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BLOW-UP OF SOLUTIONS FOR A SYSTEM OF NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT. The initial boundary value problem for a system of parabolic equations in a bounded domain is considered. We prove that, under suitable conditions on the nonlinearity and certain initial data, the lower bound for the blow-up time is determined if blow-up does occur. In addition, a criterion for blow-up to occur and conditions which ensure that blow-up does not occur are established.

1. INTRODUCTION

We consider the initial boundary value problem for the following nonlinear parabolic problems:

$$u_t - \operatorname{div}(\rho_1(|\nabla u|^2)\nabla u) = f_1(u,v) \quad \text{in } \Omega \times [0,\infty), \tag{1.1}$$

$$v_t - \operatorname{div}((\rho_2(|\nabla v|^2)\nabla v) = f_2(u,v) \quad \text{in } \Omega \times [0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.3}$$

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, ρ_i, $i = 1, 2$, are positive C^1 functions and $f_i(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, are given functions which will be specified later. $u_0(x)$, $v_0(x)$ are nonzero and nonnegative functions.

Questions related to the blow-up phenomena of the solutions for the nonlinear parabolic equations and systems have attracted considerable attention in recent years. A natural question concerning the blow-up properties is about whether the solution blows up and, if so, at what time t^* blow-up occurs. In this direction, there is a vast literature to deal with the blow-up time when the solution does blow up at finite time t^* [1, 2, 3, 4, 5, 6, 7, 8, 10, 12], [15, page 3]. Yet, this blow-up time can seldom be determined explicitly. Indeed, the methods used in the study of blow-up very often have yielded only upper bound for t^*. However, a lower bound on blow-up time is more important in some applied problems because of the explosive nature of the solution. To the authors knowledge, some of the first work on lower bounds for t^* was by Weissler [16, 17]. Recently, a number of papers

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deriving lower bounds for t^* in various problems have appeared, beginning with the paper of Payne and Schaefer [13]. Payne et al. [14] considered the single equation

$$u_t - \operatorname{div}\big(\rho(|\nabla u|^2)\nabla u\big) = f(u).$$

Under certain conditions on the nonlineartities, they obtained a lower bound for blow-up time if blow-up does occur. Additionally, a criterion for blow-up and conditions which ensure that blow-up does not occur are obtained.

Motivated by previous works, in this study, we establish the lower bound and the upper bound for problem (1.1)-(1.4) when blow-up does occur. Besides, the nonblow-up properties for a class of problem (1.1)-(1.4) are also investigated. Our proof technique closely follows the arguments of [14], with some modifications being needed for our problems. The paper is organized as follows. In section 2, under suitable conditions on $\rho_i,\ f_i,\ i = 1, 2$, the lower bound for the blow-up time is established if blow-up occurs when Ω is a bounded domain in $\mathbb{R}^3$. In Section 3, the nonblow-up phenomena are investigated. Finally, the sufficient condition which guarantees the blow-up occurs is obtained and an upper bound for the blow-up time is also given.

2. Lower bound for the blow-up time

In this section, we focus our attention to the lower bound time t^* for the blow-up time of the solutions to problem (1.1)-(1.4). For this purpose, we give the assumptions on ρ_i and f_i, $i = 1, 2$ as follows.

(A1) $\rho_i(s)$, $i = 1, 2$ are nonnegative C^1 function for $s > 0$ satisfying

$$\rho_1(s) \ge b_1 + b_2 s^{q_1}, \quad \rho_2(s) \ge b_3 + b_3 s^{q_2}, \quad q_1,\ q_2,\ b_i > 0, \quad i = 1-4.$$

(A2) Concerning the functions $f_1(u, v)$ and $f_2(u, v)$, we take (see [9])

$$f_1(u,v) = \Big(a|u+v|^{m-1}(u+v) + b|u|^{\frac{m-3}{2}}|v|^{\frac{m+1}{2}}u\Big), \tag{2.1}$$

$$f_2(u,v) = \Big(a|u+v|^{m-1}(u+v) + b|v|^{\frac{m-3}{2}}|u|^{\frac{m+1}{2}}v\Big), \tag{2.2}$$

where a, $b > 0$ are constants and m satisfies

$$m > 1, \text{ if } N = 1, 2 \quad \text{or} \quad 1 < m \le \frac{N+2}{N-2}, \text{ if } N \ge 3.$$

One can easily verify that

$$uf_1(u,v) + vf_2(u,v) = (m+1)F(u,v), \quad \forall (u,v) \in \mathbb{R}^2,$$

where

$$F(u,v) = \frac{1}{m+1}\Big(a|u+v|^{m+1} + 2b|uv|^{\frac{m+1}{2}}\Big).$$

As in [9], we still have the following result.

Lemma 2.1. *There exists a positive constant β such that, for $p > 0$,*

$$u^p f_1(u,v) + v^p f_2(u,v) \le \beta(|u|^{p+m} + |v|^{p+m}), \quad \forall (u,v) \in \mathbb{R}^2.$$

We define

$$\begin{aligned}\phi(t) &= \int_\Omega u^{2(n-1)(q_1+1)+2}dx + \int_\Omega v^{2(n-1)(q_2+1)+2}dx \\ &= \int_\Omega u^{\sigma_1}dx + \int_\Omega v^{\sigma_2}dx,\end{aligned} \tag{2.3}$$

where $\sigma_1 = 2(n-1)(q_1+1)+2$, $\sigma_2 = 2(n-1)(q_2+1)+2$ *and* n *is a positive constant satisfying*

$$
\begin{aligned}
n > \max\Big\{&\frac{3(m-1)-2q_1}{2(q_1+1)}, \frac{3(m-1)-2q_2}{2(q_2+1)}, \frac{3(m-1)-2(3q_1-2q_2)}{2(3q_1-2q_2+1)}, \\
&\frac{3(m-1)-2(3q_2-2q_1)}{2(3q_2-2q_1+1)}\Big\}.
\end{aligned} \tag{2.4}
$$

Theorem 2.2. *Suppose that* (A1), (A2), (2.4) *hold and* $\Omega \subset \mathbb{R}^3$ *is a bounded domain. Assume further that* $m-1 > 2\max(q_1, q_2) > 0$ *and* $q_1 > \frac{2}{3}q_2 > \frac{4}{9}q_1 > 0$. *Let* (u, v) *be the nonnegative solution of problem* (1.1)-(1.4), *which become unbounded in the measure* ϕ *at time* t^*, *then* t^* *is bounded below as*

$$
t^* \geq \int_{\phi(0)}^{\infty} \frac{1}{\sum_{i=1}^{4} k_i \phi(s)^{\mu_i}} ds,
$$

where $k_i > 0$ *and* $\mu_i > 0$, $i = 1-4$ *are constnats given in the proof.*

Proof. Differentiating (2.3) and using (1.1)-(1.2), (A1) and Lemma 2.1, we obtain

$$
\begin{aligned}
\phi'(t) &= \sigma_1 \int_\Omega u^{\sigma_1-1} u_t dx + \sigma_2 \int_\Omega v^{\sigma_2-1} v_t dx \\
&= -\sigma_1(\sigma_1-1) \int_\Omega u^{\sigma_1-2} \rho_1(|\nabla u|^2)|\nabla u|^2 dx + \sigma_1 \int_\Omega u^{\sigma_1-1} f_1(u,v) dx \\
&\quad - \sigma_2(\sigma_2-1) \int_\Omega v^{\sigma_2-2} \rho_2(|\nabla v|^2)|\nabla v|^2 dx + \sigma_2 \int_\Omega v^{\sigma_2-1} f_2(u,v) dx \\
&\leq -\sigma_1(\sigma_1-1) \int_\Omega u^{\sigma_1-2} |\nabla u|^2 (b_1 + b_2|\nabla u|^{2q_1}) dx \\
&\quad + \beta\sigma_1 \int_\Omega (u^{m+\sigma_1-1} + v^{m+\sigma_1-1}) dx \\
&\quad - \sigma_2(\sigma_2-1) \int_\Omega v^{\sigma_2-2} |\nabla v|^2 (b_3 + b_4|\nabla v|^{2q_2}) dx \\
&\quad + \beta\sigma_2 \int_\Omega (u^{m+\sigma_2-1} + v^{m+\sigma_2-1}) dx.
\end{aligned} \tag{2.5}
$$

Dropping the terms $\sigma_1(\sigma_1-1)b_1 \int_\Omega u^{\sigma_1-2}|\nabla u|^2 dx$ and $\sigma_2(\sigma_2-1)b_3 \int_\Omega v^{\sigma_2-2}|\nabla v|^2 dx$ on the right-hand side of (2.5) and using $|\nabla w^n|^2 = n^2 w^{2(n-1)}|\nabla w|^2$, we deduce that

$$
\begin{aligned}
\phi'(t) \leq &-\frac{\sigma_1(\sigma_1-1)b_2}{n^{2(q_1+1)}} \int_\Omega |\nabla u^n|^{2(q_1+1)} dx + \beta\sigma_1 \int_\Omega (u^{m+\sigma_1-1} + v^{m+\sigma_1-1}) dx \\
&- \frac{\sigma_2(\sigma_2-1)b_4}{n^{2(q_2+1)}} \int_\Omega |\nabla v^n|^{2(q_2+1)} dx + \beta\sigma_2 \int_\Omega (u^{m+\sigma_2-1} + v^{m+\sigma_2-1}) dx.
\end{aligned}
$$

For simplicity, setting $w_1 = u^n$, $w_2 = v^n$ and $\gamma_i = m - 1 - 2q_i > 0$, $i = 1, 2$, then we obtain

$$\begin{aligned} \phi'(t) \le & -\frac{\sigma_1(\sigma_1 - 1)b_2}{n^{2(q_1+1)}} \int_\Omega |\nabla w_1|^{2(q_1+1)} dx \\ & + \beta\sigma_1 \int_\Omega (w_1^{2(q_1+1)+\frac{\gamma_1}{n}} + w_2^{2(q_1+1)+\frac{\gamma_1}{n}}) dx \\ & - \frac{\sigma_2(\sigma_2 - 1)b_4}{n^{2(q_2+1)}} \int_\Omega |\nabla w_2|^{2(q_2+1)} dx + \beta\sigma_2 \int_\Omega w_1^{2(q_2+1)+\frac{\gamma_2}{n}} dx \\ & + \beta\sigma_2 \int_\Omega w_2^{2(q_2+1)+\frac{\gamma_2}{n}} dx. \end{aligned} \tag{2.6}$$

Next, we will estimate the right-hand side of (2.6). It follows from [14, (2.12)] that

$$\int_\Omega w_1^{2(q_1+1)+\frac{\gamma_1}{n}} dx \le K_1 \Big(\int_\Omega |\nabla w_1|^{2(q_1+1)} dx \Big)^{2/3} \Big(\int_\Omega w_1^{q_1+1+\frac{3\gamma_1}{2n}} dx \Big)^{2/3}, \tag{2.7}$$

where $K_1 = \alpha \lambda_1^{-\frac{4q_1+1}{6}} (q_1 + 1)^{\frac{4(q_1+1)}{3}}$, $\alpha = 4^{1/3} \cdot 3^{-1/2} \cdot \pi^{-2/3}$ and λ_1 is the first eigenvalue in the fixed membrane problem

$$\Delta w + \lambda w = 0, \quad w > 0 \text{ in } \Omega, \quad \text{and} \quad w = 0 \text{ on } \partial\Omega.$$

By using Hölder inequality and (2.3), we obtain

$$\begin{aligned} \int_\Omega w_1^{q_1+1+\frac{3\gamma_1}{2n}} dx &= \int_\Omega u^{n(q_1+1)+\frac{3\gamma_1}{2}} dx \\ &\le \Big(\int_\Omega u^{\sigma_1} dx \Big)^{\mu_1} \cdot |\Omega|^{1-\mu_1} \\ &\le \phi(t)^{\mu_1} \cdot |\Omega|^{1-\mu_1}, \end{aligned} \tag{2.8}$$

which together with (2.7) implies

$$\int_\Omega w_1^{2(q_1+1)+\frac{\gamma_1}{n}} dx \le K_1 |\Omega|^{\frac{2(1-\mu_1)}{3}} \phi(t)^{\frac{2\mu_1}{3}} (\int_\Omega |\nabla w_1|^{2(q_1+1)} dx)^{2/3}.$$

with $\mu_1 = \frac{2n(q_1+1)+3\gamma_1}{2\sigma_1}$, we note that $\mu_1 < 1$ in view of (2.4). Further, thanks to the inequality

$$x^r y^s \le rx + sy, \quad r + s = 1, \quad x, y \ge 0, \tag{2.9}$$

we obtain, for $\alpha_1 > 0$,

$$\int_\Omega w_1^{2(q_1+1)+\frac{\gamma_1}{n}} dx \le K_1 |\Omega|^{\frac{2(1-\mu_1)}{3}} \Big[\frac{1}{3\alpha_1^2} \phi(t)^{2\mu_1} + \frac{2\alpha_1}{3} \int_\Omega |\nabla w_1|^{2(q_1+1)} dx \Big]. \tag{2.10}$$

and similarly

$$\int_\Omega w_2^{2(q_2+1)+\frac{\gamma_2}{n}} dx \le K_2 |\Omega|^{\frac{2(1-\mu_2)}{3}} \Big[\frac{1}{3\alpha_2^2} \phi(t)^{2\mu_2} + \frac{2\alpha_2}{3} \int_\Omega |\nabla w_2|^{2(q_2+1)} dx \Big], \tag{2.11}$$

where $\alpha_2 > 0$, $K_2 = \alpha \lambda_1^{-\frac{4q_2+1}{6}} (q_2 + 1)^{\frac{4(q_2+1)}{3}}$ and $\mu_2 = \frac{2n(q_2+1)+3\gamma_2}{2\sigma_2} < 1$.

To estimate the other two terms in the right hand side of (2.6), we use Hölder inequality and the following result (see [14, (2.7)-(2.10)])

$$\int_\Omega w^{4(q+1)} dx \le \alpha^3 (q + 1)^{4(q+1)} \lambda_1^{-\frac{4q+1}{2}} \Big(\int_\Omega |\nabla w|^{2(q+1)} dx \Big)^2, \quad q > 0, \tag{2.12}$$

to obtain

$$\begin{aligned}
&\int_\Omega w_2^{2(q_1+1)+\frac{\gamma_1}{n}}\,dx \\
&= \int_\Omega w_2^{\frac{4(q_2+1)}{3}} \cdot w_2^{2(q_1+1)-\frac{4(q_2+1)}{3}+\frac{\gamma_1}{n}}\,dx \\
&\le \Big(\int_\Omega w_2^{4(q_2+1)}dx\Big)^{1/3}\Big(\int_\Omega w_2^{3q_1-2q_2+1+\frac{3\gamma_1}{2n}}\,dx\Big)^{2/3} \\
&\le K_2\Big(\int_\Omega |\nabla w_2|^{2(q_2+1)}dx\Big)^{2/3}\Big(\int_\Omega w_2^{3q_1-2q_2+1+\frac{3\gamma_1}{2n}}\,dx\Big)^{2/3}.
\end{aligned} \tag{2.13}$$

As in deriving (2.8), we see that

$$\begin{aligned}
\int_\Omega w_2^{3q_1-2q_2+1+\frac{3\gamma_1}{2n}}\,dx &= \int_\Omega v^{n(3q_1-2q_2+1)+\frac{3\gamma_1}{2}}\,dx \\
&\le \Big(\int_\Omega v^{\sigma_2}dx\Big)^{\mu_3}\cdot|\Omega|^{1-\mu_3} \\
&\le \phi(t)^{\mu_3}\cdot|\Omega|^{1-\mu_3}
\end{aligned} \tag{2.14}$$

where $\mu_3 = \frac{2n(3q_1-2q_2+1)+3\gamma_1}{2\sigma_2} < 1$. Substituting (2.14) into (2.13) and using (2.9) once more, we obtain, for $\alpha_3 > 0$,

$$\int_\Omega w_2^{2(q_1+1)+\frac{\gamma_1}{n}}\,dx \le K_2|\Omega|^{\frac{2(1-\mu_3)}{3}}\Big[\frac{1}{3\alpha_3^2}\phi(t)^{2\mu_3} + \frac{2\alpha_3}{3}\int_\Omega|\nabla w_2|^{2(q_2+1)}dx\Big]. \tag{2.15}$$

and similarly

$$\int_\Omega w_1^{2(q_2+1)+\frac{\gamma_2}{n}}\,dx \le K_1|\Omega|^{\frac{2(1-\mu_4)}{3}}\Big[\frac{1}{3\alpha_4^2}\phi(t)^{2\mu_4} + \frac{2\alpha_4}{3}\int_\Omega|\nabla w_1|^{2(q_1+1)}dx\Big], \tag{2.16}$$

where $\alpha_4 > 0$ and $\mu_4 = \frac{2n(3q_2-2q_1+1)+3\gamma_2}{2\sigma_1} < 1$. Combining (2.10), (2.11), (2.15) and (2.16) with (2.6), we conclude that

$$\begin{aligned}
\phi'(t) \le &-C_1\int_\Omega|\nabla w_1|^{2(q_1+1)}dx - C_2\int_\Omega|\nabla w_2|^{2(q_2+1)}dx \\
&+ k_1\phi(t)^{2\mu_1} + k_2\phi(t)^{2\mu_2} + k_3\phi(t)^{2\mu_3} + k_4\phi(t)^{2\mu_4},
\end{aligned}$$

where

$$\begin{gathered}
C_1 = \frac{\sigma_1(\sigma_1-1)b_2}{n^{2(q_1+1)}} - \frac{2\alpha_1K_1\beta\sigma_1}{3}|\Omega|^{\frac{2(1-\mu_1)}{3}} - \frac{2\alpha_4K_1\beta\sigma_2}{3}|\Omega|^{\frac{2(1-\mu_4)}{3}}, \\
C_2 = \frac{\sigma_2(\sigma_2-1)b_4}{n^{2(q_2+1)}} - \frac{2\alpha_2K_2\beta\sigma_1}{3}|\Omega|^{\frac{2(1-\mu_2)}{3}} - \frac{2\alpha_3K_2\beta\sigma_2}{3}|\Omega|^{\frac{2(1-\mu_3)}{3}}, \\
k_1 = \frac{K_1|\Omega|^{\frac{2(1-\mu_1)}{3}}\beta\sigma_1}{3\alpha_1^2}, \quad k_2 = \frac{K_2|\Omega|^{\frac{2(1-\mu_2)}{3}}\beta\sigma_2}{3\alpha_2^2}, \\
k_3 = \frac{K_2|\Omega|^{\frac{2(1-\mu_3)}{3}}\beta\sigma_1}{3\alpha_3^2}, \quad k_4 = \frac{K_1|\Omega|^{\frac{2(1-\mu_4)}{3}}\beta\sigma_2}{3\alpha_4^2}.
\end{gathered}$$

Now, setting $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$, and choosing α_1, α_3 such that $C_1 = 0$ and $C_2 = 0$, hence, we have

$$\phi'(t) \le g(\phi), \tag{2.17}$$

where

$$g(s) = k_1s^{2\mu_1} + k_2s^{2\mu_2} + k_3s^{2\mu_3} + k_4s^{2\mu_4}.$$

An integration of (2.17) from 0 to t leads to

$$\int_{\phi(0)}^{\phi(t)} \frac{ds}{g(s)} \le t,$$

so that if (u, v) blows up in the measure of ϕ as $t \to t^*$, we derive the lower bound

$$\int_{\phi(0)}^{\infty} \frac{ds}{g(s)} \le t^*,$$

and Theorem 2.2 is proved. Clearly, the integral is bounded since $2\mu_1 > 1$. $\square$

3. Non blow-up case

In this section, we consider the non blow-up property of problem (1.1)-(1.4) when $2\max(q_1, q_2) > m - 1 > 0$. To achieve this, we define the auxiliary function

$$\phi(t) = \frac{1}{2}\int_\Omega u^2 dx + \frac{1}{2}\int_\Omega v^2 dx. \tag{3.1}$$

Theorem 3.1. *Suppose that* (A1), (A2) *hold and that* $2\max(q_1, q_2) > m - 1 > 0$. *Let* (u, v) *be the nonnegative solution of problem* (1.1)-(1.4), *then* (u, v) *can not blow up in the measure* ϕ *in finite time.*

Proof. From(3.1), (1.1), (1.2) and (A2), we have

$$\begin{aligned}
\phi'(t) &= \int_\Omega u u_t dx + \int_\Omega v v_t dx \\
&\le -\int_\Omega |\nabla u|^2 (b_1 + b_2|\nabla u|^{2q_1}) dx - \int_\Omega |\nabla v|^2 (b_3 + b_4 |\nabla v|^{2q_2}) dx \\
&\quad + \beta \int_\Omega (u^{m+1} + v^{m+1}) dx \\
&\le \int_\Omega (\beta u^{m+1} - b_2 |\nabla u|^{2(q_1+1)}) dx + \int_\Omega (\beta v^{m+1} - b_4 |\nabla v|^{2(q_2+1)}) dx \\
&\le \int_\Omega \Big(\beta u^{m+1} - b_2 (\frac{\lambda_1}{(q_1+1)^2})^{q_1+1} u^{2(q_1+1)}\Big) dx \\
&\quad + \int_\Omega \big(\beta v^{m+1} - b_4 (\frac{\lambda_1}{(q_2+1)^2})^{q_2+1} v^{2(q_2+1)}\big) dx,
\end{aligned} \tag{3.2}$$

where the last inequality is obtained by using [14, (2.10]. For $q > 0$,

$$\int_\Omega w^{2(q+1)} dx \le (\frac{(q+1)^2}{\lambda_1})^{q+1} \int_\Omega |\nabla w|^{2(q+1)} dx,$$

where λ_1 is the first eigenvalue in the fixed membrane problem, as defined in Section 2. Employing Hölder inequality, we have

$$\int_\Omega u^{m+1} dx \le \Big(\int_\Omega u^{2(q_1+1)} dx\Big)^{\frac{m+1}{2(q_1+1)}} \cdot |\Omega|^{\frac{2q_1-m+1}{2(q_1+1)}}, \tag{3.3}$$

$$\int_\Omega v^{m+1} dx \le \Big(\int_\Omega v^{2(q_2+1)} dx\Big)^{\frac{m+1}{2(q_{21}+1)}} \cdot |\Omega|^{\frac{2q_2-m+1}{2(q_2+1)}}, \tag{3.4}$$

$$\int_\Omega u^2 dx \le \Big(\int_\Omega u^{m+1} dx\Big)^{\frac{2}{m+1}} \cdot |\Omega|^{\frac{m-1}{m+1}}. \tag{3.5}$$

Inserting (3.3)-(3.5) into (3.2), we see that

$$\begin{aligned}\phi'(t) \le &\int_\Omega u^{m+1}dx(\beta - M_1(\int_\Omega u^2dx)^{\frac{2q_1-m+1}{2}})dx\\ &+2\int_\Omega v^{m+1}dx(\beta - M_2(\int_\Omega v^2dx)^{\frac{2q_2-m+1}{2}})dx\end{aligned} \tag{3.6}$$

where

$$M_1 = b_2(\frac{\lambda_1}{(q_1+1)^2})^{q_1+1}|\Omega|^{-\frac{2q_1-m+1}{2}}, \quad M_2 = b_4(\frac{\lambda_1}{(q_2+1)^2})^{q_2+1}|\Omega|^{-\frac{2q_2-m+1}{2}}.$$

Apparently, if (u,v) blows up in the ϕ measure at some time t then $\phi'(t)$ would be negative which leads to a contradiction. Thus, the solution (u,v) can not blow up in the measure ϕ. The proof is complete. □

4. Criterion for blow-up

In this section, we investigate the blow up properties of solutions for (1.1)-(1.4) with

$$\rho_1(s) = b_1 + b_2 s^{q_1}, \quad \rho_2(s) = b_3 + b_3 s^{q_2}, \quad q_1, q_2, b_i > 0, \quad i = 1-4. \tag{4.1}$$

For this purpose, we first define

$$\phi(t) = \frac{1}{2}\int_\Omega u^2dx + \frac{1}{2}\int_\Omega v^2dx \tag{4.2}$$

and

$$\begin{aligned}\psi(t) = &-\frac{b_1}{2}\|\nabla u\|_2^2 - \frac{b_2}{2(q_1+1)}\int_\Omega |\nabla u|^{2(q_1+1)}dx - \frac{b_3}{2}\|\nabla v\|_2^2\\ &-\frac{b_4}{2(q_2+1)}\int_\Omega |\nabla v|^{2(q_2+1)}dx + \int_\Omega F(u,v)dx,\end{aligned} \tag{4.3}$$

where $\|\cdot\|_2$ is the $L^2(\Omega)$-norm.

Theorem 4.1. *Suppose that* (4.1) *and* (A2) *hold. Assume further that* $m-1 > 2\max(q_1,q_2) \ge 0$ *and* $\psi(0) > 0$. *If (u,v) is the non-negative solution of problem* (1.1)-(1.4), *then the solution blows up at finite time* t^* *with*

$$t^* \le \frac{\phi(0)^{-2m-1}}{(2m+1)(m+1)}.$$

Proof. From (4.1)-(4.3), we have

$$\begin{aligned}\phi'(t) &= -\int_\Omega |\nabla u|^2(b_1 + b_2|\nabla u|^{2q_1})dx - \int_\Omega |\nabla v|^2(b_3 + b_4|\nabla v|^{2q_2})dx\\ &\quad + (m+1)\int_\Omega F(u,v)dx\\ &\ge (m+1)\Big[-\frac{b_1}{2}\int_\Omega |\nabla u|^2dx - \frac{b_2}{2(q_1+1)}\int_\Omega |\nabla u|^{2(q_1+1)}dx\\ &\quad -\frac{b_3}{2}\|\nabla v\|_2^2 - \frac{b_4}{2(q_2+1)}\int_\Omega |\nabla v|^{2(q_2+1)}dx + \int_\Omega F(u,v)dx\Big]\\ &= (m+1)\psi(t),\end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \psi'(t) &= -b_1 \int_\Omega \nabla u \cdot \nabla u_t dx - b_2 \int_\Omega |\nabla u|^{2q_1} \nabla u \cdot \nabla u_t dx - b_3 \int_\Omega \nabla v \cdot \nabla v_t dx \\ &\quad - b_4 \int_\Omega |\nabla v|^{2q_2} \nabla v \cdot \nabla v_t dx - \int_\Omega a|u+v|^{m-1}(u+v)(u_t+v_t) dx \\ &\quad - b \int_\Omega \big(|u|^{\frac{m-3}{2}} |v|^{\frac{m+1}{2}} u u_t + |v|^{\frac{m-3}{2}} |u|^{\frac{m+1}{2}} v v_t \big) dx \\ &= \int_\Omega (u_t^2 + v_t^2) dx \geq 0. \end{aligned} \tag{4.5}$$

This, together with $\psi(0) > 0$, implies that $\psi(t) \geq \psi(0) > 0$, for $t \geq 0$. By using Hölder inequality, Schwarz inequality, (4.2) and (4.5), we obtain

$$\begin{aligned} (\phi'(t))^2 &= \Big(\int_\Omega u u_t dx + \int_\Omega v v_t dx \Big)^2 \\ &\leq \|u\|_2^2 \|u_t\|_2^2 + \|v\|_2^2 \|v_t\|_2^2 + \|u\|_2^2 \|v_t\|_2^2 + \|u\|_2^2 \|u_t\|_2^2 \\ &= \frac{1}{2} \phi(t) \psi'(t). \end{aligned} \tag{4.6}$$

Then, using (4.4) and (4.6), we deduce that

$$\phi'(t)\psi(t) \leq \frac{1}{m+1} (\phi'(t))^2 \leq \frac{1}{2(m+1)} \phi \psi'(t),$$

which implies that

$$(\psi(t)\phi(t)^{-2m-2})' \geq 0. \tag{4.7}$$

An integration of (4.7) from 0 to t gives to

$$\psi(t)\phi(t)^{-2m-2} \geq \psi(0)\phi(0)^{-2m-2} \equiv M. \tag{4.8}$$

Combining (4.4) with (4.8) and integrating the resultant differential inequality, we have

$$\phi(t)^{-2m-1} \leq \phi(0)^{-2m-1} - (2m+1)(m+1)Mt \tag{4.9}$$

Since $\phi(0) > 0$, (4.9) shows that ϕ becomes infinite in a finite time

$$t^* \leq T = \frac{\phi(0)^{-2m-1}}{(2m+1)(m+1)}.$$

This completes the proof. $\square$

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