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GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR PARABOLIC SYSTEMS WITH NONLINEAR NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this article we study a nonlinear parabolic system with nonlinear nonlocal boundary conditions. We prove the uniqueness of the solutions and establish the conditions for global solutions and non-global solutions. It is interesting to observe that the weight function for the nonlocal Dirichlet boundary conditions plays a crucial role in determining whether the solutions are global or blow up in finite time.

1. Introduction

In this article, we consider the parabolic system with nonlinear nonlocal boundary conditions

$$
\begin{gathered}
u_t = \Delta u + v^p, \quad x \in \Omega,\ t > 0, \\
v_t = \Delta v + u^q, \quad x \in \Omega,\ t > 0, \\
u(x,t) = \int_\Omega f(x,y) u^r(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0, \\
v(x,t) = \int_\Omega g(x,y) v^r(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), v(x,0) = v_0(x), \quad x \in \Omega,
\end{gathered}
\tag{1.1}
$$

where Ω is a bounded domain in $R^N (N \geq 1)$ with smooth boundary $\partial\Omega$, and $p, q, r > 0$. The functions f and g are nonnegative, continuous, defined for $x \in \partial\Omega$, $y \in \Omega$ and $t \geq 0$. The initial data $u_0(x)$ and $v_0(x)$ are nonnegative continuous functions satisfying the boundary conditions at $t = 0$.

Over the previous twenty years, many physical phenomena were formulated into nonlocal mathematical models (see [1, 2, 3, 7, 13]). There has been a considerable amount of literature dealing with the properties of solutions to local semilinear parabolic equation or systems of heat equations with homogeneous Diriclet boundary conditions or with nonlinear boundary conditions (see [5, 9, 10, 11, 16, 17, 18, 19, 20, 21, 22, 24, 26] and references therein). However, there are some important phenomena formulated as parabolic equations which are coupled with nonlocal

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boundary conditions in mathematical modeling such as thermoelasticity theory (see [4, 6, 25]).

The problem of nonlocal boundary value for linear parabolic equations is of the type

$$\begin{gathered} u_t - Au = c(x)u, \quad x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega K(x,y)u(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{gathered} \tag{1.2}$$

with uniformly elliptic operator

$$A = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i}$$

and $c(x) \le 0$ was studied by Friedman [12]. The global existence and monotonic decay of the solution of problem (1.2) was obtained under the condition $\int_\Omega |K(x,y)|\,dy < 1$ for all $x \in \partial\Omega$. And later the problem (1.2) with Au replaced by Δu and the linear term $c(x)u$ replaced by the nonlinear term $g(x,u)$ was discussed by Deng [8]. The comparison principle and the local existence were established. On the basis of Deng's work, Seo [23] investigated the above problem with $g(x,u) = g(u)$, by using the upper and lower solution's technique, he gained the blow-up condition of positive solution, and in the special case $g(u) = u^p$ or $g(u) = e^u$ he also derived the blow-up rate estimates.

For more general discussions on the dynamics of parabolic problems with nonlocal boundary conditions, Pao [20] consider the problem

$$\begin{gathered} u_t - Lu = f(x,u), \quad x \in \Omega,\ t > 0, \\ Bu = \int_\Omega K(x,y)u(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{gathered} \tag{1.3}$$

where

$$Lu = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^n b_i(x)\frac{\partial u}{x_i}, \quad Bu = \alpha_0 \frac{\partial u}{\partial \nu} + u.$$

The scalar problems with both nonlocal sources and nonlocal boundary conditions have been studied as well. For example, the problem

$$\begin{gathered} u_t - \Delta u = \int_\Omega g(u)\,dy, \quad x \in \Omega,\ t > 0, \\ Bu = \int_\Omega K(x,y)u(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{gathered} \tag{1.4}$$

was studied by Lin and Liu [16], where $\int_\Omega g(u)\,dy \equiv \int_\Omega g(u(y,t))\,dy$, and Zheng and Kong [26] established global existence condition for solution to a nonlocal parabolic system subject to nonlocal Dirichlet boundary conditions

$$\begin{gathered} u_t = \Delta u - u^m(x,t)\int_\Omega v^n(y,t)\,dy, \quad x \in \Omega,\ t>0, \\ v_t = \Delta v - v^q(x,t)\int_\Omega u^p(y,t)\,dy, \quad x \in \Omega,\ t>0, \\ u = \int_\Omega \varphi(x,y)u(y,t)\,dy, \quad v = \int_\Omega \psi(x,y)v(y,t)\,dy, \quad x \in \partial\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad x\in\Omega. \end{gathered} \tag{1.5}$$

Recently, Gladkov and Kim [14] studied the heat equation with nonlinear nonlocal boundary condition,

$$\begin{gathered} u_t = \Delta u + c(x,t)u^p, \quad x\in\Omega,\ t>0, \\ u(x,t) = \int_\Omega k(x,y,t)u^l(y,t)\,dy, \quad x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad x\in\Omega\,. \end{gathered} \tag{1.6}$$

The comparison principle, the uniqueness of solution with any initial data for $\min(p,l)\geq 1$ and with nontrivial initial data otherwise, non-uniqueness of solution with trivial initial data for $p<1$ or $l<1$, and local existence theorem had been proved. And in [15] they presented some criteria for the existence of global behavior of the coefficients $c(x,t)$ and $k(x,y,t)$ as t tends to infinity.

Motivated by the above works, we are interested in the blow-up properties of problem (1.1). The aim of this paper is to establish the global existence and finite time blow-up conditions for the solution of problem (1.1).

Before stating our main results, we state the following assumptions on the kernels $f(x,y)$, $g(x,y)$ and the initial data $u_0(x)$, $v_0(x)$:

- (H1) $f(x,y)$ and $g(x,y)$ are continuous and nonnegative functions on $\partial\Omega\times\bar\Omega$ and satisfy
$$\int_\Omega f(x,y)\,dy>0, \quad \int_\Omega g(x,y)\,dy>0 \quad \text{for all } x\in\partial\Omega.$$
- (H2) $u_0(x), v_0(x)\in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$, $u_0(x)>0$ and $v_0(x)>0$ in Ω satisfy
$$u_0(x)=\int_\Omega f(x,y)u_0^r(y)\,dy, \quad v_0(x)=\int_\Omega g(x,y)v_0^r(y)\,dy \quad \text{on } \partial\Omega.$$

This article is organized as follows. Section 2 is devoted to dealing with the comparison principle and the local existence in time for problem (1.1). In Section 3 we give the global existence for $p,q\leq 1$. The blow-up conditions for $p,q>1$ with large initial data will be established in Section 4. In Section 5, we discuss the blow-up solutions and the existence of global solutions for $p>1>q$ or $q>1>p$.

2. The comparison principle and existence of local solutions

In this section we start with the definition of supersolution and subsolution of problem (1.1). Then we present some material needed in the proof of our main results. For convenience, We set $Q_T=\Omega\times(0,T]$, $S_T=\partial\Omega\times(0,T]$, $Q_t=\Omega\times(0,t]$, $0<t\leq T<\infty$, where $\overline{Q}_T$ and $\overline{Q}_t$ are their respective closures.

Definition 2.1. A pair of nonnegative functions $\tilde{u}, \tilde{v} \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is called a subsolution of (1.1) if

$$\begin{gathered}
\tilde{u}_t \le \Delta \tilde{u} + \tilde{v}^p, \quad (x,t) \in Q_T, \\
\tilde{v}_t \le \Delta \tilde{v} + \tilde{u}^q, \quad (x,t) \in Q_T, \\
\tilde{u}(x,t) \le \int_\Omega f(x,y) \tilde{u}^r(y,t)\, dy, \quad (x,t) \in S_T, \\
\tilde{v}(x,t) \le \int_\Omega g(x,y) \tilde{v}^r(y,t)\, dy, \quad (x,t) \in S_T, \\
\tilde{u}(x,0) \le u_0(x), \tilde{v} \le v_0(x), \quad x \in \Omega.
\end{gathered} \tag{2.1}$$

and a pair of functions $\hat{u}, \hat{v} \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a supersolution of problem (1.1) if $\hat{u}, \hat{v} \ge 0$ and it satisfies inequalities in (2.1) in the reverse order. Furthermore, we say that u and v are solutions of problem (1.1) in $\bar{Q}_T$ if they are both subsolutions and supersolutions of (1.1) in $\bar{Q}_T$.

Definition 2.2. We say a pair of nonnegative functions $\tilde{u}, \tilde{v}$ is a strict subsolution of (1.1) in Q_T if it is a subsolution and the equalities in boundary conditions in (2.1) is strict. Similarly, a strict supersolution is defined by the opposite inequalities.

The following lemma and comparison principle play a crucial role in our discussions.

Lemma 2.3. *Let u_0, v_0 be nontrivial functions in $\overline{\Omega}$ and assume that the assumptions* (H1)–(H2) *hold. Suppose that $(\hat{u}, \hat{v})$ is a supersolution of* (1.1) *in Q_T, then $\hat{u} > 0$, $\hat{v} > 0$ in $\overline{Q}_T$ for all $t > 0$.*

Proof. Since u_0 is a nontrivial nonnegative function and $\hat{u}_t - \Delta \hat{u} \ge \hat{v}^p \ge 0$, a minimum of u over $\bar{Q}_T$ should be attained only at a parabolic boundary point by the strong maximum principle. Thus, $\hat{u}(x,t) > 0$ in $\Omega \times (0,T]$. By (1.1) and (H1), we have $\hat{u}(x,t) > 0$ for $x \in \partial\Omega$, $0 < t \le T$. Similarly, $\hat{v}(x,t) > 0$ in $\overline{Q}_T$ for all $t > 0$ can be proved. □

Lemma 2.4. *Let $(\hat{u}, \hat{v})$ and $(\tilde{u}, \tilde{v})$ be a nonnegative supersolution and a nonnegative subsolution of* (1.1) *in Q_T, respectively, and $\hat{u}(x,0) > \tilde{u}(x,0), \hat{v}(x,0) > \tilde{v}(x,0)$ in $\overline{\Omega}$. Suppose* (H1) *holds or $(\hat{u}, \hat{v})$ is a strict supersolution. Then $\hat{u} > \tilde{u}$ and $\hat{v} > \tilde{v}$ hold in $\overline{Q}_T$.*

Proof. Set $\varphi(x,t) = \hat{u} - \tilde{u}$, $\psi = \hat{v} - \tilde{v}$, then φ, ψ satisfy

$$\begin{gathered}
\varphi_t - \Delta \varphi \ge \hat{v}^p - \tilde{v}^q \ge \rho_1(\theta_v)\psi, \quad (x,t) \in Q_T, \\
\psi_t - \Delta \psi \ge \hat{u}^p - \tilde{u}^q \ge \rho_2(\theta_u)\varphi, \quad (x,t) \in Q_T,
\end{gathered} \tag{2.2}$$

with the following boundary and initial conditions

$$\begin{gathered}
\varphi(x,t) \ge \int_\Omega f(x,y) h_1(\theta_u) \varphi(y,t)\, dy, \quad (x,t) \in S_T, \\
\psi(x,t) \ge \int_\Omega g(x,y) h_2(\theta_v) \psi(y,t)\, dy, \quad (x,t) \in S_T,
\end{gathered} \tag{2.3}$$

$$\varphi(x,0) > 0, \quad \psi(x,0) > 0, \quad x \in \bar{\Omega}, \tag{2.4}$$

where θ_u is between $\hat{u}$ and $\tilde{u}$, and θ_v is between $\hat{v}$ and $\tilde{v}$.

Since $\varphi(x,0), \psi(x,0) > 0$, by continuity, there exists $\delta > 0$ such that $\varphi, \psi > 0$ for $(x,t) \in \bar{\Omega} \times (0,\delta)$. Suppose for a contradiction that $t_0 = \sup\{t \in (0,T), \varphi, \psi >$

0on $\bar\Omega\times(0,t)\} < T$. Then $\varphi,\psi\ge 0$ on $\bar Q_{\bar t_0}$, and at least one of φ,ψ vanishes at (x_0,t_0) for some $x_0\in\bar\Omega$. Without loss of generality, we suppose $\varphi(x_0,t_0)=0$. Let $G(x,y;t)$ denote the Green's function for

$$Lu=u_t-\Delta u,\quad x\in\Omega,\ t>0$$

with boundary conditions

$$u=0,\quad x\in\partial\Omega,\ t>0.$$

Then for $y\in\partial\Omega$, $G(x,y;t)=0$ and $\frac{\partial G(x,y;t)}{\partial n}<0$. Applying $G(x,y;t)$ to (2.2), we have

$$\begin{aligned}\varphi(x,t)\ge&\int_\Omega G(x,y;t)\varphi(y,0)\,dy\\&+\int_0^t\int_\Omega G(x,y;t-\eta)\rho_1(\theta_v)\psi(y,\eta)\,dy\,d\eta\\&-\int_0^t\int_{\partial\Omega}\frac{\partial G(x,\xi;t-\eta)}{\partial n}\int_\Omega f(\xi,y)h_1(\theta_u)\varphi(y,\eta)\,dy\,d\xi\,d\eta.\end{aligned}$$

Since $\varphi,\psi>0$ for all $x\in\bar\Omega$, $0<t<t_0$, and $f(x,y)>0$, we find that

$$\varphi(x,t_0)\ge\int_\Omega G(x,y;t)\varphi(y,0)\,dy>0.$$

In particular, $\varphi(x_0,t_0)>0$, which contradicts our previous assumption. □

Remark 2.5. If $\int_\Omega f(x,y)\,dy\le 1$, $\int_\Omega g(x,y)\,dy\le 1$, we need only $\hat u(x,0)\ge\tilde u(x,0)$, $\hat v(x,0)\ge\tilde v(x,0)$ in $\bar\Omega$, since for any $\varepsilon>0$, $\varphi(x,t)=\hat u+\varepsilon-\tilde u$, $\psi=\hat v+\varepsilon-\tilde v$ satisfy all inequalities in (2.2)-(2.4), then we have $\hat u+\varepsilon>\tilde u$, $\hat v+\varepsilon>\tilde v$, and it follows that $\hat u\ge\tilde u$, $\hat v\ge\tilde v$.

Let ε_m be decreasing to 0 such that $0<\varepsilon_m<1$. For $\varepsilon=\varepsilon_m$, let $u_{0\varepsilon},v_{0\varepsilon}$ be the functions with the following properties: $u_{0\varepsilon},v_{0\varepsilon}\in C(\bar\Omega)$, $u_{0\varepsilon}>\varepsilon,v_{0\varepsilon}>\varepsilon$, $u_{0\varepsilon_i}>u_{0\varepsilon_j}$, $v_{0\varepsilon_i}>v_{0\varepsilon_j}$ for $\varepsilon_i>\varepsilon_j$, $u_{0\varepsilon}\to u_0(x)$, $v_{0\varepsilon}\to v_0(x)$ as $\varepsilon\to 0$, and $u_{0\varepsilon}(x)=\int_\Omega f(x,y)u_{0\varepsilon}^r(y)\,dy$, $v_{0\varepsilon}(x)=\int_\Omega g(x,y)v_{0\varepsilon}^r(y)\,dy$. Since the nonlinearities in (1.1) do not satisfy the Lipschitz condition, we need to consider the following auxiliary problem:

$$\begin{gathered}u_t=\Delta u+v^p,\quad x\in\Omega,\ t>0,\\ v_t=\Delta v+u^q,\quad x\in\Omega,\ t>0,\\ u(x,t)=\int_\Omega f(x,y)u^r(y,t)\,dy+\varepsilon,\quad x\in\partial\Omega,\ t>0,\\ v(x,t)=\int_\Omega g(x,y)v^r(y,t)\,dy+\varepsilon,\quad x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad x\in\Omega,\end{gathered}\tag{2.5}$$

where $\varepsilon=\varepsilon_m$. The notion of a solution u_ε for problem (2.5) can be defined in a similar way as in Definition 2.1.

Theorem 2.6. *There exists T^* ($0<T^*\le\infty$) such that (2.5) has a unique solution $(u(x,t),v(x,t))\in C(\bar\Omega\times[0,T^*))\cap C^{2,1}(\Omega\times(0,T^*))$.*

Let $\varepsilon_2 > \varepsilon_1$. Obviously, $u_{\varepsilon_2(x,t)}, v_{\varepsilon_2}(x,t)$ is a pair of strict supersolution of (2.5) with $\varepsilon = \varepsilon_1$. By the comparison principle for problem (2.5) we have that $u_{\varepsilon_1} < u_{\varepsilon_2}$, $v_{\varepsilon_1} < v_{\varepsilon_2}$. Taking $\varepsilon \to 0$, we get

$$u_M(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t), \quad v_M(x,t) = \lim_{\varepsilon\to 0} v_\varepsilon(x,t)$$

with $u_M(x,t) \geq 0, v_M(x,t) \geq 0$. It is easy to check that $(u_M(x,t), v_M(x,t))$ are solutions of (1.1). Let $\bar{u}, \bar{v}$ be any solution of (1.1), then by comparison principle $u_\varepsilon \geq \bar{u}$, $v_\varepsilon \geq \bar{v}$. Taking $\varepsilon \to 0$, we conclude that $u_M \geq \bar{u}$, $v_M \geq \bar{v}$. So we have the existence of a local solution.

Theorem 2.7. *There exists T $(0 < T \leq \infty)$ such that* (1.1) *has a maximal solution* $(u(x,t), v(x,t)) \in C(\bar{\Omega}\times[0,T)) \cap C^{2,1}(\Omega\times(0,T))$.

We now give the uniqueness for solutions to (1.1).

Theorem 2.8. *Assume that* (H1), (H2) *hold. If $p, q, r \geq 1$, or if $p, q < 1$ or $r < 1$, then* (1.1) *has a unique solution* $(u(x,t), v(x,t)) \in C(\bar{\Omega}\times[0,T)) \cap C^{2,1}(\Omega\times(0,T))$.

Lemma 2.9. *Assume* (H1) *holds,* $\min\{\max\{p,q\}, r\} \leq 1$, *and* $u_0(x) \equiv 0$, $v_0(x) \equiv 0$. *Then maximal solution u_M, v_M of* (1.1) *is strictly positive for $x \in \bar{\Omega}$ and all positive time, as long as it exists.*

Proof of Theorem 2.8. Case 1: $p, q, r \geq 1$. Assume that (1.1) has the maximal solution $(u_M(x,t), v_M(x,t))$ and another solution $(u(x,t), v(x,t))$. Then there exists $t_0 \geq 0$, such that $(u_M(x,t), v_M(x,t)) \equiv (u(x,t), v(x,t))$ for $x \in \bar{\Omega}$, $0 \leq t \leq t_0$ and $(u_M(x,t), v_M(x,t)) \not\equiv (u(x,t), v(x,t))$ for $x \in \bar{\Omega}$, $t_0 \leq t \leq t_0+\gamma$ with $\gamma \in (0, T-t_0)$. We can assume that $t_0 = 0$. Let $G(x,y;t)$ denote the Green's function for the heat equation

$$u_t - \Delta u = 0, \quad x \in \Omega,\ t > 0$$

with a boundary condition $u = 0$ for $x \in \partial\Omega$, $t > 0$. Then from (1.1) we obtain

$$\begin{aligned} u_m(x,t) - u(x,t) &= \int_0^t \int_\Omega G(x,y;t-\eta)\rho_1(\theta_v)(v_m - v)\,dy\,d\eta \\ &\quad - \int_0^t \int_{\partial\Omega} \frac{\partial G(x,\xi;t-\eta)}{\partial n} \int_\Omega f(\xi,y)h_1(\theta_u)(u_m - u)\,dy\,d\xi\ d\eta, \end{aligned}$$

where $\rho_1(\theta_v)$ and $h_1(\theta_u)$ are continuous functions in $\bar{Q}_T$. Due to the assumptions of in this theorem and Lemma 2.3, we have

$$u_m(x,t) - u(x,t) \leq \sigma_1(T)\{\sup_{Q_T}(v_m(x,t) - v(x,t)) + \sup_{Q_T}(u_m(x,t) - u(x,t))\},$$

where

$$\begin{aligned} \sigma_1(T) &= \sup_{Q_T} \int_0^t \int_\Omega G(x,y;t-\eta)\rho_1(\theta_v)\,dy\,d\eta \\ &\quad + \sup_{Q_T} \int_0^t \int_{\partial\Omega} -\frac{\partial G(x,\xi;t-\eta)}{\partial n} \int_\Omega f(\xi,y)h_1(\theta_u)\,dy\,d\xi\,d\eta. \end{aligned}$$

Similarly we can prove that

$$v_m(x,t) - v(x,t) \leq \sigma_2(T)\{\sup_{Q_T}(u_m(x,t) - u(x,t)) + \sup_{Q_T}(v_m(x,t) - v(x,t))\}.$$

Choosing T so small that $\sigma_1(T) + \sigma_2(T) < 1$, we prove the uniqueness of solution for (1.1) in Q_T.

Case 2: $p, q < 1$ or $r < 1$. We distinguish three cases: $r < 1$ and $p, q \le 1$; $r < 1$ and $p, q > 1$; or $r \ge 1$ and $p, q < 1$.

To prove the uniqueness it suffices to show that if (u, v) is any solution of (1.1), then

$$u(x,t) \ge u_M(x,t), \quad v(x,t) \ge v_M(x,t). \tag{2.6}$$

Let $r < 1$ and $p, q \le 1$. Set $w = u_M - u$, $z = v_M - v$. Since $r < 1$ and $p, q \le 1$, it is easy to verify that

$$\begin{gathered}
w_t \le \Delta w + z^p, \quad (x,t) \in Q_{T_1}, \\
z_t \le \Delta z + w^q, \quad (x,t) \in Q_{T_1}, \\
w(x,t) \le \int_\Omega f(x,y) w^r(y,t)\,dy, \quad (x,t) \in S_{T_1}, \\
z(x,t) \le \int_\Omega g(x,y) z^r(y,t)\,dy, \quad (x,t) \in S_{T_1}, \\
w(x,0) \equiv 0, \quad z(x,0) \equiv 0, \quad x \in \Omega.
\end{gathered}$$

By Lemma 2.9 there exists a unique solution $w^0(x,t) > 0, z^0(x,t)$ for $x \in \bar{\Omega}$, $0 < t < T_2$ satisfying the equations in (1.1) and the boundary conditions

$$w^0(x,0) = 0, \quad z^0(x,0) = 0.$$

In a similar way as that used in Lemma 2.4, we can prove that $w^0(x,t) \ge w(x,t)$, $z^0(x,t) \ge z(x,t)$ and $u_M(x,t) \ge w^0(x,t)$, $v_M(x,t) \ge z^0(x,t)$. We put $a(x,t) = w^0(x,t) - w(x,t)$, $b(x,t) = z^0(x,t) - z(x,t)$ and obtain

$$\begin{gathered}
a_t \ge \Delta a + b^p, \quad (x,t) \in Q_{T_3}, \\
b_t \ge \Delta b + a^q, \quad (x,t) \in Q_{T_3}, \\
a(x,t) \ge \int_\Omega f(x,y) a^r(y,t)\,dy, \quad (x,t) \in S_{T_3}, \\
b(x,t) \ge \int_\Omega g(x,y) b^r(y,t)\,dy, \quad (x,t) \in S_{T_3}, \\
a(x,0) \equiv 0, \quad b(x,0) \equiv 0, \quad x \in \Omega.
\end{gathered}$$

It is not difficult to prove that $a(x,t) > 0, b(x,t) > 0$ in Q_T. Thus by the comparison principle we conclude that $a(x,t) \ge w^0(x,t)$, $b(x,t) \ge z^0(x,t)$. This implies (2.6) and the theorem holds under the assumptions $r < 1$ and $p, q \le 1$. For the other cases we can discuss in a similar way. □

3. Existence of global solutions for $p, q \le 1$

Theorem 3.1. *Let $p, q \le 1$. Then for all $r \le 1$, the solution (u, v) of* (1.1) *exists globally for any nonnegative initial data.*

Proof. We first suppose $0 < r < 1$. By the conditions for $f(x,y), g(x,y)$, there exists a constant $M > 1$ such that $f(x,y) \le M$ and $g(x,y) \le M$ for all $(x,y) \in \partial\Omega \times \bar{\Omega}$. It is easy to see that the pair of functions $(\hat{u}, \hat{v}) = (Ce^{\beta t}, Ce^{\beta t})$ is a strict supersolution of (1.1) if $\beta \ge M$ and $C \ge \max\{\sup_{\overline{\Omega}} u_0(x), \sup_{\overline{\Omega}} v_0(x), (M|\Omega|)^{\frac{1}{1-r}}\}$.

In the case $r = 1$, the pair of function $(\hat{u}, \hat{v}) = (Ce^{\beta t}, Ce^{\beta t})$ is a strict supersolution of (1.1) if and only if

$$\int_\Omega f(x,y)\,dy < 1, \int_\Omega g(x,y)\,dy < 1 \quad \text{for all } x \in \partial\Omega. \tag{3.1}$$

Therefore, when (3.1) is not valid, we need to construct another supersolution. Denote by $\varphi(x)$ the eigenfunction corresponding to the first eigenvalue λ_1^Ω of the elliptic problem

$$-\Delta\varphi = \lambda\varphi, \quad x \in \Omega; \quad \varphi = 0, \quad x \in \partial\Omega. \tag{3.2}$$

and that for $0 < \varepsilon < 1$ satisfies

$$M \int_\Omega \frac{1}{\varphi(y) + \varepsilon}\, dy \le 1.$$

Now we set $\hat{u} = \frac{Ce^{\gamma t}}{\varphi(x)+\varepsilon}$, $\hat{v} = \frac{Ce^{\gamma t}}{\varphi(x)+\varepsilon}$. A simple computation shows

$$\begin{aligned} \hat{u}_t - \Delta\hat{u} - \hat{v}^p &\ge \gamma\hat{u} - \hat{u}\Big(\frac{\lambda_1^\Omega\varphi}{\varphi(x)+\varepsilon} + \frac{2|\nabla\varphi|^2}{(\varphi(x)+\varepsilon)^2}\Big) - \hat{v} \ge 0, \\ \hat{v}_t - \Delta\hat{v} - \hat{u}^q &\ge \gamma\hat{v} - \hat{v}\Big(\frac{\lambda_1^\Omega\varphi}{\varphi(x)+\varepsilon} + \frac{2|\nabla\varphi|^2}{(\varphi(x)+\varepsilon)^2}\Big) - \hat{u} \ge 0, \end{aligned} \tag{3.3}$$

if we choose

$$C \ge \max\{\sup_{\overline{\Omega}}(\varphi(x)+\varepsilon), \sup_{\overline{\Omega}} u_0(x) \sup_{\overline{\Omega}}(\varphi(x)+\varepsilon), \sup_{\overline{\Omega}} v_0(x) \sup_{\overline{\Omega}}(\varphi(x)+\varepsilon)\},$$

and

$$\gamma \ge \lambda_1^\Omega + \sup_{\overline{\Omega}} \frac{2|\nabla\varphi|^2}{(\varphi+\varepsilon)^2} + 1.$$

It is clear from (3.3) and the choice of C, γ that $(\hat{u}, \hat{v})$ is a strict supersolution of (1.1). Thus the solution of problem (1.1) exists globally. □

4. Blow-up in finite time for $p, q > 1$

In this section we will get some blow-up results for (1.1). First, we give the following lemma which is crucial in proving the blow-up results.

Lemma 4.1. *There exists a positive solution ϕ for the elliptic eigenvalue problem*

$$-\Delta\phi = \lambda\phi, \quad x \in \Omega, \quad \min_{x\in\overline{\Omega}} \phi = 1. \tag{4.1}$$

Proof. Since Ω is a bounded domain in R^N $(N \ge 1)$ with smooth boundary $\partial\Omega$, we can choose another bounded domain Ω_1 such that $\Omega \subset\subset \Omega_1$. Let $\lambda_1^{\Omega_1}$ be the first eigenvalue of following elliptic eigenvalue problem

$$-\Delta\phi = \lambda\phi, \quad x \in \Omega_1, \quad \phi = 0, \quad x \in \partial\Omega_1$$

and ϕ_1 is the corresponding eigenfunction with $\phi_1 > 0$. Then $\phi_1 \ge \delta > 0$ in $\overline{\Omega}$. Set $\phi = \frac{1}{\delta}\phi_1$, then ϕ is the function satisfying (4.1). □

Remark 4.2. From the continuity of eigenvalue to the domain Ω, we can choose Ω_1 such that $\lambda_1^\Omega > \lambda_1^{\Omega_1} > \lambda_1^\Omega - \varepsilon$ for some constant ε, sufficiently small.

Now we turn to the blow-up conclusions. We denote

$$K = \max_{x\in\overline{\Omega}} \phi(x), \quad h_0 = \min\{\min_{x\in\partial\Omega} \int_\Omega f(x,y)\, dy, \min_{x\in\partial\Omega} \int_\Omega g(x,y)\, dy\}.$$

From Lemma 4.1 and the assumption (H1) we can see $K > 1$ and $h_0 > 0$.

Theorem 4.3. *Let $p, q > 1$. Then if $r > 1$ and $h_0|\Omega| > K^{\min\{p,q\}}$, the solution (u, v) of (1.1) blows up in finite time for the large initial data.*

Proof. Without lose of generality, we can assume that $p \geq q$. Set $\tilde{u} = s^l(t)\phi^q(x), \tilde{v} = s^l(t)\phi^q(x)$, where l is a constant and satisfies $p \geq q > l$, $r > l$. The function ϕ is defined as in Lemma 4.1. Let $s(t)$ be the solution to the problem

$$s'(t) = -\lambda_1^{\Omega_1} q s(t) + s^l(t), \quad t > 0, \quad s(0) = s_0. \tag{4.2}$$

with initial data $s_0 \geq \max\{(\lambda_1^{\Omega_1} q)^{\frac{1}{l-1}}, 1\}$. It is easy to see that $s(t) \geq 1$ and blows up in finite time T_{s_0}. A direct computation yields

$$\begin{aligned} \tilde{u}_t - \Delta\tilde{u} - \tilde{v}^p &= l s^{l-1}(t)s'(t)\phi^q - s^l(t)[-\lambda_1 q\phi^q + q(q-1)\phi^{q-2}|\nabla\phi|^2] - s^{lp}(t)\phi^{pq} \\ &\leq l s^{l-1} s'(t)\phi^q + s^l(t)\lambda_1 q\phi^q - s^l(t)\phi^q \\ &= l s^{l-1}(t)(s'(t) + \lambda_1 q s(t) - s^l(t))\phi^q = 0, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \tilde{v}_t - \Delta\tilde{v} - \tilde{u}^q &= l s^{\alpha-1} s'(t)\phi^q - s^l(t)[-\lambda_1 q\phi^q + q(q-1)\phi^{q-2}|\nabla\phi|^2] - s^{lq}(t)\phi^{q^2} \\ &\leq l s^{l-1} s'(t)\phi^q + s^l(t)\lambda_1 q\phi^q - s^l\phi^q \\ &= l s^{l-1}(t)(s'(t) + \lambda_1 q s(t) - s^l(t))\phi^q = 0, \end{aligned} \tag{4.4}$$

in $\Omega \times (0, T_{s_0})$, and for $x \in \partial\Omega \times (0, T_{s_0})$. So we have

$$\begin{aligned} \tilde{u}(x,t) &\leq s^l(t)K^q \leq h_0 s^l(t)|\Omega| \\ &\leq \int_\Omega f(x,y)s^{lr}(t)\phi^{qr}(y)\,dy \\ &= \int_\Omega f(x,y)\tilde{u}^r(y,t)\,dy, \\ \tilde{v}(x,t) &\leq s^l(t)K^q \leq h_0 s^l(t)|\Omega| \\ &\leq \int_\Omega g(x,y)s^{lr}(t)\phi^{qr}(y)\,dy \\ &= \int_\Omega g(x,y)\tilde{v}^r(y,t)\,dy. \end{aligned} \tag{4.5}$$

From (4.3)-(4.5) we can see that $(\tilde{u}, \tilde{v})$ is a subsolution provided the initial data so large that $s^l(0)\phi^q(x) \leq u_0(x), s^l(0)\phi^q(x) \leq v_0(x)$ for $x \in \Omega$. Thus by Lemma 2.4, the solution (u,v) of problem (1.1) blows up because $(\tilde{u}, \tilde{v})$ blows up in finite time. □

5. Blow-up and global solution for $p > 1 > q$ or $q > 1 > p$

Theorem 5.1. *Let $p > 1 > q$ or $q > 1 > p$. If $pq < 1$, $r \leq 1$, and*

$$\int_\Omega f(x,y)\,dy \leq 1, \quad \int_\Omega g(x,y)\,dy \leq 1$$

for all $x \in \partial\Omega$, then the solution (u,v) of (1.1) *exists globally for sufficiently small initial data.*

Proof. We assume $p > 1 > q$. Since $pq < 1$, there exists a constant l such that $0 < pq \leq l < 1$. Let $s(t)$ be the unique solution to the problem

$$s'(t) = s^l(t), \quad t > 0, \quad s(0) = s_0$$

with initial data $s_0 > 1$. It is easy to see that $s(t) > 1$ and exists globally. Set $\hat{u} = s^{p+1}(t)$ and $\hat{v} = s^{q+1}(t)$. Then we have

$$\begin{aligned}\hat{u}_t - \Delta\hat{u} - \hat{v}^p &= (p+1)s^p(t)s'(t) - s^{(q+1)p}(t)\\ &= (p+1)s^p(t)s^l(t) - s^{(q+1)p}(t)\\ &> s^{p+l}(t) - s^{(q+1)p}(t) \geq 0,\end{aligned} \tag{5.1}$$

$$\begin{aligned}\hat{v}_t - \Delta\hat{v} - \hat{u}^q &= (q+1)s^q(t)s'(t) - s^{(p+1)q}(t)\\ &= (q+1)s^q(t)s^l(t) - s^{(p+1)q}(t)\\ &> s^{q+l}(t) - s^{(p+1)q}(t) \geq 0,\end{aligned} \tag{5.2}$$

in $\Omega \times (0,\infty)$, and for $x \in \partial\Omega \times (0,\infty)$. So we have

$$\begin{aligned}\hat{u} = s^{p+1}(t) \geq \int_\Omega f(x,y)s^{(p+1)r}(t)\,dy = \int_\Omega f(x,y)\hat{u}^r\,dy,\\ \hat{v} = s^{q+1}(t) \geq \int_\Omega g(x,y)s^{(q+1)r}(t)\,dy = \int_\Omega g(x,y)\hat{v}^r\,dy.\end{aligned} \tag{5.3}$$

From (5.1)–(5.3) we see that $(\hat{u},\hat{v})$ is a supersolution provided that $s_0^{p+1} \geq u_0(x)$, $s_0^{q+1} \geq v_0(x)$ for $x \in \Omega$. The case $q > 1 > p$ can be treated by exchanging the roles of u and v in the above case. □

Theorem 5.2. *Let $p > 1 > q$ or $q > 1 > p$. If $pq > 1$, $r \geq 1$, and*

$$\int_\Omega f(x,y)\,dy \geq 1, \int_\Omega g(x,y)\,dy \geq 1$$

for all $x \in \partial\Omega$, then the solution (u,v) of (1.1) blows up in finite time for sufficiently large initial data.

Proof. We assume $p > 1 > q$. Since $pq > 1$, there exists a constant l such that $1 < l \leq pq$. Let $s(t)$ be the unique solution to the problem

$$s'(t) = s^l(t), \quad t > 0, \quad s(0) = s_0$$

with the initial data $s_0 > 1$. It is easy to see that $s(t) > 1$ and blows up in finite time. Set $\tilde{u} = s^{p+1}(t)$, $\tilde{v} = s^{q+1}(t)$, then we have

$$\begin{aligned}\tilde{u}_t - \Delta\tilde{u} - \tilde{v}^p &= (p+1)s^p(t)s'(t) - s^{(q+1)p}(t)\\ &= (p+1)s^p(t)s^l(t) - s^{(q+1)p}(t)\\ &< s^{p+l}(t) - s^{(q+1)p}(t) \leq 0,\end{aligned} \tag{5.4}$$

$$\begin{aligned}\tilde{v}_t - \Delta\tilde{v} - \tilde{u}^q &= (q+1)s^q(t)s'(t) - s^{(p+1)q}(t)\\ &= (q+1)s^q(t)s^l(t) - s^{(p+1)q}(t)\\ &< s^{q+l}(t) - s^{(p+1)q}(t) \leq 0,\end{aligned} \tag{5.5}$$

in $\Omega \times (0,\infty)$, and for $x \in \partial\Omega \times (0,\infty)$. So we have

$$\begin{aligned}\tilde{u} = s^{p+1}(t) \leq \int_\Omega f(x,y)s^{(p+1)r}(t)\,dy = \int_\Omega f(x,y)\tilde{u}^r\,dy,\\ \tilde{v} = s^{q+1}(t) \leq \int_\Omega g(x,y)s^{(q+1)r}(t)\,dy = \int_\Omega g(x,y)\tilde{v}^r\,dy.\end{aligned} \tag{5.6}$$

From (5.4)–(5.6) we can see that $(\tilde{u},\tilde{v})$ is a supersolution provided $s_0^{p+1} \le u_0(x)$, $s_0^{q+1} \le v_0(x)$ for $x \in \Omega$. Since $(\tilde{u},\tilde{v})$ blows up in finite time. The case $q > 1 > p$ can be treated by exchanging the roles of u and v in the above case. The proof is complete. □

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