$, respectively. Now, it is sufficient to choose $r_1>1$ satisfying $A<\frac{1}{r_1}<\min\{\frac{2\alpha}{N(2-\gamma)},\frac{\alpha}{2-\gamma+\alpha},\frac{2\alpha}{N}\}$.

Lemma 2.1. *Assume the conditions* (1.4)-(1.8). *Let $r_2=\frac{\alpha r_1}{\tilde\alpha}$, $\beta_1=\alpha-\frac{N}{2r_1}$, $\beta_2=\tilde\alpha-\frac{N}{2r_2}$, $\frac{1}{\eta_1}=\frac{1}{pr_1}(\frac{2-\gamma}{\alpha}+1)$, $\frac{1}{\eta_2}=\frac{1}{pr_1}(\frac{2-\gamma}{\alpha}+\frac{\tilde\alpha}{\alpha})$, $\theta_1=\frac{2-\gamma+\alpha-p\tilde\alpha}{p(\alpha-\tilde\alpha)}$, and $\theta_2=\frac{2-\gamma+(1-p)\tilde\alpha}{p(\alpha-\tilde\alpha)}$. For $i=1,2$ we have*

(i) *$\eta_i\in[r_1,r_2]$ and $\eta_i\in(p,r_ip)$.*
(ii) $\frac{p}{\eta_1}-\frac{1}{r_1}=\frac{p}{\eta_2}-\frac{1}{r_2}=\frac{2-\gamma}{r_1\alpha}<\frac{2}{N}$.
(iii) $\theta_i\in[0,1]$, $\frac{1}{\eta_i}=\frac{\theta_i}{r_1}+\frac{(1-\theta_i)}{r_2}$.

(iv) $\frac{1}{p}a > \theta_i\beta_1 + (1-\theta_i)\beta_2$, *with*

$$\theta_1\beta_1 + (1-\theta_1)\beta_2 = \frac{1}{p}(2-\gamma+\alpha)(1-\frac{N}{2r_1\alpha}),$$
$$\theta_2\beta_1 + (1-\theta_2)\beta_2 = \frac{1}{p}(2-\gamma+\tilde{\alpha})(1-\frac{N}{2r_1\alpha}).$$

(v) $2-\gamma+\beta_i - \frac{N}{2}(\frac{p}{\eta_i}-\frac{1}{r_i}) - p[\beta_1\theta_i + \beta_2(1-\theta_i)] = 0$.

Proof. (i) From (1.7), we see that $\eta_1 \geq r_1$ and $\eta_2 \leq r_2$. Since $\tilde{\alpha} \leq \alpha$, it follows from (1.7) and (1.8) that

$$2-\gamma+\tilde{\alpha} \leq p\alpha, \;\; p\tilde{\alpha} \leq 2-\gamma+\alpha \tag{2.1}$$

respectively. From here, $\eta_2 \geq r_1$ and $\eta_1 \leq r_2$. The condition $r_1 > (2-\gamma)/\alpha + 1$ of Proposition 1.2(i) and $\gamma < 2$ ensure that $\eta_1 \in (p, r_1 p)$. Moreover, since $r_1 > (2-\gamma)/\alpha + 1 \geq (2-\gamma+\tilde{\alpha})/\alpha$ and $\gamma < 2$, we conclude that $\eta_2 \in (p, r_2 p)$.

Item (ii) follows from Proposition 1.2(i).

(iii) From (1.7) and (1.8) we get $\theta_1 \leq 1$ and $\theta_2 \geq 0$ respectively, and from (2.1) we see that $\theta_2 \leq 1$ and $\theta_1 \geq 0$ respectively.

We obtain (iv) from Proposition 1.2(ii). □

Proof of Theorem 1.3. The proof is based on a contraction mapping argument. Let E be the set of Bochner measurable functions $u : (0,\infty) \to L^{r_1}(\mathbb{R}^N) \cap L^{r_2}(\mathbb{R}^N)$, such that $\|u\|_E = \sup_{t>0}\{t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|u(t)\|_{r_2}\} < \infty$, where $\beta_1 = \alpha - N/(2r_1), \beta_2 = \tilde{\alpha} - N/(2r_2)$. The space E is a Banach space. Let $M > 0$ and K be the closed ball of radius M in E.

Let $\Phi_\varphi : K \to E$ be the mapping defined by

$$\Phi_\varphi(u)(t) = e^{t\Delta}\varphi + \int_0^t e^{(t-s)\Delta} \int_0^s k(s,\sigma)|u|^{p-1}u(\sigma)d\sigma ds. \tag{2.2}$$

We will prove that Φ_φ is a strict contraction mapping on K. Let φ, ψ satisfying (1.10) and $u, v \in K$. We will use several times the smoothing effect for the heat semigroup: if $1 \leq s \leq r \leq \infty$ and $\varphi \in L^r$, then

$$\|e^{t\Delta}\varphi\|_r \leq t^{-\frac{N}{2}(\frac{1}{s}-\frac{1}{r})}\|\varphi\|_s$$

for all $t > 0$. From (2.2), we deduce

$$\begin{aligned}
&t^{\beta_1}\|\Phi_\varphi(u)(t) - \Phi_\psi(v)(t)\|_{r_1} \leq t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1} \\
&+ pt^{\beta_1}\int_0^t \|e^{(t-s)\Delta}\int_0^s |k(s,\sigma)|(|u|^{p-1}+|v|^{p-1})|u(\sigma)-v(\sigma)|\|_{r_1} d\sigma \\
&\leq t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1} + pt^{\beta_1}\int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{\eta_1}-\frac{1}{r_1})} \\
&\quad \times \int_0^s |k(s,\sigma)|(\|u\|_{\eta_1}^{p-1} + \|v\|_{\eta_1}^{p-1})\|u(\sigma)-v(\sigma)\|_{\eta_1} d\sigma ds.
\end{aligned} \tag{2.3}$$

From Lemma 2.1,(i) and (iii), and an interpolation inequality

$$\|u\|_{\eta_1} \leq \|u\|_{r_1}^{\theta_1}\|u\|_{r_2}^{1-\theta_1}$$

where $\frac{1}{\eta_1} = \frac{\theta_1}{r_1} + \frac{1-\theta_1}{r_2}$. Replacing this inequality into (2.3) we obtain

$$\begin{aligned} &t^{\beta_1}\|\Phi_\varphi(u)(t) - \Phi_\psi(v)(t)\|_{r_1} \\ &\le t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1} + 2M^{p-1}p\|u-v\|_E t^{\beta_1} \\ &\quad\times \int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{\eta_1}-\frac{1}{r_1})} \int_0^s |k(s,\sigma)|\sigma^{-p[\theta_1\beta_1+(1-\theta_1)\beta_2]} d\sigma ds. \end{aligned} \tag{2.4}$$

From (K4), there exist $\eta_0, \nu > 0$ such that $\eta^l|k(1,\eta)| < \nu$ for $\eta \in (0,\eta_0)$. Thus, if $\theta_1\beta_1 + \beta_2(1-\theta_1) = \Theta_1$, we have

$$\begin{aligned} \int_0^s |k(s,\sigma)|\sigma^{-p\Theta_1} d\sigma &= s^{1-\gamma-p\Theta_1} \int_0^1 |k(1,\sigma)|\sigma^{-p\Theta_1} d\sigma \\ &\le s^{1-\gamma-p\Theta_1}\Big[\nu \int_0^{\eta_0} \sigma^{-l-p\Theta_1} d\sigma + \eta_0^{-p\Theta_1} \int_{\eta_0}^1 |k(1,\sigma)| d\sigma\Big] \\ &= C_1 s^{1-\gamma-p\Theta_1}, \end{aligned} \tag{2.5}$$

where

$$C_1 = \nu \int_0^{\eta_0} \sigma^{-l-p\Theta_1} d\sigma + \eta_0^{-p\Theta_1} \int_{\eta_0}^1 |k(1,\sigma)| d\sigma. \tag{2.6}$$

Since $p\Theta_1 < a$ (see Lemma 2.1(iv)) and k satisfies (K3), we conclude that $C_1 < \infty$.

From (2.4), (2.5) and properties (iv) and (v) of Lemma 2.1,

$$\begin{aligned} &t^{\beta_1}\|\Phi_\varphi u(t) - \Phi_\psi v(t)\|_{r_1} \\ &\le t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1} \\ &\quad + 2C_1 M^{p-1} p t^{\beta_1}\|u-v\|_E \int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{\eta_1}-\frac{1}{r_1})} s^{1-\gamma-p\Theta_1} ds \\ &\le t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1} + C_1'\|u-v\|_E, \end{aligned} \tag{2.7}$$

where $C_1' = 2C_1M^{p-1}p\int_0^1(1-s)^{-\frac{N}{2}(\frac{p}{\eta_1}-\frac{1}{r_1})} s^{1-\gamma-p\Theta_1} ds$. From Lemma 2.1, (ii) and (iv), we see that $C_1' < \infty$. Similarly, one can prove that

$$t^{\beta_2}\|\Phi_\varphi u(t) - \Phi_\psi v(t)\|_{r_2} \le t^{\beta_2}\|e^{t\Delta}(\varphi-\psi)\|_{r_2} + C_2'\|u-v\|_E, \tag{2.8}$$

where

$$\begin{gathered} C_2' = 2C_2M^{p-1}p\int_0^1 (1-s)^{-\frac{N}{2}(\frac{p}{\eta_2}-\frac{1}{r_2})} s^{1-\gamma-p\Theta_2} ds < \infty, \\ C_2 = \nu \int_0^{\eta_0} \sigma^{-l-p\Theta_2} d\sigma + \eta_0^{-p\Theta_1} \int_{\eta_0}^1 |k(1,\sigma)| d\sigma < \infty, \\ \Theta_2 = \frac{1}{p}(1 - \frac{N}{2r_1\alpha})(2-\gamma+\tilde{\alpha}). \end{gathered}$$

From (2.7) and (2.8) we obtain

$$\|\Phi_\varphi(u)(t) - \Phi_\psi(v)(t)\|_E \le \mathcal{N}(\varphi-\psi) + C\|u-v\|_E, \tag{2.9}$$

where

$$C = \max\{C_1', C_2'\}. \tag{2.10}$$

Setting $\psi = 0, v = 0$ in (2.9) we get $\|\Phi_\varphi(u)\|_E \le \mathcal{N}(\varphi) + C\|u\|_E$. Since φ satisfies (1.10) and $R + CM \le M$, we conclude that $\Phi_\varphi u \in K$. Moreover, since $C < 1$ we conclude from (2.9) that Φ_φ is a strict contraction from K into itself, so Φ_φ has a unique fixed point in K.

The continuous dependence (1.11) follows clearly from (2.9). To show (1.13), let

$$\|u-v\|_{E,\delta} = \sup_{t>0}\{t^{\beta_1+\delta}\|u(t)\|_{r_1}, t^{\beta_2+\delta}\|v(t)\|_{r_2}\}. \tag{2.11}$$

Proceeding as (2.3) we obtain

$$\begin{aligned}
&t^{\beta_1+\delta}\|u(t)-v(t)\|_{r_1}\\
&\le t^{\beta_1+\delta}\|e^{t\Delta}(\varphi-\psi)\|_{r_1} + 2pM^{p-1}t^{\beta_1+\delta}\int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{\eta_1}-\frac{1}{r_1})}\\
&\quad\times\int_0^s |k(s,\sigma)|\sigma^{-[\theta_1\beta_1+(1-\theta_1)\beta_2](p-1)}\|u-v\|_{\eta_1}d\sigma ds\\
&\le t^{\beta_1+\delta}\|e^{t\Delta}(\varphi-\psi)\|_{r_1} + 2pM^{p-1}\sup_{\sigma\in(0,t)}\{\sigma^{\beta_1+\delta}\|u(\sigma)\|_{r_1}, \sigma^{\beta_2+\delta}\|v(\sigma)\|_{r_2}\}\\
&\quad\times t^{\beta_1+\delta}\int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{\eta_1}-\frac{1}{r_1})}\int_0^s |k(s,\sigma)|\sigma^{-p[\theta_1\beta_1+(1-\theta_1)\beta_2]-\delta}d\sigma dt
\end{aligned} \tag{2.12}$$

For $0<\delta<1-l-p\Theta_1$, arguing as in (2.5), we have

$$\begin{aligned}
&\int_0^s |k(s,\sigma)|\sigma^{-p[\theta_1\beta_1+\theta_2\beta_2]-\delta}d\sigma\\
&= s^{1-\gamma-p\Theta_1-\delta}\int_0^1 |k(1,\sigma)|\sigma^{-p\Theta_1-\delta}\\
&\le s^{1-\gamma-p\Theta_1-\delta}\Big[\nu\int_0^{\eta_0}\sigma^{-l-p\Theta_1-\delta}d\sigma + \eta_0^{-p\Theta_1-\delta}\int_{\eta_0}^1 |k(1,\sigma)|d\sigma\Big]\\
&= C_{1,\delta}s^{1-\gamma-p\Theta_1-\delta},
\end{aligned}$$

where

$$C_{1,\delta} = \nu\int_0^{\eta_0}\sigma^{-l-p\Theta_1-\delta}d\sigma + \eta_0^{-p\Theta_1-\delta}\int_{\eta_0}^1 |k(1,\sigma)|d\sigma < \infty.$$

Therefore, from (2.12) we obtain

$$\begin{aligned}
&t^{\beta_1+\delta}\|u(t)-v(t)\|_{r_1}\\
&\le t^{\beta_1+\delta}\|e^{t\Delta}(\varphi-\psi)\|_{r_1} + C'_{1,\delta}\sup_{\sigma\in(0,t)}\{\sigma^{\beta_1+\delta}\|u(\sigma)\|_{r_1}, \sigma^{\beta_2+\delta}\|v(\sigma)\|_{r_2}\},
\end{aligned} \tag{2.13}$$

and

$$C'_{1,\delta} = 2pM^{p-1}C_{1,\delta}. \tag{2.14}$$

Similarly, for $0<\delta<1-l-p\Theta_2$, one can to obtain

$$\begin{aligned}
&t^{\beta_2+\delta}\|u(t)-v(t)\|_{r_2}\\
&\le t^{\beta_2+\delta}\|e^{t\Delta}\varphi-\psi\|_{r_2} + C'_{2,\delta}\sup_{\sigma\in(0,t)}\{\sigma^{\beta_1+\delta}\|u(\sigma)\|_{r_1}, \sigma^{\beta_2+\delta}\|v(\sigma)\|_{r_2}\},
\end{aligned} \tag{2.15}$$

where

$$C_{2,\delta} = \nu\int_0^{\eta_0}\sigma^{-l-p\Theta_2-\delta}d\sigma + \eta_0^{-p\Theta_2-\delta}\int_{\eta_0}^1 k(1,\sigma)d\sigma < \infty$$

and $C'_{2,\delta} = 2pM^{p-1}C_{2,\delta}$.

From (2.13) and (2.15) it follows that

$$(1-C_\delta)\|u-v\|_{E,\delta} \le \mathcal{N}_\delta(\varphi-\psi),$$

where

$$C_\delta = \max\{C'_{1,\delta}, C'_{2,\delta}\}. \tag{2.16}$$

□

3. Asymptotic behavior

The next result will be used in the proof of Theorem 1.5(1).

Lemma 3.1. *Let $l < 1$, $\gamma < 2$, $p > 1$ and $a = \min\{1-l, 2-\gamma\}$. Assume (1.7) and let α satisfying (1.4), (1.5) and (1.7). Let $\tilde{\eta}$ satisfying (1.8). For $\delta > 0$ we define $\eta' \geq 1$ by $\frac{1}{\eta'_1} = \frac{1}{pr_1}\big(\frac{2-\gamma+\delta}{\alpha}+1\big)$, and $\theta'_1 = \frac{2-\gamma+\delta+\alpha-p\tilde{\alpha}}{p(\alpha-\tilde{\alpha})}$.*

If $\frac{2-\gamma}{p-1} < \alpha$, then there exists $\delta_0 > 0$ small such that for all $\delta \in (0, \delta_0]$:

(i) *$\eta'_1 \in [r_1, r_2]$ and $\eta'_1 \in (p, r_1 p)$, where $r_2 = (\alpha r_1)/\tilde{\alpha}$.*

(ii) $\frac{N}{2}\big(\frac{p}{\eta'_1} - \frac{1}{r_1}\big) = \frac{N}{2r_1\alpha}(2-\gamma+\delta) < 1$.

(iii) $\theta_1 \in [0,1]$, $\frac{1}{\eta'_1} = \frac{\theta'_1}{r_1} + \frac{1-\theta'_1}{r_2}$.

(iv) *If $\beta_1 = \alpha - \frac{N}{2r_1}$ and $\beta_2 = \tilde{\alpha} - \frac{N}{2r_2}$, then*

$$a > p[\beta_1\theta'_1 + \beta_2(1-\theta'_1)] = (2-\gamma+\alpha+\delta)\Big(1 - \frac{N}{2r_1\alpha}\Big).$$

(v) $2-\gamma+\beta_1+\delta - \frac{N}{2}\big(\frac{p}{\eta'_1} - \frac{1}{r_1}\big) - p[\beta_1\theta'_1 + \beta_2(1-\theta'_1)] = 0$.

Proof. Since $\alpha > (2-\gamma)/(p-1)$, (1.4) and (1.5) hold, it follows from Proposition 1.2 that there exists $\delta_0 > 0$ small so that such that $\alpha > (2-\gamma+\delta_0)/(p-1)$, $r_1 > \frac{N}{2\alpha}(2-\gamma+\delta_0)$, $r_1 > (2-\gamma+\delta_0)/\alpha + 1$ and $(2-\gamma+\alpha+\delta_0)(1-N/(2r_1\alpha)) < a$. The rest of the proof follows similarly as the proof of Lemma 2.1. □

Proof of Theorem 1.5. (i) Let $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ be satisfying (1.10) and let u be the corresponding solution given by Theorem 1.3. We have that

$$\sup_{t>0}\{t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|u(t)\|_{r_2}\} \leq M.$$

Arguing as in (2.12), (2.5) and (2.6), we conclude conclude that

$$\begin{aligned} t^{\beta_1+\delta}\|u(t) - e^{t\Delta}\varphi_h\|_{r_1} &\leq t^{\beta_1+\delta}\|e^{t\Delta}(\varphi-\varphi_h)\|_{r_1} + 2pM^p t^{\beta_1+\delta}\int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{\eta'_1}-\frac{1}{r_1})} \\ &\quad\times \int_0^s |k(s,\sigma)|\sigma^{-p[\theta'_1\beta_1+(1-\theta'_1)\beta_2]}d\sigma dt \\ &\leq t^{\beta_1+\delta}\|e^{t\Delta}(\varphi-\varphi_h)\|_{r_1} + C'_\delta, \end{aligned}$$

where $C'_\delta = 2pM^pC_\delta$, $C_\delta = \nu\int_0^{\eta_0}\sigma^{-l-p\Theta_\delta}d\sigma + \eta_0^{-p\Theta_\delta}\int_{\eta_0}^1 |k(1,\sigma)|d\sigma$ and $\Theta_\delta = \frac{1}{p}(1-\frac{N}{2}\frac{P_1}{r_1})(2-\gamma+\frac{1}{P_1}+\delta)$. From the above result and (1.14) we have the desired conclusion.

(ii) For $\lambda > 0$, we define $z(t,x) = \lambda^{(4-2\gamma)/(p-1)}w(\lambda^2 t, \lambda x)$ for all $t > 0, x \in \mathbb{R}^N$. Clearly z is a solution of (1.1). We claim that $\sup_{t>0} t^{\beta_1}\|z\|_{r_1} \leq M$. To see this, we observe that

$$\begin{aligned} t^{\beta_1}\|z\|_{r_1} &= t^{\beta_1}\lambda^{\frac{4-2\gamma}{p-1}}\|w(\lambda^2 t, \lambda\cdot)\|_{r_1} \\ &= t^{\beta_1}\lambda^{\frac{4-2\gamma}{p-1}-\frac{N}{r_1}}\|w(\lambda^2 t)\|_{r_1} \\ &= (\lambda^2 t)^{\beta_1}\|w(\lambda^2 t)\|_{r_1}. \end{aligned}$$

Since $z(0) = \varphi_h$, we have from (1.11) that $w = z$; that is, w is self-similar. The conclusion now follows from (1.13) and the Remark 1.4(i). $\square$

4. Non existence of global solutions

Proof of Theorem 1.7. Let B_R be the open ball in $\mathbb{R}^N$ with radius $R > 0$. Let $\lambda_R > 0$ and $\rho_R > 0$ be the first eigenvalue and the first normalized (i.e. $\int_{B_R} \rho_R = 1$) eigenfunction of $-\Delta$ on B_R with zero Dirichlet boundary condition.

Set $w_R(t) = \int_{B_R} u(t)\rho_R$. Then by Green's identity and Jensen's inequality we obtain

$$(w_R)_t + \lambda_R w_R \geq \int_0^t k(t,s) w_R^p(s) ds. \tag{4.1}$$

Set $\phi_R(t) = \phi(t/R^2)$ for all $t \geq 0$. Multiplying (4.1) by ϕ_R and integrating on $[0, TR^2]$, we have

$$\begin{aligned} -w_R(0) + \lambda_R \int_0^{TR^2} w_R(t)\phi_R(t) dt &\geq \int_0^{TR^2} \int_0^t k(t,s) w_R^p(s) ds\, \phi_R(t) dt \\ &= \int_0^{TR^2} I_R(s) w_R^p(s) ds, \end{aligned} \tag{4.2}$$

where

$$I_R(s) = \int_s^{TR^2} k(t,s)\phi_R(t) dt.$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} \int_0^{TR^2} w_R(t)\phi_R(t) dt &= \int_0^{TR^2} w_R(t) I_R(t)^{1/p} I_R(t)^{-1/p} \phi_R(t) dt \\ &\leq \Big\{ \int_0^{TR^2} w_R^p I_R(t) dt \Big\}^{1/p} \underbrace{\Big\{ \int_0^{TR^2} I_R(t)^{-p'/p} \phi_R^{p'}(t) dt \Big\}^{1/p'}}_{II}. \end{aligned} \tag{4.3}$$

Since

$$I_R(R^2 s) = \int_{R^2 s}^{TR^2} k(t,s)\phi(t/R^2) dt = (R^2)^{1-\gamma} \int_s^T k(t,s)\phi(t) dt = (R^2)^{1-\gamma} I_1(s),$$

we have

$$\begin{aligned} II^{p'} &= R^2 \int_0^T I_R(R^2 t)^{-p'/p} \phi_R^{p'}(R^2 t) dt \\ &= (R^2)^{1-(p'/p)(1-\gamma)} \int_0^T I_1(t)^{-p'/p} \phi^{p'}(t) dt \\ &= C(T)(R^2)^{1-(p'/p)(1-\gamma)}, \end{aligned} \tag{4.4}$$

where $C(T) = \int_0^T \phi^{p'}(t) I_1(t)^{-p'/p} dt < \infty$ by (1.15). From (4.2)–(4.4) it follows that

$$\begin{aligned} &\lambda_R \Big\{ \int_0^{TR^2} w_R^p(t) I_R(t) dt \Big\}^{1/p} C(T)^{1/p'} (R^2)^{\frac{1}{p'} - \frac{1-\gamma}{p}} \\ &\geq \int_0^{TR^2} I_R(s) w_R^p(s) ds + w_R(0), \end{aligned}$$

and by Young's inequality,

$$\frac{1}{p}\int_0^{TR^2} w_R^p(t)I_R(t)dt + \frac{1}{p'}\lambda_R^{p'}C(T)(R^2)^{1-\frac{(1-\gamma)p'}{p}} \geq w_R(0) + \int_0^{TR^2} I_R(t)w_R^p(t)dt.$$

Thus,

$$\frac{1}{p'}\lambda_R^{p'}C(T)(R^2)^{1-\frac{(1-\gamma)p'}{p}} \geq w_R(0).$$

Since $\lambda_R = \lambda_1/R^2$ we concluded that

$$w_R(0) \leq C(T)(\frac{\lambda_1^{p'}}{p'})(R^2)^{-p'+1-\frac{(1-\gamma)p'}{p}} = C'(T)(R^2)^{-\frac{2-\gamma}{p-1}}, \tag{4.5}$$

where $C'(T) = [C(T)\lambda_1^{p'}]/p'$. On the other hand, for $\epsilon \in (0,1)$ small

$$\begin{aligned} w_R(0) &= \int_{B_R} u_0(x)\rho_R(x)dx \\ &\geq \Big(\inf_{R\geq|x|\geq\epsilon R} u_0(x)\Big)\int_{\{\epsilon R\leq|x|\leq R\}}\rho_R(x)dx \\ &\geq \Big(\inf_{R\geq|x|\geq\epsilon R} u_0(x)\Big)\int_{\{\epsilon\leq|x|\leq 1\}}\rho_1(x)dx. \end{aligned}$$

Thus, from (4.5), it follows that

$$C'(T) \geq \Big(\inf_{R\geq|x|\geq\epsilon R}|x|^{2(2-\gamma)/(p-1)}u_0(x)\Big)\int_{\{\epsilon\leq|x|\leq 1\}}\rho_1(x)dx.$$

Putting, $\epsilon = \kappa/R > 0$ and letting $R\to\infty$ we have $\inf_{|x\geq\kappa}|x|^{2(\frac{2-\gamma}{p-1})}u_0(x) \leq C'(T)$. Since $C'(T) < \infty$ and κ is arbitrary the conclusion follows. $\square$

References

[1] H. Bellout; *Blow-up of solution of parabolic equations with nonlinear memory*, J. Diff. Eq. 70, (1987), 42-68.

[2] M. Cannone; *A generalization of a theorem by Kato on Navier-Stokes equations*, Mat. Ib. 13, (1997) 515-541.

[3] M. Cannone, F. Planchon; *Self-similar solutions for the Navier-Stokes equations in* $\mathbb{R}^3$, Comm. partial Diff. Eqs 21 (1996) 179-193.

[4] T. Cazenave, F. Dickstein, F. Weissler; *An equation whose Fujita critical exponent is not given by scaling*, Nonlinear Anal. 68, (2008) 862-874.

[5] T. Cazenave, F. Weissler; *Asymptotically self similar global solutions of the nonlinear Schrodinger and heat equations*, Math. Z. (1998) 83-120.

[6] A. Fino; *Contributions aux problèmes d'évolution*, Doctoral theses, Université de La Rochelle, France, 2009.

[7] A. Fino, M. Kirane; *Qualitative properties of solutions to a space time fractional evolution equation.* Quarterly of Applied Mathematics, 70, No 1, (2012), 133-157.

[8] H. Fujita, T. Kato; *On the Navier-Stokes initial value problem*, Arch. Rat. Mech. Anal. 16, (1964) 269-315.

[9] T. Kato, H. Fujita; *On the non stationary Navier-Stokes system*, Rend. Sem. Math. Univ. Padova 32 (1962) 243-260.

[10] M. Loayza; *Global existence and blow up results for a heat equation with nonlinear nonlocal term*, Diff. and Int. Eq. 25, (2012) , 665-683.

[11] P. Souplet; *Blow-up in nonlocal reaction-diffusion equations*, SIAM J. Math. Anal. 29, (1998) 1301-1334.

[12] P. Souplet; *Nonexistence of global solution to some differential inequalities of the second order and applications*, Portugaliae Mathematica 52, (1995) 289-299.

[13] S. Snoussi, S. Tayachi; *Asymptotic self-similar behavior of solutions for a semilinear parabolic system*, Comm. Cont. Math. Vol 3, No 3 (2001), 363-392.

[14] F. Weissler; *Existence and nonexistence of global solutions for a semilinear heat equation.* Israel J. Math. 38 (1981), 29-40.

[15] F. Weissler; *Local existence and nonexistence for semilinear parabolic equations in L^p.* Indiana Univ. Math. J. 29 (1980), 79-102.

Miguel Loayza
Departamento de Matemática, Universidade Federal de Pernambuco, 50740-540, Recife, PE, Brazil
E-mail address: miguel@dmat.ufpe.br