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WEAK HETEROCLINIC SOLUTIONS OF ANISOTROPIC DIFFERENCE EQUATIONS WITH VARIABLE EXPONENT

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Abstract. In this article, we prove the existence of heteroclinic solutions for a family of anisotropic difference equations. The proof of the main result is based on a minimization method, a change of variables and a discrete Hölder type inequality.

1. Introduction

In this article we study the existence of heteroclinic solutions for the nonlinear discrete anisotropic problem

$$\begin{gathered}-\Delta(a(k-1,\Delta u(k-1)))+g(k,u(k))=f(k),\quad k\in\mathbb{Z}^*\\ u(0)=0,\quad \lim_{k\to-\infty}u(k)=-1,\quad \lim_{k\to+\infty}u(k)=1,\end{gathered}\tag{1.1}$$

where $\Delta u(k)=u(k+1)-u(k)$ is the forward difference operator.

The study of heteroclinic connections for boundary value problems had a certain impulse in recent years, motivated by applications in various biological, physical and chemical models, such has phase-transition, physical processes in which the variable transits from an unstable equilibrium to a stable one, or front-propagation in reaction-diffusion equations. Indeed, heteroclinic solutions are often called transitional solutions (see [2, 6] and the references therein).

In this article, we show that the solvability of (1.1) is connected to the behavior of $g(k,s)$ as $k\in\mathbb{Z}^+$ and as $k\in\mathbb{Z}^-$. Problem (1.1) involves variable exponents due to their use in image restoration (see [3]), in electrorheological and thermorheological fluids dynamic (see [4, 7, 8]). The paper is organized as follows: In section 2, we introduce hypotheses on f, g and a, we define the functional spaces and some of their useful properties and in section 3, we prove the existence of heteroclinic solutions of (1.1).

2. Auxiliary results

For the rest of this article, we will use the notation:

$$p^+=\sup_{k\in\mathbb{Z}}p(k),\quad p^-=\inf_{k\in\mathbb{Z}}p(k).$$

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We assume that

$$p(.) : \mathbb{Z} \to (1, +\infty) \quad \text{and} \quad 1 < p^- \le p(.) < p^+ < +\infty. \tag{2.1}$$

We introduce the spaces:

$$\begin{aligned}
l^1 &= \{u : \mathbb{Z} \to \mathbb{R}, \|u\|_{l^1} := \sum_{k\in\mathbb{Z}} |u(k)| < \infty\},\\
l_0^1 &= \{u : \mathbb{Z} \to \mathbb{R}; u(0) = 0 \text{ and } \|u\|_{l_0^1} := \sum_{k\in\mathbb{Z}} |u(k)| < \infty\},\\
l_0^{p(.)} &= \{u : \mathbb{Z} \to \mathbb{R}; u(0) = 0 \text{ and } \rho_{p(.)}(u) := \sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} < \infty\},\\
l_{0,+}^{p(.)} &= \{u : \mathbb{Z}^+ \to \mathbb{R}; u(0) = 0 \text{ and } \rho_{p_+(.)}(u) := \sum_{k\in\mathbb{Z}^+} |u(k)|^{p(k)} < \infty\},\\
l_{0,-}^{p(.)} &= \{u : \mathbb{Z}^- \to \mathbb{R}; u(0) = 0 \text{ and } \rho_{p_-(.)}(u) := \sum_{k\in\mathbb{Z}^-} |u(k)|^{p(k)} < \infty\},\\
\mathcal{W}_{0,+}^{1,p(.)} &= \{u : \mathbb{Z}^+ \to \mathbb{R}; u(0) = 0 \text{ and } \rho_{1,p_+(.)}(u) := \sum_{k\in\mathbb{Z}^+} |u(k)|^{p(k)}\\
&\quad + \sum_{k\in\mathbb{Z}^+} |\Delta u(k)|^{p(k)} < \infty\}\\
&= \{u : \mathbb{Z}^+ \to \mathbb{R}; u \in l_+^{p(.)},\ \Delta u(k) \in l_+^{p(.)} \text{ and } u(0) = 0\}\\
\mathcal{W}_{0,-}^{1,p(.)} &= \{u : \mathbb{Z}^- \to \mathbb{R}; u(0) = 0 \text{ and } \rho_{1,p_-(.)}(u) := \sum_{k\in\mathbb{Z}^-} |u(k)|^{p(k)}\\
&\quad + \sum_{k\in\mathbb{Z}^-} |\Delta u(k)|^{p(k)} < \infty\}\\
&= \{u : \mathbb{Z}^- \to \mathbb{R}; u \in l_-^{p(.)},\ \Delta u(k) \in l_-^{p(.)} \text{ and } u(0) = 0\}.
\end{aligned}$$

On $l_{0,+}^{p(.)}$ we introduce the Luxemburg norm

$$\|u\|_{p_+(.)} := \inf\big\{\lambda > 0 : \sum_{k\in\mathbb{Z}^+} |\frac{u(k)}{\lambda}|^{p(k)} \le 1\big\}$$

and we deduce that

$$\begin{aligned}
\|u\|_{1,p_+(.)} &:= \inf\big\{\lambda > 0; \sum_{k\in\mathbb{Z}^+} |\frac{u(k)}{\lambda}|^{p(k)} + \sum_{k\in\mathbb{Z}^+} |\frac{\Delta u(k)}{\lambda}|^{p(k)} \le 1\big\}\\
&= \|u\|_{p_+(.)} + \|\Delta u\|_{p_+(.)}
\end{aligned}$$

is a norm on the space $\mathcal{W}_{0,+}^{1,p(.)}$. We replace $\mathbb{Z}^+$ by $\mathbb{Z}^-$ to get the norms on $l_{0,-}^{p(.)}$ and $\mathcal{W}_{0,-}^{1,p(.)}$ denoted respectively $\|\cdot\|_{p_-(.)}$ and $\|\cdot\|_{1,p_-(.)}$.

For the data f, g and a, we assume the following:

$$\begin{gathered}
a(k,.) : \mathbb{R} \to \mathbb{R} \text{ for all } k \in \mathbb{Z} \text{ and there exists a mapping } A : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}\\
\text{such that } a(k,\xi) = \frac{\partial}{\partial\xi}A(k,\xi) \text{ for all } k \in \mathbb{Z} \text{ and } A(k,0) = 0 \text{ for all } k \in \mathbb{Z}.
\end{gathered} \tag{2.2}$$

$$|\xi|^{p(k)} \le a(k,\xi)\xi \le p(k)A(k,\xi) \quad \forall k \in \mathbb{Z} \text{ and } \xi \in \mathbb{R}. \tag{2.3}$$

There exists a positive constant C_1 such that

$$|a(k,\xi)| \le C_1(j(k) + |\xi|^{p(k)-1}), \tag{2.4}$$

for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$ where $j \in l^{p'(.)}$ with $\frac{1}{p(k)} + \frac{1}{p'(k)} = 1$.

$$f \in l^1, \tag{2.5}$$

$$g(k,t) = |t-1|^{p(k)-2}(t-1)\chi_{\mathbb{Z}^+}(k) + |t+1|^{p(k)-2}(t+1)\chi_{\mathbb{Z}^-}(k), \tag{2.6}$$

where $\chi_A(k) = 1$ if $k \in A$ and $\chi_A(k) = 0$ if $k \notin A$.

Remark 2.1. Note that $l_{0,+}^{p(.)} \subset l_0^{p(.)}$, $l_{0,-}^{p(.)} \subset l_0^{p(.)}$, $\mathcal{W}_{0,+}^{1,p(.)} \subset \mathcal{W}_0^{1,p(.)}$ and $\mathcal{W}_{0,-}^{1,p(.)} \subset \mathcal{W}_0^{1,p(.)}$.

If $u \in l_{0,+}^{p(.)}$ (or $u \in l_{0,-}^{p(.)}$ or $u \in l_0^{p(.)}$) then $\lim_{k\to+\infty} u(k) = 0$ (or $\lim_{k\to-\infty} u(k) = 0$ or $\lim_{|k|\to+\infty} u(k) = 0$). Indeed, for instance, if $u \in l_{0,+}^{p(.)}$ then $\sum_{k\in\mathbb{Z}^+} |u(k)|^{p(k)} < \infty$. Let

$$\sum_{k\in\mathbb{Z}^+} |u(k)|^{p(k)} = \sum_{k\in S_1} |u(k)|^{p(k)} + \sum_{k\in S_2} |u(k)|^{p(k)},$$

where $S_1 = \{k \in \mathbb{Z}^+; |u(k)| < 1\}$ and $S_2 = \{k \in \mathbb{Z}^+; |u(k)| \ge 1\}$. The set S_2 is necessarily finite, and $|u(k)| < \infty$ for any $k \in S_2$ since $u \in l_{0,+}^{p(.)}$. We also have that

$$\sum_{k\in S_1} |u(k)|^{p^+} \le \sum_{k\in\mathbb{Z}^+} |u(k)|^{p(k)},$$

then $\sum_{k\in S_1} |u(k)|^{p^+} < \infty$. As S_2 is a finite set then $\sum_{k\in S_2} |u(k)|^{p^+} < \infty$, which implies that

$$\sum_{k\in\mathbb{Z}^+} |u(k)|^{p^+} < \infty.$$

Thus, $\lim_{k\to+\infty} u(k) = 0$.

We now give useful properties of the spaces defined above which are similar to those in [5].

Proposition 2.2. *Assume that* (2.1) *is fulfilled. Then* $l_0^1 \subset l_0^{p(.)}$.

Proposition 2.3. *Under conditions* (2.1), $\rho_{p+(.)}$ *satisfies*

(a) $\rho_{p+(.)}(u+v) \le 2^{p+}(\rho_{p+(.)}(u) + \rho_{p+(.)}(v))$, *for all* $u, v \in l_{0,+}^{p(.)}$.

(b) *For* $u \in l_{0,+}^{p(.)}$, *if* $\lambda > 1$ *we have*

$$\rho_{p+(.)}(u) \le \lambda\rho_{p+(.)}(u) \le \lambda^{p^-}\rho_{p+(.)}(u) \le \rho_{p+(.)}(\lambda u) \le \lambda^{p^+}\rho_{p+(.)}(u)$$

and if $0 < \lambda < 1$, *we have*

$$\lambda^{p^+}\rho_{p+(.)}(u) \le \rho_{p+(.)}(\lambda u) \le \lambda^{p^-}\rho_{p+(.)}(u) \le \lambda\rho_{p+(.)}(u) \le \rho_{p+(.)}(u).$$

(c) *For every fixed* $u \in l_{0,+}^{p(.)} \setminus \{0\}$, $\rho_{p+(.)}(\lambda u)$ *is a continuous convex even function in* λ *and it increases strictly when* $\lambda \in [0,\infty)$.

Proposition 2.4. *Let* $u \in l_{0,+}^{p(.)} \setminus \{0\}$, *then* $\|u\|_{p+(.)} = a$ *if and only if* $\rho_{p+(.)}(\frac{u}{a}) = 1$.

Proposition 2.5. *If* $u \in l_{0,+}^{p(.)}$ *and* $p^+ < +\infty$, *then the following properties hold:*

(1) $\|u\|_{p+(.)} < 1$ $(= 1;\ > 1)$ *if and only if* $\rho_{p+(.)}(u) < 1$ $(= 1; > 1)$;

(2) $\|u\|_{p+(.)} > 1$ *implies* $\|u\|_{p+(.)}^{p^-} \le \rho_{p+(.)}(u) \le \|u\|_{p+(.)}^{p+}$;

(3) $\|u\|_{p_+(.)} < 1$ *implies* $\|u\|_{p_+(.)}^{p^+} \leq \rho_{p_+(.)}(u) \leq \|u\|_{p_+(.)}^{p-}$;
(4) $\|u_n\|_{p_+(.)} \to 0$ *if and only if* $\rho_{p_+(.)}(u_n) \to 0$ *as* $n \to +\infty$.

Proposition 2.6. *Let* $u \in \mathcal{W}_{0,+}^{1,p(.)} \setminus \{0\}$. *Then* $\|u\|_{1,p_+(.)} = a$ *if and only if* $\rho_{1,p_+(.)}(u/a) = 1$.

Proposition 2.7. *If* $u \in \mathcal{W}_{0,+}^{1,p(.)}$ *and* $p^+ < +\infty$, *then the following properties hold:*
(1) $\|u\|_{1,p_+(.)} < 1$ $(= 1;\ > 1)$ *if and only if* $\rho_{1,p_+(.)}(u) < 1$ $(= 1; > 1)$;
(2) $\|u\|_{1,p_+(.)} > 1$ *implies* $\|u\|_{1,p_+(.)}^{p^-} \leq \rho_{1,p_+(.)}(u) \leq \|u\|_{1,p_+(.)}^{p^+}$;
(3) $\|u\|_{1,p_+(.)} < 1$ *implies* $\|u\|_{1,p_+(.)}^{p^+} \leq \rho_{1,p_+(.)}(u) \leq \|u\|_{1,p_+(.)}^{p^-}$;
(4) $\|u_n\|_{1,p_+(.)} \to 0$ *if and only if* $\rho_{1,p_+(.)}(u_n) \to 0$ *as* $n \to +\infty$.

Theorem 2.8 (Discrete Hölder type inequality)**.** *Let* $u \in l_+^{p(.)}$ *and* $v \in l_+^{q(.)}$ *be such that* $\frac{1}{p(k)} + \frac{1}{q(k)} = 1$ *for all* $k \in \mathbb{Z}+$, *then*

$$\sum_{k=0}^{+\infty} |uv| \leq (\frac{1}{p^-} + \frac{1}{q^-})\|u\|_{p_+(.)}\|v\|_{q_+(.)}.$$

Remark 2.9. All the properties above hold for the spaces $l^{p(.)}$, $l_-^{p(.)}$ and $\mathcal{W}_{0,-}^{1,p(.)}$.

3. Existence of weak heteroclinic solutions

In this section, we study the existence of weak heteroclinic solutions of problem (1.1).

Definition 3.1. A weak heteroclinic solution of (1.1) is a function $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\sum_{k\in\mathbb{Z}} a(k-1, \Delta u(k-1))\Delta v(k-1) + \sum_{k\in\mathbb{Z}} g(k, u(k))v(k) = \sum_{k\in\mathbb{Z}} f(k)v(k), \tag{3.1}$$

for any $v : \mathbb{Z} \to \mathbb{R}$, with $u(0) = 0$, $\lim_{k\to+\infty} u(k) = 1$ and $\lim_{k\to-\infty} u(k) = -1$.

Theorem 3.2. *Assume that* (2.1)–(2.6) *hold. Then, there exists at least one weak heteroclinic solution of* (1.1).

To prove Theorem 3.2, we first prove that the problem

$$\begin{gathered} -\Delta(a(k-1, \Delta u(k-1))) + |u(k)|^{p(k)-2}u(k) = f(k), \quad k \in \mathbb{Z}_*^+ \\ u(0) = 0, \quad \lim_{k\to+\infty} u(k) = 0, \end{gathered} \tag{3.2}$$

admits a weak solution in the following sense.

Definition 3.3. A weak solution of (3.2) is a function $u \in \mathcal{W}_{0,+}^{1,p(.)}$ such that

$$\sum_{k=1}^{+\infty} a(k-1, \Delta u(k-1))\Delta v(k-1) + \sum_{k=1}^{+\infty} |u(k)|^{p(k)-2}u(k)v(k) = \sum_{k=1}^{+\infty} f(k)v(k), \tag{3.3}$$

for any $v \in \mathcal{W}_{0,+}^{1,p(.)}$.

We have the following result.

Theorem 3.4. *Assume that* (2.1)–(2.5) *hold. Then, there exists at least one weak solution of problem* (3.2).

The energy functional corresponding to problem (3.2) is $J : \mathcal{W}_{0,+}^{1,p(.)} \to \mathbb{R}$ defined by

$$J(u) = \sum_{k=1}^{+\infty} A(k-1, \Delta u(k-1)) + \sum_{k=1}^{+\infty} \frac{1}{p(k)} |u(k)|^{p(k)} - \sum_{k=1}^{+\infty} f(k)u(k). \tag{3.4}$$

We first present some basic properties of J.

Proposition 3.5. *The functional J is well-defined on the space $\mathcal{W}_{0,+}^{1,p(.)}$ and is of class $C^1(\mathcal{W}_{0,+}^{1,p(.)}, \mathbb{R})$, with the derivative given by*

$$\begin{aligned} \langle J'(u), v\rangle &= \sum_{k=1}^{+\infty} a(k-1, \Delta u(k-1))\Delta v(k-1) \\ &\quad + \sum_{k=1}^{+\infty} |u(k)|^{p(k)-2} u(k)v(k) - \sum_{k=1}^{+\infty} f(k)v(k), \end{aligned} \tag{3.5}$$

for all $u, v \in \mathcal{W}_{0,+}^{1,p(.)}$.

Proof. We denote

$$I(u) = \sum_{k=1}^{+\infty} A(k-1, \Delta u(k-1)), \quad L(u) = \sum_{k=1}^{+\infty} \frac{1}{p(k)} |u(k)|^{p(k)}, \quad \Lambda(u) = \sum_{k=1}^{+\infty} f(k)u(k).$$

Using Young inequality, from assumptions (2.2) and (2.4) it follows that

$$\begin{aligned} |I(u)| &= \Big|\sum_{k=1}^{+\infty} A(k-1, \Delta u(k-1))\Big| \\ &\le \sum_{k=1}^{+\infty} |A(k-1, \Delta u(k-1))| \\ &\le \sum_{k=1}^{+\infty} C_1 \Big(j(k-1) + \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)-1} \Big) |\Delta u(k-1)| \\ &\le \sum_{k=1}^{+\infty} C_1 j(k-1)|\Delta u(k-1)| + \sum_{k=1}^{+\infty} \frac{C_1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} < \infty, \end{aligned}$$

$$|L(u)| \le \frac{1}{p_-} \sum_{k=1}^{+\infty} |u(k)|^{p(k)} < \infty,$$

$$|\Lambda(u)| = \Big|\sum_{k=1}^{+\infty} f(k)u(k)\Big| \le \sum_{k=1}^{+\infty} |f(k)||u(k)| < \infty.$$

Therefore, J is well-defined.

Clearly I, L and Λ are in $C^1(\mathcal{W}_{0,+}^{1,p(.)}, \mathbb{R})$. Let us now choose $u, v \in \mathcal{W}_{0,+}^{1,p(.)}$. We have

$$\langle I'(u), v\rangle = \lim_{\delta\to 0^+} \frac{I(u+\delta v) - I(u)}{\delta}, \quad \langle L'(u), v\rangle = \lim_{\delta\to 0^+} \frac{L(u+\delta v) - L(u)}{\delta},$$

$$\langle \Lambda'(u), v\rangle = \lim_{\delta\to 0^+} \frac{\Lambda(u+\delta v) - \Lambda(u)}{\delta}.$$

Let us denote $g_\delta = \frac{A(k-1,\Delta u(k-1)+\delta\Delta v(k-1))-A(k-1,\Delta u(k-1))}{\delta}$. Using Young inequality,

$$\sum_{k=1}^{+\infty}|g_\delta| \le \frac{1}{\delta}\sum_{k=1}^{+\infty}|A(k-1,\Delta u(k-1)+\delta\Delta v(k-1))| + \frac{1}{\delta}\sum_{k=1}^{+\infty}|A(k-1,\Delta u(k-1))| < +\infty.$$

Thus,

$$\begin{aligned}
&\lim_{\delta\to 0^+}\frac{I(u+\delta v)-I(u)}{\delta}\\
&= \lim_{\delta\to 0^+}\sum_{k=1}^{+\infty}\frac{A(k-1,\Delta u(k-1)+\delta\Delta v(k-1))-A(k-1,\Delta u(k-1))}{\delta}\\
&= \sum_{k=1}^{+\infty}\lim_{\delta\to 0^+}\frac{A(k-1,\Delta u(k-1)+\delta\Delta v(k-1))-A(k-1,\Delta u(k-1))}{\delta}\\
&= \sum_{k=1}^{+\infty}a(k-1,\Delta u(k-1))\Delta v(k-1).
\end{aligned}$$

By the same method, we deduce that

$$\begin{aligned}
\lim_{\delta\to 0^+}\frac{L(u+\delta v)-L(u)}{\delta} &= \lim_{\delta\to 0^+}\sum_{k=1}^{+\infty}\frac{|u(k)+\delta v(k)|^{p(k)}-|u(k)|^{p(k)}}{p(k)\delta}\\
&= \sum_{k=1}^{+\infty}\lim_{\delta\to 0^+}\frac{|u(k)+\delta v(k)|^{p(k)}-|u(k)|^{p(k)}}{p(k)\delta}\\
&= \sum_{k=1}^{+\infty}|u(k)|^{p(k)-2}u(k)v(k)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\delta\to 0^+}\frac{\Lambda(u+\delta v)-\Lambda(u)}{\delta} &= \lim_{\delta\to 0^+}\sum_{k=1}^{+\infty}\frac{f(k)(u(k)+\delta v(k))-f(k)u(k)}{\delta}\\
&= \sum_{k=1}^{+\infty}\lim_{\delta\to 0^+}\frac{f(k)(u(k)+\delta v(k))-f(k)u(k)}{\delta}\\
&= \sum_{k=1}^{+\infty}f(k)v(k)
\end{aligned}$$

□

Lemma 3.6. *The functional I is weakly lower semi-continuous.*

Proof. From (2.2), I is convex with respect to the second variable. Thus, by [1, corollary III.8], it is sufficient to show that I is lower semi-continuous. For this, we fix $u \in \mathcal{W}_{0,+}^{1,p(.)}$ and $\epsilon > 0$. Since I is convex, we deduce that for any $v \in \mathcal{W}_{0,+}^{1,p(.)}$,

$$\begin{aligned}
I(v) &\ge I(u) + \langle I'(u), v-u\rangle\\
&\ge I(u) + \sum_{k=1}^{+\infty}a(k-1,\Delta u(k-1))(\Delta v(k-1)-\Delta u(k-1))
\end{aligned}$$

$$
\begin{aligned}
&\geq I(u) - C(\frac{1}{p^-} + \frac{1}{p'^-})\|g\|_{p'_+(.)}\|\Delta(u-v)\|_{p_+(.)},\\
&\quad \text{with } g(k) = j(k) + |\Delta u(k)|^{p(k)-1}\\
&\geq I(u) - K\Big(\|u-v\|_{p_+(.)} + \|\Delta(u-v)\|_{p_+(.)}\Big)\\
&\geq I(u) - K\|u-v\|_{1,p_+(.)}\\
&\geq I(u) - \epsilon,
\end{aligned}
$$

for all $v \in \mathcal{W}^{1,p(.)}_{0,+}$ with $\|u-v\|_{1,p_+(.)} < \delta = \epsilon/K$. Hence, we conclude that I is weakly lower semi-continuous. □

Proposition 3.7. *The functional J is bounded from below, coercive and weakly lower semi-continuous.*

Proof. By Lemma 3.6, J is weakly lower semi-continuous. We shall only prove the coerciveness of the energy functional J and its boundedness from below.

$$
\begin{aligned}
J(u) &= \sum_{k=1}^{+\infty} A(k-1,\Delta u(k-1)) + \sum_{k=1}^{+\infty}\frac{1}{p(k)}|u(k)|^{p(k)} - \sum_{k=1}^{+\infty} f(k)u(k)\\
&\geq \sum_{k=1}^{+\infty}\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \sum_{k=1}^{+\infty}\frac{1}{p(k)}|u(k)|^{p(k)} - \sum_{k=1}^{+\infty}|f(k)u(k)|\\
&\geq \frac{1}{p^+}(\sum_{s=1}^{+\infty}|\Delta u(s)|^{p(s)} + \sum_{k=1}^{+\infty}|u(k)|^{p(k)}) - \sum_{k=1}^{+\infty}|f(k)||u(k)|\\
&\geq \frac{1}{p^+}\rho_{1,p_+(.)}(u) - c_0\|f\|_{p'_+(.)}\|u\|_{p_+(.)}\\
&\geq \frac{1}{p^+}\rho_{1,p_+(.)}(u) - K\|u\|_{1,p_+(.)}.
\end{aligned}
$$

To prove the coerciveness of J, we may assume that $\|u\|_{1,p_+(.)} > 1$ and we get from the above inequality that

$$
J(u) \geq \frac{1}{p^+}\|u\|^{p^-}_{1,p_+(.)} - K\|u\|_{1,p_+(.)}.
$$

Thus,

$$
J(u) \to +\infty \quad \text{as } \|u\|_{1,p_+(.)} \to +\infty.
$$

As $J(u) \to +\infty$ when $\|u\|_{1,p_+(.)} \to +\infty$, then for $\|u\|_{1,p_+(.)} > 1$, there exists $c \in \mathbb{R}$ such that $J(u) \geq c$. For $\|u\|_{1,p_+(.)} \leq 1$, we have

$$
J(u) \geq \frac{1}{p^+}\rho_{1,p_+(.)}(u) - K\|u\|_{1,p_+(.)} \geq -K\|u\|_{1,p_+(.)} \geq -K > -\infty.
$$

Thus J is bounded below. □

We can now give the proof of the main result.

Proof of Theorem 3.4. By Proposition 3.7, J has a minimizer which is a weak solution of (3.2) □

Now, we consider the problem

$$
\begin{gathered}
-\Delta(a(k-1,\Delta u(k-1))) + |u(k)|^{p(k)-2}u(k) = f(k), \quad k \in \mathbb{Z}^- \\
u(0) = 0, \quad \lim_{k\to-\infty} u(k) = 0.
\end{gathered} \tag{3.6}
$$

A weak solution of problem (3.6) is defined as follows.

Definition 3.8. A weak solution of (3.6) is a function $u \in \mathcal{W}_{0,-}^{1,p(.)}$ such that

$$
\sum_{k=-\infty}^{0} a(k-1,\Delta u(k-1))\Delta v(k-1) + \sum_{k=-\infty}^{0} |u(k)|^{p(k)-2}u(k)v(k) = \sum_{k=-\infty}^{0} f(k)v(k), \tag{3.7}
$$

for any $v \in \mathcal{W}_{0,-}^{1,p(.)}$.

By mimicking the proof of Theorem 3.4, we prove the following result.

Theorem 3.9. *Assume that* (2.1)–(2.5) *hold. Then, there exists at least one weak solution of problem* (3.6).

Let us now show the existence of weak heteroclinic solutions of problem (1.1).

Proof of Theorem 3.2. We define $v_1 = u_1 + 1$, where u_1 is a weak solution of problem (3.2) and $v_2 = u_2 - 1$, where u_2 is a weak solution of problem (3.6). Therefore, we deduce that

$$
u = v_1\chi_{\mathbb{Z}^+} + v_2\chi_{\mathbb{Z}^-}
$$

is an heteroclinic solution of problem (1.1). □

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