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REGULARITY CRITERIA OF SUPERCRITICAL BETA-GENERALIZED QUASI-GEOSTROPHIC EQUATIONS IN TERMS OF PARTIAL DERIVATIVES

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Abstract. We study the two-dimensional beta-generalized supercritical quasi-geostrophic equation, and in particular show that to obtain global regularity results, one needs to bound only its partial derivative. Results may be generalized to similar active scalars, e.g. the porous media equation.

1. Introduction and statement of results

We study the two-dimensional β-generalized surface quasi-geostrophic equation (β-SQG)

$$\begin{gathered}\frac{\partial\theta}{\partial t}+(u\cdot\nabla)\theta+\nu\Lambda^{2\alpha}\theta=0\\ \theta(x,0)=\theta_0(x),\quad u=\Lambda^{1-2\beta}(-\mathcal{R}_2\theta,\mathcal{R}_1\theta)\end{gathered}\tag{1.1}$$

and the three-dimensional porous media equation

$$\begin{gathered}\frac{\partial\theta}{\partial t}+(u\cdot\nabla)\theta+\nu\Lambda^{2\alpha}\theta=0\\ \theta(x,0)=\theta_0(x),\quad u=-\kappa(\nabla p+g\gamma\theta),\quad \nabla\cdot u=0\end{gathered}\tag{1.2}$$

In both equations, θ is a scalar function and $\Lambda=(-\Delta)^{1/2}$ with $\alpha\in(0,1/2),\beta\in[1/2,1)$ as fixed parameters. In (1.1), θ represents temperature and we denoted Riesz transform by $\mathcal{R}$. In (1.2), θ represents density, κ the matrix medium permeability divided by viscosity in different directions respectively, g the acceleration due to gravity, vector γ the last canonical vector e_3. Moreover, p is the pressure and u the liquid discharge by Darcy's law (flux per unit area). Hereafter, without loss of generality we set $\nu=1$ and denote $\partial_t=\frac{\partial}{\partial t}$ and $\partial_i=\frac{\partial}{\partial x_i}, i=1,2,3$. We denote by L^p the Lebesgue space equipped with the norm $\|\cdot\|_{L^p}$ while those of Sobolev space H^s with $\|\cdot\|_{H^s}$ and a horizontal gradient operator by $\nabla_h=(\partial_1,\partial_2)$.

The β-SQG was formally introduced in [19]. At $\beta=1/2$, the equation reduces to an important model in geophysical fluid dynamics used in meteorology and oceanography (cf. [26]). The physical and mathematical similarity between the β-SQG at $\nu=0,\beta=\frac{1}{2}$ and the three-dimensional Euler equation as well as the

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β-SQG at $\nu > 0$, $\alpha = \beta = \frac{1}{2}$ and the Navier-Stokes equation (NSE) are discussed in [8]. The latter case is also physically relevant, modeling the Eckmann pumping effect in the boundary layer near the surface. We also note that while the Navier-Stokes and the Stokes systems are both microscopic equations, Darcy's law yields a macroscopic description of a flow in the porous medium (cf. [1]).

Due to the rescaling of the solutions to the β-SQG and the fact that the L^p-norms of the solutions for $1 \leq p \leq \infty$ are non-increasing (cf. [13]), we consider the case $\alpha + \beta \in (1, 3/2)$ the subcritical case, the case $\alpha + \beta = 1$ the critical case, and the case $\alpha + \beta \in (1/2, 1)$ the supercritical case. We do so similarly for the system (1.2) with $\beta = 1/2$.

The existence of global weak solution to (1.1) in case $\alpha > 0, \beta = 1/2$ was established in [27], followed by the existence of the global unique smooth solution in case $\beta = 1/2$, $\alpha > 1/2$ in [10]. The same result in the case $\beta = 1/2$, $\alpha = 1/2$ was completed in [22] while the authors in [2] showed that solutions with initial data in L^2 are locally smooth for any space dimension. These works were subsequently generalized to allow $\beta \neq 1/2$ in [7, 24, 25, 36]; in particular, in [19] the author proved the global regularity of the unique solution in the critical case for α and β in the range of our consideration. More recently, the third and fourth proofs in the case $\alpha = \beta = 1/2$ appeared in [9, 20].

Concerning the system (1.2), the authors in [3] showed the existence of the unique global solution in the critical and subcritical cases. In the supercritical case, they obtained local results in $H^s, s > (5 - 2\alpha)/2$ and extended to be global under the condition $\|\theta_0\|_{H^s} \leq c\nu$ for some fixed small constant $c > 0$. In [33], the author considered the supercritical case and obtained local results using iterative process while in [39], the authors considered the critical case and obtained the global existence and uniqueness of the solution in critical Besov spaces $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$ with $1 \leq p \leq \infty$ by the method introduced in [22]. Finally, in [35] the author considered a modified porous media equation, analogously to the modified QG in [7] and obtained global regularity results in the critical case with initial data $\theta_0 \in H^s, s > 5/2$.

Regularity criteria results for β-SQG started in [8]: at $\beta = 1/2$, $\nu = 0$,

$$\limsup_{t \nearrow T} \|\theta(t)\|_{H^m} < \infty \text{ if and only if } \int_0^T \|\nabla^\perp \theta(t)\|_{L^\infty} dt < \infty$$

Extension and improvements of this type of results were followed by many (e.g. [4, 5, 11, 12, 14, 16, 29, 31, 32, 38, 40]). In particular, the author in [4] obtained a Serrin-type regularity criteria which showed that at $\beta = 1/2, \alpha \in (0, \frac{1}{2}]$, if the solution $\theta(x,t)$ satisfied

$$\int_0^T \|\nabla^\perp \theta\|_{L^p}^r dt < \infty \quad \text{for some } p, r \text{ such that } \frac{2}{p} + \frac{2\alpha}{r} \leq 2\alpha, \quad \frac{1}{\alpha} < p < \infty,$$

then there is no singularity up to time T. These results are in harmony with the finding of [8] that $\nabla^\perp \theta$ plays an analogous role for the system (1.1) as the vorticity of the three-dimensional Euler equation does. Recently, the author in [34] provided a first regularity criteria that relies only on a partial derivative of θ in the critical case. In the proof, the fact that $\alpha + \beta = 1$ was crucial and hence in the supercritical case, it was not clear whether the global regularity relies only on a bound of a partial derivative of any order. In this paper we provide an affirmative answer to this question. Let us present our results:

Theorem 1.1. *Let $\alpha \in (0, \frac{1}{2}), \beta \in [1/2, 1)$ so that $\alpha + \beta \in (\frac{1}{2}, 1)$. If the solution $\theta(x,t)$ to (1.1) in $\mathbb{R}^2$ or $\mathbb{T}^2$ satisfied*

$$\int_0^T \|\partial_1 \theta\|_{L^p}^r dt < \infty \tag{1.3}$$

for some p, r such that

$$\frac{2}{p} + \frac{2\alpha}{r} \le 2\alpha, \quad \frac{2(1-\alpha)}{\alpha^2} < p < \infty \quad \text{if } \beta = 1/2$$
$$\frac{2}{p} + \frac{2\alpha}{r} \le 2\alpha + 2\beta - 1, \quad \frac{2(1-\alpha)}{\alpha^2 + 2\beta - 1} < p < \frac{2(1-\alpha)}{2\beta - 1} \quad \text{if } \beta > 1/2$$

then there is no singularity up to time T. Also, if

$$\sup_{t \in [0,T]} \|\partial_1 \theta\|_{L^{\frac{2}{2\alpha + 2\beta - 1}}}$$

is sufficiently small, then there is no singularity up to time T.

Theorem 1.2. *Let $\alpha \in (0, 1/2)$. If the solution $\theta(x,t)$ to (1.2) in $\mathbb{R}^3$ or $\mathbb{T}^3$ satisfies*

$$\int_0^T \|\nabla_h \theta\|_{L^p}^r dt < \infty \tag{1.4}$$

for some p, r such that

$$\frac{3}{p} + \frac{2\alpha}{r} \le 2\alpha, \quad \frac{9 - 6\alpha}{2\alpha^2} < p < \infty,$$

then there is no singularity up to time T. Also, if

$$\sup_{t \in [0,T]} \|\nabla_h \theta(t)\|_{L^{\frac{3}{2\alpha}}}$$

is sufficiently small, then there is no singularity up to time T.

Next theorem was partially inspired by the work in [28] where the author obtained a regularity criteria for the NSE in terms of $\operatorname{div}(u/|u|)$ (see also [6]). We make use of the fact that that due to the incompressibility of u,

$$-\sum_{i=1}^{2} \partial_1 u_i \partial_{1i} \theta = (\partial_1 \theta)^2 \sum_{i=1}^{2} \partial_i \left(\frac{\partial_1 u_i}{\partial_1 \theta} \right) = (\partial_1 \theta)^2 \operatorname{div} \left(\frac{\partial_1 u}{\partial_1 \theta} \right)$$

Theorem 1.3. *Let $\alpha \in (0, 1/2), \beta \in [1/2, 1)$ so that $\alpha + \beta \in (1/2, 1)$. If the solution $\theta(x,t)$ to (1.1) in $\mathbb{R}^2$ or $\mathbb{T}^2$ satisfies*

$$\int_0^T \| \operatorname{div}(\frac{\partial_1 u}{\partial_1 \theta}) \|_{L^p}^r dt < \infty \tag{1.5}$$

for some p, r such that

$$\frac{2}{p} + \frac{2\alpha}{r} \le 2\alpha, \quad \frac{2(1-\alpha)^2}{\alpha^3} < p < \infty \quad \text{if } \beta = 1/2$$
$$\frac{2}{p} + \frac{2\alpha}{r} \le 2\alpha, \quad \frac{2(1-\alpha)^2}{\alpha(\alpha^2 + 2\beta - 1)} < p < \frac{2(1-\alpha)^2}{\alpha(2\beta - 1)} \quad \text{if } \beta > 1/2$$

then there is no singularity up to time T. Also, if

$$\sup_{t\in[0,T]} \| \operatorname{div}(\frac{\partial_1 u}{\partial_1 \theta})\|_{L^{1/\alpha}} < \infty,$$

then there exists $C = C(\alpha, \beta)$ such that

$$\|\theta_0\|_{L^\infty} < C$$

which implies that there is no singularity up to time T.

Remark 1.4. (1) The Roles of x_1, x_2, x_3 in the hypothesis may be switched.

(2) Theorem 1.1 may be seen as a component reduction type results in comparison to the [4, Theorem 1.1]. In the case of $\mathbb{T}^3$, one may also compare our Theorem 1.2 with [39, Theorem 1.3] (wee also [3, Remark 2.4]). Upon succeeding the reduction component type result, it is common that the range of p, r become restrictive (cf. e.g. [23]). From the proof, it becomes clear that one can also extend Theorems 1.1 and 1.3 for dispersive SQG introduced in [21].

(3) In fact, in the case of β-SQG with $\beta = \frac{1}{2}$, because $\widehat{\Lambda u_2}(\xi) = \widehat{\partial_1 \theta}(\xi)$ we have shown that

$$\int_0^T \|\Lambda u_2\|_{L^p}^r dt < \infty$$

under the same condition of Theorem 1.1 implies global regularity. It is not difficult to show that there exists a constant $c > 0$ such that

$$\|\Lambda u_2\|_{L^p} \leq c\|\nabla u_2\|_{L^p}$$

holds for $p \in (1, \infty)$. Hence, we have also shown that

$$\int_0^T \|\nabla u_2\|_{L^p}^r dt < \infty$$

implies global regularity of the solution. In relevance, we refer readers to [29] in which the author obtained a regularity criteria that involves ∇u.

(4) Results of this type exist for NSE (e.g. [23]). However, to the best of our knowledge, there does not exist any regularity criteria in terms of partial derivatives for the generalized NSE with fractional Laplacian with power arbitrary close to zero.

(5) In general for systems like NSE or MHD, one relies on a decomposition of nonlinear term such as the [23, Lemma 2.2] that makes use of the incompressibility of the solution itself. The key observation in this paper is that if we start with the L^p estimate of a partial derivative for $p > 2$, then because the vector u dotted with a gradient within the nonlinear term is incompressible, the partial derivatives within the nonlinear term may be shifted to miss one direction.

(6) It is not clear if we can obtain analogous results for different range of α and β as considered in [25].

In the next section, we prove Theorems 1.1 and 1.3. The proof of Theorem 1.2 is similar to that of Theorem 1.1 and will be addressed in [37] with different results; hence we just sketch it in the Appendix.

2. Proofs

To prove Theorem 1.1, we need the following proposition.

Proposition 2.1. *Let $\alpha \in (0, \frac{1}{2}), \beta \in [1/2, 1)$ so that $\alpha + \beta \in (\frac{1}{2}, 1)$. If the solution $\theta(x,t)$ to (1.1) in $[0,T]$ satisfies (1.3), then*

$$\sup_{t\in[0,T]} \|\partial_1\theta(t)\|_{L^{\frac{2\alpha p}{2+p(1-2\beta)}}}^{\frac{2\alpha p}{2+p(1-2\beta)}} + \|\partial_2\theta(t)\|_{L^{\frac{2\alpha p}{2+p(1-2\beta)}}}^{\frac{2\alpha p}{2+p(1-2\beta)}}$$
$$+ \int_0^T \|\partial_1\theta\|_{L^{(\frac{\alpha}{1-\alpha})[\frac{2p}{2+p(1-2\beta)}]}}^{\frac{2\alpha p}{2+p(1-2\beta)}} + \|\partial_2\theta\|_{L^{(\frac{\alpha}{1-\alpha})[\frac{2p}{2+p(1-2\beta)}]}}^{\frac{2\alpha p}{2+p(1-2\beta)}} dt < \infty$$

Proof. We fix p and r that satisfy (1.3) and then define

$$q = \frac{2\alpha p}{2+p(1-2\beta)}$$

One can check that $q \in (2, \infty)$. We apply ∂_1 on (1.1), multiply by $q|\partial_1\theta|^{q-2}\partial_1\theta$ and integrate in space to obtain

$$\partial_t\|\partial_1\theta\|_{L^q}^q + q\int(\Lambda^{2\alpha}\partial_1\theta)|\partial_1\theta|^{q-2}\partial_1\theta = -q\int\partial_1((u\cdot\nabla)\theta)|\partial_1\theta|^{q-2}\partial_1\theta$$

□

We use the following lemma from [17].

Lemma 2.2. *For $\alpha \in [0, 1]$, $x \in \mathbb{R}^2$, $\mathbb{T}^2, \Lambda^{2\alpha}\theta \in L^s$, $s \geq 2$, we have*

$$2\int|\Lambda^\alpha\theta^{\frac{s}{2}}|^2 \leq s\int\Lambda^{2\alpha}\theta|\theta|^{s-2}\theta\,.$$

Proof of Theorem 1.1. By this lemma and the homogeneous Sobolev embedding $\dot{H}^\alpha \hookrightarrow L^{\frac{2}{1-\alpha}}$, we have

$$c(q,\alpha)\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}^q \leq 2\int|\Lambda^\alpha(\partial_1\theta)^{q/2}|^2 \leq q\int(\Lambda^{2\alpha}\partial_1\theta)|\partial_1\theta|^{q-2}\partial_1\theta$$

Expanding the right hand side now, we have

$$q\int\partial_1((u\cdot\nabla)\theta)|\partial_1\theta|^{q-2}\partial_1\theta = q\int\partial_1 u\cdot\nabla\theta|\partial_1\theta|^{q-2}\partial_1\theta + u\cdot\nabla\partial_1\theta|\partial_1\theta|^{q-2}\partial_1\theta$$

The second term vanishes by the incompressibility of u. We further bound by

$$\begin{aligned}
&-q\int\partial_1 u_1\partial_1\theta|\partial_1\theta|^{q-2}\partial_1\theta + \partial_1 u_2\partial_2\theta|\partial_1\theta|^{q-2}\partial_1\theta\\
&\leq c\Big(\|\partial_1 u_1\|_{L^{\frac{2q}{2(1-\alpha)-q(2\beta-1)}}}\|\partial_1\theta\|_{L^q}^{q-1}\|\partial_1\theta\|_{L^{\frac{2q}{2\alpha+q(2\beta-1)}}}\\
&\quad + \|\partial_1 u_2\|_{L^{\frac{2q}{2(1-\alpha)-q(2\beta-1)}}}\|\partial_2\theta\|_{L^q}\|\partial_1\theta\|_{L^q}^{q-2}\|\partial_1\theta\|_{L^{\frac{2q}{2\alpha+q(2\beta-1)}}}\Big)\\
&\leq c\Big(\|\partial_1\mathcal{R}_2\theta\|_{L^{\frac{q}{1-\alpha}}}\|\partial_1\theta\|_{L^q}^{q-1}\|\partial_1\theta\|_{L^{\frac{2q}{2\alpha+q(2\beta-1)}}}\\
&\quad + \|\partial_1\mathcal{R}_1\theta\|_{L^{\frac{q}{1-\alpha}}}\|\partial_2\theta\|_{L^q}\|\partial_1\theta\|_{L^q}^{q-2}\|\partial_1\theta\|_{L^{\frac{2q}{2\alpha+q(2\beta-1)}}}\Big)
\end{aligned}$$

by Hölder's inequality and the homogeneous Sobolev embedding

$$\dot{W}^{2\beta-1,\frac{q}{1-\alpha}} \hookrightarrow L^{\frac{2q}{2(1-\alpha)-q(2\beta-1)}}.$$

Now we use continuity of the Riesz transform in $L^s, s \in (1,\infty)$ for a bound from above,

$$c\Big(\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}\|\partial_1\theta\|_{L^q}^{q-1}\|\partial_1\theta\|_{L^{\frac{2q}{2\alpha+q(2\beta-1)}}}$$

$$
\begin{aligned}
&+\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}\|\partial_2\theta\|_{L^q}\|\partial_1\theta\|_{L^q}^{q-2}\|\partial_1\theta\|_{L^{\frac{2q}{2\alpha+q(2\beta-1)}}}\Big)\\
&\le \epsilon\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}^q+c\|\partial_1\theta\|_{L^{\frac{2q}{2\alpha+q(2\beta-1)}}}^{\frac{q}{q-1}}(\|\partial_1\theta\|_{L^q}^q+\|\partial_2\theta\|_{L^q}^q)
\end{aligned}
$$

for any $\epsilon>0$ where we used Young's inequality twice. Next, applying ∂_2 on (1.1) and following similar procedure as above leads to

$$
\begin{aligned}
&\partial_t\|\partial_2\theta\|_{L^q}^q+c(q,\alpha)\|\partial_2\theta\|_{L^{\frac{q}{1-\alpha}}}^q\\
&\le -q\int \partial_2u_1\partial_1\theta|\partial_2\theta|^{q-2}\partial_2\theta+\partial_2u_2\partial_2\theta|\partial_2\theta|^{q-2}\partial_2\theta\\
&= -q\int \partial_2u_1\partial_1\theta|\partial_2\theta|^{q-2}\partial_2\theta-\partial_1u_1\partial_2\theta|\partial_2\theta|^{q-2}\partial_2\theta\\
&\le \epsilon\|\partial_2\theta\|_{L^{\frac{q}{1-\alpha}}}^q+c\|\partial_1\theta\|_{L^{\frac{2q}{2\alpha+q(2\beta-1)}}}^{\frac{q}{q-1}}\|\partial_2\theta\|_{L^q}^q
\end{aligned}
$$

where the only difference from the previous estimate was the equality which made use of the incompressibility of u so that $\partial_1u_1=-\partial_2u_2$.

In summary, for $\epsilon>0$ sufficiently small we have

$$
\begin{aligned}
&\partial_t(\|\partial_1\theta\|_{L^q}^q+\|\partial_2\theta\|_{L^q}^q)+\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}^q+\|\partial_2\theta\|_{L^{\frac{q}{1-\alpha}}}^q\\
&\le c\|\partial_1\theta\|_{L^{\frac{2q}{2\alpha+q(2\beta-1)}}}^{\frac{q}{q-1}}(\|\partial_1\theta\|_{L^q}^q+\|\partial_2\theta\|_{L^q}^q)
\end{aligned}
$$

Thus, by Gronwall's inequality and (1.3),

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|\partial_1\theta(t)\|_{L^q}^q+\|\partial_2\theta(t)\|_{L^q}^q+\int_0^T\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}^q+\|\partial_2\theta\|_{L^{\frac{q}{1-\alpha}}}^q\,dt\\
&\le c(\|\partial_1\theta_0\|_{L^q}^q+\|\partial_2\theta_0\|_{L^q}^q)e^{\int_0^T\|\partial_1\theta\|_{L^p}^r\,dt}<\infty
\end{aligned}
$$

This completes the proof. $\square$

We will use the following well-known commutator estimate due to [18].

Lemma 2.3. *Let $p\in(1,\infty)$, $s>0$, $f,g\in W^{s,p}$. Then there exists some constant $c>0$ independent of f,g such that for $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$, $p_2,p_3\in(1,\infty)$*

$$
\|\Lambda^s(fg)-f\Lambda^sg\|_{L^p}\le c(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}}+\|\Lambda^sf\|_{L^{p_3}}\|g\|_{L^{p_4}})
$$

Proof of Theorem 1.1. For any $s\in\mathbb{R}^+$, applying Λ^s on (1.1), taking an L^2-inner product with $\Lambda^s\theta$ we have

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^{\alpha+s}\theta\|_{L^2}^2\\
&\le c\Big(\|\nabla u\|_{L^{\frac{2}{\alpha}}}\|\Lambda^{s-1}\nabla\theta\|_{L^2}+\|\Lambda^su\|_{L^{\frac{1}{1-\beta}}}\|\nabla\theta\|_{L^{\frac{2}{2\beta+\alpha-1}}}\Big)\|\Lambda^s\theta\|_{L^{\frac{2}{1-\alpha}}}\\
&\le c\|\nabla\theta\|_{L^{\frac{2}{\alpha+2\beta-1}}}\|\Lambda^s\theta\|_{L^2}\|\Lambda^{s+\alpha}\theta\|_{L^2}
\end{aligned}
$$

by the homogeneous Sobolev embeddings

$$
\dot W^{2\beta-1,\frac{2}{\alpha+2\beta-1}}\hookrightarrow L^{\frac{2}{\alpha}},\quad \dot H^{2\beta-1}\hookrightarrow L^{\frac{1}{1-\beta}},\quad \dot H^{\alpha}\hookrightarrow L^{\frac{2}{1-\alpha}}
$$

By Young's and Gagliardo-Nirenberg inequalities we have the bound

$$
\frac{1}{2}\|\Lambda^{s+\alpha}\theta\|_{L^2}^2+c\|\theta\|_{L^\infty}^{2(\frac{2\alpha+2\beta-1}{1+\alpha})}\|\Lambda^{2+\alpha}\theta\|_{L^2}^{2(\frac{2-\alpha-2\beta}{1+\alpha})}\|\Lambda^s\theta\|_{L^2}^2\,.
$$

Let us use the fact that (cf. [13])

$$\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$$

Gronwall's inequality implies that to complete the proof it suffices to show that

$$\int_0^T \|\Lambda^{2+\alpha}\theta\|_{L^2}^2 dt < \infty$$

We apply Δ to (1.1) and take an L^2-inner product with $\Delta\theta$ to estimate

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\Delta\theta\|_{L^2}^2 + \|\Lambda^\alpha\Delta\theta\|_{L^2}^2 \\
&\le c\Big(\|\nabla u\|_{L^{\frac{2}{\epsilon\alpha}}}\|\Lambda\nabla\theta\|_{L^2} + \|\Lambda^2 u\|_{L^{\frac{1}{1-\beta}}}\|\nabla\theta\|_{L^{\frac{2}{\epsilon\alpha+2\beta-1}}}\Big)\|\Delta\theta\|_{L^{\frac{2}{1-\epsilon\alpha}}}
\end{aligned}$$

by Lemma 2.3 where we choose

$$\epsilon = \frac{2+p(1-2\beta)-2\alpha}{p\alpha^2}$$

One can readily check using (1.3) and the range of α and β that $\epsilon \in (0,1)$. Now by the homogeneous Sobolev embeddings of

$$\dot{W}^{2\beta-1,\frac{2}{\epsilon\alpha+2\beta-1}} \hookrightarrow L^{\frac{2}{\epsilon\alpha}}, \quad \dot{H}^{2\beta-1} \hookrightarrow L^{\frac{1}{1-\beta}}$$

and Gagliardo-Nirenberg inequality, followed by Young's inequality we bound the right-hand side as

$$c\|\nabla\theta\|_{L^{\frac{2}{2\beta+\epsilon\alpha-1}}}\|\Delta\theta\|_{L^2}^{2-\epsilon}\|\Lambda^{2+\alpha}\theta\|_{L^2}^{\epsilon} \le \frac{1}{2}\|\Lambda^\alpha\Delta\theta\|_{L^2}^2 + c\|\nabla\theta\|_{L^{\frac{2}{2\beta+\epsilon\alpha-1}}}^{\frac{2}{2-\epsilon}}\|\Delta\theta\|_{L^2}^2$$

Hence, absorbing the dissipative term, Gronwall's inequality implies

$$\|\Delta\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha\Delta\theta\|_{L^2}^2 dt \le c\|\Delta\theta_0\|_{L^2}^2 e^{\int_0^T \|\nabla\theta\|_{L^{\frac{2}{2\beta+\epsilon\alpha-1}}}^{\frac{2}{2-\epsilon}} dt}$$

for any $t \in [0,T]$. Next, we will use the following elementary inequality several times: for $a,b \ge 0$,

$$(a+b)^s \le 2^s(a^s+b^s), \quad \forall s \ge 0 \tag{2.1}$$

Using this inequality, we observe that

$$\begin{aligned}
&\int_0^T \|\nabla\theta\|_{L^{\frac{2}{2\beta+\epsilon\alpha-1}}}^{\frac{2}{2-\epsilon}} dt \\
&\le \int_0^T \Big(\int 2^{\frac{1}{2\beta+\epsilon\alpha-1}}\big[|\partial_1\theta|^{\frac{2}{2\beta+\epsilon\alpha-1}} + |\partial_2\theta|^{\frac{2}{2\beta+\epsilon\alpha-1}}\big]\Big)^{\frac{2\beta+\epsilon\alpha-1}{2-\epsilon}} dt \\
&\le 2^{\frac{2\beta+\epsilon\alpha}{2-\epsilon}}\int_0^T \|\partial_1\theta\|_{L^{\frac{2}{2\beta+\epsilon\alpha-1}}}^{\frac{2}{2-\epsilon}} + \|\partial_2\theta\|_{L^{\frac{2}{2\beta+\epsilon\alpha-1}}}^{\frac{2}{2-\epsilon}} dt \\
&\le c(\alpha,\beta,\epsilon,T)\int_0^T \|\partial_1\theta\|_{L^{(\frac{\alpha}{1-\alpha})\frac{2p}{2+p(1-2\beta)}}}^{\frac{2\alpha p}{2+p(1-2\beta)}} + \|\partial_2\theta\|_{L^{(\frac{\alpha}{1-\alpha})\frac{2p}{2+p(1-2\beta)}}}^{\frac{2\alpha p}{2+p(1-2\beta)}} dt < \infty\,,
\end{aligned}$$

due to Proposition 2.1. This completes the proof of the first claim.

Now we prove the second claim. By the first claim, it suffices to show that

$$\int_0^T \|\partial_2\theta\|_{L^p}^r dt < \infty$$

for some p, r that satisfy(1.3). We fix p in such a range, apply ∂_2 on (1.1), multiply by $p|\partial_2\theta|^{p-2}\partial_1\theta$ and integrate in space to estimate

$$\partial_t\|\partial_2\theta\|_{L^p}^p + p\int (\Lambda^{2\alpha}\partial_2\theta)|\partial_2\theta|^{p-2}\partial_2\theta = -p\int \partial_2((u\cdot\nabla)\theta)|\partial_2\theta|^{p-2}\partial_2\theta$$

Since $p \geq 2$, by Lemma 2.2 and the same homogeneous Sobolev embedding, as before, we see that due to the incompressibility of u,

$$\begin{aligned}
&\partial_t\|\partial_2\theta\|_{L^p}^p + c(p,\alpha)\|\partial_2\theta\|_{L^{\frac{p}{1-\alpha}}}^p \\
&\leq -p\int \partial_2 u_1\partial_1\theta|\partial_2\theta|^{p-2}\partial_2\theta + p\int \partial_1 u_1\partial_2\theta|\partial_2\theta|^{p-2}\partial_2\theta \\
&\leq p\|\partial_2 u_1\|_{L^{\frac{2p}{2(1-\alpha)-p(2\beta-1)}}}\|\partial_1\theta\|_{L^{\frac{2}{2\alpha+2\beta-1}}}\|\partial_2\theta\|_{L^{\frac{p}{1-\alpha}}}^{p-1} + p\|\partial_1 u_1\|_{L^{\frac{1}{\alpha}}}\|\partial_2\theta\|_{L^{\frac{p}{1-\alpha}}}^p
\end{aligned}$$

by Hölder's inequality. Now Sobolev's embeddings

$$\dot{W}^{2\beta-1,\frac{2}{2\alpha+2\beta-1}} \hookrightarrow L^{\frac{1}{\alpha}}, \quad \dot{W}^{2\beta-1,\frac{p}{1-\alpha}} \hookrightarrow L^{\frac{2p}{2(1-\alpha)-p(2\beta-1)}}$$

we obtain

$$\partial_t\|\partial_2\theta\|_{L^p}^p \leq \Big(c\sup_{t\in[0,T]}\|\partial_1\theta\|_{L^{\frac{2}{2\alpha+2\beta-1}}} - c(p,\alpha)\Big)\|\partial_2\theta\|_{L^{\frac{p}{1-\alpha}}}^p$$

This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.3. We fix p and r that satisfy (1.5). It is important for the proof below to note that

$$\frac{2(1-\alpha)^2}{\alpha(\alpha^2+2\beta-1)} > \frac{2}{\alpha(2\alpha+2\beta-1)}$$

Now define $q = p\alpha$ for which one can verify that $q \in (2,\infty)$. We apply ∂_1 on (1.1), multiply by $q|\partial_1\theta|^{q-2}\partial_1\theta$ and integrate in space to obtain

$$\partial_t\|\partial_1\theta\|_{L^q}^q + q\int (\Lambda^{2\alpha}\partial_1\theta)|\partial_1\theta|^{q-2}\partial_1\theta = -q\int \partial_1((u\cdot\nabla)\theta)|\partial_1\theta|^{q-2}\partial_1\theta$$

Lemma 2.2, and the same homogeneous Sobolev embedding, as before, lead to

$$\begin{aligned}
\partial_t\|\partial_1\theta\|_{L^q}^q + c(q,\alpha)\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}^q &\leq -q\int \partial_1((u\cdot\nabla)\theta)|\partial_1\theta|^{q-2}\partial_1\theta \\
&= -q\sum_{i=1}^2\int \partial_1 u_i\partial_i\theta|\partial_1\theta|^{q-2}\partial_1\theta
\end{aligned}$$

due to the incompressibility of u. We integrate by parts once more and obtain

$$\partial_t\|\partial_1\theta\|_{L^q}^q + c(q,\alpha)\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}^q \leq c\|\theta\|_{L^\infty}\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}\|\partial_1\theta\|_{L^q}^{q-1}\|\operatorname{div}\big(\frac{\partial_1 u}{\partial_1\theta}\big)\|_{L^{\frac{q}{\alpha}}}$$

by Hölder's inequality. Now Young's inequality gives

$$\partial_t\|\partial_1\theta\|_{L^q}^q + c(q,\alpha)\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}^q \leq \epsilon\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}^q + c\|\theta\|_{L^\infty}^{\frac{q}{q-1}}\|\partial_1\theta\|_{L^q}^q\|\operatorname{div}\big(\frac{\partial_1 u}{\partial_1\theta}\big)\|_{L^{\frac{q}{\alpha}}}^{\frac{q}{q-1}}$$

As shown in [13], for $\epsilon > 0$ sufficiently small Gronwall's inequality implies

$$\|\partial_1\theta(T)\|_{L^q}^q + \int_0^T\|\partial_1\theta\|_{L^{\frac{q}{1-\alpha}}}^q\,dt \leq c\|\partial_1\theta_0\|_{L^q}^q e^{\int_0^T\|\operatorname{div}\big(\frac{\partial_1 u}{\partial_1\theta}\big)\|_{L^{\frac{q}{\alpha}}}^{\frac{q}{q-1}}dt}$$

By the definition of q and (1.5), we have

$$\int_0^T \|\partial_1\theta\|^q_{L^{\frac{q}{1-\alpha}}} dt < \infty$$

That is, we have proved the hypothesis of Theorem 1.1. This completes the proof of the first claim.

Next, we prove the second claim. We take p that satisfies (1.3). An identical process as above gives

$$\partial_t\|\partial_1\theta\|^p_{L^p} + c(p,\alpha)\|\partial_1\theta\|^p_{L^{\frac{p}{1-\alpha}}} \le p(p-1)\|\theta_0\|_{L^\infty}\|\partial_1\theta\|^p_{L^{\frac{p}{1-\alpha}}}\|\operatorname{div}(\frac{\partial_1 u}{\partial_1\theta})\|_{L^{\frac{1}{\alpha}}}$$

Thus,

$$\partial_t\|\partial_1\theta\|^p_{L^p} \le \Big(p(p-1)\|\theta_0\|_{L^\infty}\sup_{t\in[0,T]}\|\operatorname{div}(\frac{\partial_1 u}{\partial_1\theta})(t)\|_{L^{\frac{1}{\alpha}}} - c(p,\alpha)\Big)\|\partial_1\theta\|^p_{L^{\frac{p}{1-\alpha}}}$$

This completes the proof of Theorem 1.3. □

3. Appendix

In this section we prove Theorem 1.2. It was shown in [3] that u of (1.2) may be decomposed as $u = c(\theta + \mathcal{P}\theta)$ for some constant $c > 0$ where $\mathcal{P}$ is a type of singular integral operator bounded in $L^p, p \in (1,\infty)$. Immediately, we have

$$\|u\|_{L^p} \le c\|\theta\|_{L^p}, \quad p \in (1,\infty). \tag{3.1}$$

Proposition 3.1. *Let $\alpha \in (0,1/2)$. If the solution $\theta(x,t)$ to* (1.2) *in $[0,T]$ satisfies* (1.4)*, then*

$$\sup_{t\in[0,T]}\|\partial_3\theta(t)\|^{\frac{2p\alpha}{3}}_{L^{\frac{2p\alpha}{3}}} + \int_0^T \|\partial_3\theta\|^{\frac{2p\alpha}{3}}_{L^{\frac{2p\alpha}{3-2\alpha}}} dt < \infty$$

Proof. We fix p and r that satisfy (1.4) and define $q = \frac{2p\alpha}{3}$ for which it is easy to check that $q \in (2,\infty)$. We apply ∂_3 to (1.2), multiply by $q|\partial_3\theta|^{q-2}\partial_3\theta$, integrate in space to obtain

$$\partial_t\|\partial_3\theta\|^q_{L^q} + q\int(\Lambda^{2\alpha}\partial_3\theta)|\partial_3\theta|^{q-2}\partial_3\theta = -q\int\partial_3((u\cdot\nabla)\theta)|\partial_3\theta|^{q-2}\partial_3\theta$$

The proof of Lemma 2.2 is generalizable to higher dimension (See [3, 30] for details); hence, by the homogeneous Sobolev embedding

$$c(q,\alpha)\|\partial_3\theta\|^q_{L^{\frac{3q}{3-2\alpha}}} \le c\|(\partial_3\theta)^{q/2}\|^2_{\dot{H}^\alpha} \le q\int(\Lambda^{2\alpha}\partial_3\theta)|\partial_3\theta|^{q-2}\partial_3\theta$$

On the right hand side, we have due to the incompressibility of u,

$$\int \partial_3 u_1\partial_1\theta|\partial_3\theta|^{q-2}\partial_3\theta + \partial_3 u_2\partial_2\theta|\partial_3\theta|^{q-2}\partial_3\theta + (-\partial_1 u_1 - \partial_2 u_2)\partial_3\theta|\partial_3\theta|^{q-2}\partial_3\theta$$

which by Hölder's and Young's inequalities and (3.1), we can bound by

$$c\|\partial_3\theta\|_{L^{\frac{3q}{3-2\alpha}}}\|\nabla_h\theta\|_{L^{\frac{3q}{2\alpha}}}\|\partial_3\theta\|^{q-1}_{L^q} \le \epsilon\|\partial_3\theta\|^q_{L^{\frac{3q}{3-2\alpha}}} + c\|\nabla_h\theta\|^{\frac{q}{q-1}}_{L^{\frac{3q}{2\alpha}}}\|\partial_3\theta\|^q_{L^q}$$

Thus, for $\epsilon > 0$ sufficiently small,

$$\partial_t\|\partial_3\theta\|^q_{L^q} + \|\partial_3\theta\|^q_{L^{\frac{3q}{3-2\alpha}}} \le c\|\nabla_h\theta\|^{\frac{q}{q-1}}_{L^{\frac{3q}{2\alpha}}}\|\partial_3\theta\|^q_{L^q}$$

Gronwall's inequality and (1.4) completes the proof. □

The following proposition can be obtained by a similar process.

Proposition 3.2. *Let $\alpha \in (0, \frac{1}{2})$. If the solution $\theta(x,t)$ to (1.2) in $[0,T]$ satisfies (1.4), then*
$$\sup_{t\in[0,T]} \|\nabla_h \theta(t)\|_{L^{\frac{2p\alpha}{3}}}^{\frac{2p\alpha}{3}} + \int_0^T \|\nabla_h \theta\|_{L^{\frac{2p\alpha}{3-2\alpha}}}^{\frac{2p\alpha}{3}} dt < \infty$$

Proof of Theorem 1.2. Using (1.4) and the range of α, one can readily check that
$$\frac{4p\alpha^2}{4p\alpha^2 - 9 + 6\alpha} \leq \frac{2p\alpha}{3}$$
With this in mind, by (2.1), Propositions 3.1 and 3.2, as before, we obtain
$$\int_0^T \|\nabla \theta\|_{L^{\frac{2p\alpha}{3-2\alpha}}}^{\frac{4p\alpha^2}{4p\alpha^2-9+6\alpha}} dt \leq c \int_0^T \|\nabla_h \theta\|_{L^{\frac{2p\alpha}{3-2\alpha}}}^{\frac{2p\alpha}{3}} + \|\partial_3 \theta\|_{L^{\frac{2p\alpha}{3-2\alpha}}}^{\frac{2p\alpha}{3}} dt < \infty$$
Now we use Lemma 2.3 to estimate (1.2), after applying Λ^2 and taking an L^2-inner product with $\Lambda^2\theta$,
$$\begin{aligned}
&\partial_t \|\Lambda^2 \theta\|_{L^2}^2 + \|\Lambda^{2+\alpha}\theta\|_{L^2}^2 \\
&\leq c(\|\nabla u\|_{L^{\frac{3}{\epsilon\alpha}}} \|\Lambda \nabla \theta\|_{L^2} + \|\Delta u\|_{L^2} \|\nabla \theta\|_{L^{\frac{3}{\epsilon\alpha}}}) \|\Delta \theta\|_{L^{\frac{6}{3-2\epsilon\alpha}}} \\
&\leq c\|\nabla \theta\|_{L^{\frac{3}{\epsilon\alpha}}} \|\Lambda^2\theta\|_{L^2}^{2-\epsilon} \|\Lambda^{2+\alpha}\theta\|_{L^2}^{\epsilon} \\
&\leq \frac{1}{2}\|\Lambda^{2+\alpha}\theta\|_{L^2}^2 + c\|\nabla\theta\|_{L^{\frac{3}{\epsilon\alpha}}}^{\frac{2}{2-\epsilon}} \|\Lambda^2\theta\|_{L^2}^2
\end{aligned}$$
by (3.1), Gagliardo-Nirenberg and Young's inequalities. We choose $\epsilon = \frac{3}{\alpha}(\frac{3-2\alpha}{2p\alpha})$ for which it is easy to check that $\epsilon \in (0,1)$. By Gronwall's inequality,
$$\sup_{t\in[0,T]} \|\Lambda^2\theta(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{2+\alpha}\theta\|_{L^2}^2 dt \leq \|\Lambda^2\theta_0\|_{L^2}^2 e^{\int_0^T \|\nabla\theta\|_{L^{\frac{2p\alpha}{3-2\alpha}}}^{\frac{4p\alpha^2}{4p\alpha^2-9+6\alpha}} dt}$$
Similarly, by Lemma 2.3 choosing the same ϵ, we see that
$$\sup_{t\in[0,T]} \|\Lambda^3\theta(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{3+\alpha}\theta\|_{L^2}^2 dt < \infty$$
In summary, by the inhomogeneous Sobolev embedding we have
$$\int_0^T \|\nabla\theta\|_{L^\infty} dt \leq c\int_0^T \|\theta\|_{H^3} dt \leq c \sup_{t\in[0,T]} \|\theta(t)\|_{H^3} T < \infty$$
This completes the proof of the first claim.

For the second claim, let us denote by $\nabla_{2,3} = (0, \partial_2, \partial_3)$. By the previous result, it suffices to show that
$$\int_0^T \|\nabla_{2,3}\theta\|_{L^p}^r dt < \infty$$
for some p, r that (1.4) is satisfied. We fix p in such a range, apply $\nabla_{2,3}$ on (1.2), multiply by $p|\nabla_{2,3}\theta|^{p-2}\nabla_{2,3}\theta$ and integrate in space to estimate as before
$$\begin{aligned}
&\partial_t \|\nabla_{2,3}\theta\|_{L^p}^p + c(p,\alpha)\|\nabla_{2,3}\theta\|_{L^{\frac{3p}{3-2\alpha}}}^p \\
&\leq -p\int \partial_2 u_1 \partial_1 \theta |\nabla_{2,3}\theta|^{p-2}\partial_2\theta + \partial_3 u_1 \partial_1\theta|\nabla_{2,3}\theta|^{p-2}\partial_3\theta \\
&\quad + \partial_2 u_2 \partial_2\theta|\nabla_{2,3}\theta|^{p-2}\partial_2\theta + \partial_3 u_2\partial_2\theta|\nabla_{2,3}\theta|^{p-2}\partial_3\theta
\end{aligned}$$

$$
\begin{aligned}
&+ \partial_2 u_3 \partial_3\theta |\nabla_{2,3}\theta|^{p-2}\partial_2\theta + (-\partial_1 u_1 - \partial_2 u_2)\partial_3\theta|\nabla_{2,3}\theta|^{p-2}\partial_3\theta \\
&\le c\|\nabla_h\theta\|_{L^{\frac{3}{2\alpha}}} \|\nabla_{2,3}\theta\|^p_{L^{\frac{3p}{3-2\alpha}}}
\end{aligned}
$$

Hence,

$$
\partial_t\|\nabla_{2,3}\theta\|^p_{L^p} \le c\Big(\sup_{t\in[0,T]} \|\nabla_h\theta(t)\|_{L^{\frac{3}{2\alpha}}} - c(p,\alpha)\Big)\|\nabla_{2,3}\theta\|^p_{L^{\frac{3p}{3-2\alpha}}}
$$

This completes the proof of Theorem 1.2. □

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