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GROUND STATE SOLUTIONS FOR SEMILINEAR PROBLEMS WITH A SOBOLEV-HARDY TERM

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Abstract. In this article, we study the existence of solutions to the problem

$$\begin{gathered}-\Delta u = \lambda u + \frac{|u|^{2_s^*-2}u}{|y|^s}, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega,\end{gathered}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 3$). We show that there is a ground state solution provided that $N = 4$ and $\lambda_m < \lambda < \lambda_{m+1}$, or that $N \geq 5$ and $\lambda_m \leq \lambda < \lambda_{m+1}$, where λ_m is the m'th eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, where $2 \leq k < N$, $N \geq 3$. Suppose that a point $(0, z_0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $(0, z_0) \in \Omega$. Without loss of generality we assume that $0 \in \Omega$. In this article, we consider the existence of solutions of the problem

$$\begin{gathered}-\Delta u = \lambda u + \frac{|u|^{2_s^*-2}u}{|y|^s}, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, and $x = (y, z) \in \Omega$, $0 < s < 2$, and $2_s^* = \frac{2(N-s)}{N-2}$ is the critical exponent related to the Hardy-Sobolev inequality

$$S\Big(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*}}{|y|^s}\, dy\, dz\Big)^{2/2_s^*} \leq \int_{\mathbb{R}^N} |\nabla u|^2\, dy\, dz, \quad \forall u \in D^{1,2}(\mathbb{R}^N), \tag{1.2}$$

where $S = S(N, k, t)$ is the best constant, see [3]. More general Hardy-Sobolev inequalities are dealt in [4] and [5]. The minimizers of problem (1.2) are solutions of the problem

$$-\Delta u = \frac{|u|^{2_s^*-2}u}{|y|^s}, \quad u > 0 \quad \text{in } \mathbb{R}^N, \quad u \in D^{1,2}(\mathbb{R}^N) \tag{1.3}$$

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up to a constant. If $s = 0$, Equation (1.2) becomes the Sobolev inequality, for which best constant was computed, and proved existence of minimizers in [2] and [16]. In the case $s = 2$, (1.2) still holds true, it is an extension of the Hardy inequality. In the more general case $0 \le s < 2$ with $k = N$, the best constant was obtained in [8], and minimizers were found in [10], which are radially symmetric. Therefore, it can be shown by using ODEs, see [10], that up to dilations and translations, minimizers take the form

$$\frac{1}{(1 + |x|^{2-s})^{\frac{N-2}{2-s}}}.$$

It is noted that equation (1.3) is invariant with respect to the scalings and z-translations; that is, u is a solution of (1.3) if only if $u_\alpha(x) = \alpha^{(N-2)/2}u(\alpha y, \alpha(z - z_0))$, $\alpha > 0$, satisfies the equation. Hence, problem (1.3) has lack of the compactness. In the case $0 < s < 2$, $2 \le k < N$, it was proved in [3] that the best constant $S > 0$, and S is achieved by the concentration-compactness principle. So problem (1.3) has a positive solution in $D^{1,2}(\mathbb{R}^N)$. Since the minimizer of problem (1.2) can not be radially symmetric, they cannot be found among solutions of ODEs, but of PDEs. This brings difficulties to find exact forms of the minimizer. In the particular case $s = 1$, problem (1.3) becomes

$$-\Delta u = \frac{u^{\frac{N}{N-2}}}{|y|}, \quad u > 0 \quad \text{in } \mathbb{R}^N, \quad u \in D^{1,2}(\mathbb{R}^N). \tag{1.4}$$

By the moving plane method, it was proved in [7] that all solutions of (1.4) are cylindrically symmetric. Thus, problem (1.4) can be reduced to an elliptic equation in the positive cone in $\mathbb{R}^2$, and it was shown in [7] that u is a solution of (1.4) if and only if

$$u(y, z) = \lambda^{(N-2)/2}V(\lambda y, \lambda(z + z_0)) \tag{1.5}$$

for some $\lambda > 0$ and $z_0 \in \mathbb{R}^{N-k}$, where

$$V(x) = V(y, z) = \frac{C_{N,k}}{((1 + |y|)^2 + |z|^2)^{(N-2)/2}} = \frac{((N-2)(k-1))^{(N-2)/2}}{((1 + |y|)^2 + |z|^2)^{(N-2)/2}}. \tag{1.6}$$

This result allows one to obtain existence results for problem (1.1) in the case $s = 1$. Denote by $0 < \lambda_1, \dots, \lambda_k, \dots$ the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition. When $0 < \lambda < \lambda_1$ and $s = 1$, it was proved in [1] and [6] that there exists a solution of problem (1.1) by the mountain pass lemma and constrained variation respectively.

In this article, we consider the existence of solutions to problem (1.1) for general case $0 < s < 2$ and λ in between λ_m and λ_{m+1} for some $m \in \mathbb{N}$. As far as we know, the exact form of the minimizer of (1.2) is known, see Mancini and Sandeep [11]. However, even without knowing it, to control $(PS)_c$ sequences so that it may avoid the energy levels where the compactness does not hold, we can always use the [6, Lemma 3.4.2], if u is the solution of (1.3), then there exist $C_2 > C_1 > 0$ such that

$$\frac{C_1}{1 + |x|^{N-2}} \le u(x) \le \frac{C_2}{1 + |x|^{N-2}}. \tag{1.7}$$

This estimate suffices to serve our purpose. Using the Nehari manifold method introduced in [12], and developed in [14], we show the following result.

Theorem 1.1. *Let $N = 4$ and $\lambda_m < \lambda < \lambda_{m+1}$ or $N \ge 5$ and $\lambda_m \le \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$, then there exists a ground state solution of problem* (1.1).

In section 2, we describe a variational framework to study the ground state solution of problem (1.1). We prove Theorem 1.1 in section 3.

2. Preliminaries

Denote by $E = H_0^1(\Omega)$ the Hilbert space with the scalar product

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v \, dx$$

and the induced norm $\|\cdot\|$. Let (φ_j, λ_j) be the eigenfunctions and eigenvalues of $-\Delta$ in Ω with zero Dirichlet boundary condition. Suppose that m is a fixed positive integer and $\lambda_m \le \lambda < \lambda_{m+1}$, we define the subspaces $E^- = \operatorname{span}\{\varphi_1, \dots, \varphi_m\}$ and $E^+ = \operatorname{span}\{\varphi_j, j \ge m\}$ of E, then $E = E^+ \oplus E^-$. The functional associated to problem (1.1) is defined by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{2}\int_\Omega |u|^2 \, dx - \frac{1}{2_s^*}\int_\Omega \frac{|u|^{2_s^*}}{|y|^s} \, dx$$

for $u \in H_0^1(\Omega)$, which is C^1 and critical points of J are solutions of problem (1.1). To find ground state solutions of (1.1), we introduce as [12] a submanifold of E. Define

$$\mathcal{N} = \{u \in E \setminus \{0\} : \langle \nabla J(u), u\rangle = 0, \nabla J(u) \in E^+\}. \tag{2.1}$$

The set $\mathcal{N}$ is the intersection of the standard Nehari manifold $\{u \in E \setminus \{0\} : \langle \nabla J(u), u\rangle = 0\}$ with the pre-image $(\nabla J)^{-1}(E^+)$.

Proposition 2.1. *The set $\mathcal{N}$ is a C^1 submanifold of E with codimension $m+1$. Moreover, every critical point of the restriction $J|_{\mathcal{N}}$ is a nontrivial critical point of the functional J.*

Proof. The result can be proved as [15], see also [13]. We sketch the proof here for reader's convenience. Let $F : E \setminus \{0\} \to \mathbb{R} \times E^-$ be a map defined by

$$F(u) = (\langle \nabla J(u), u\rangle, Q\nabla J(u)),$$

where Q is the orthogonal projection of E onto E^-, then $\mathcal{N} = F^{-1}(0)$. Consider the inner product

$$(t_1, z_1) \cdot (t_2, z_2) = t_1 t_2 + \langle z_1, z_2\rangle \quad \text{for } t_1, t_2 \in \mathbb{R},\ z_1, z_2 \in E^-.$$

We claim that for every $(t, z) \in \mathbb{R} \times E^-$, $(t, z) \ne (0, 0)$, the inequality

$$(DF(u)(tu + z)) \cdot (t, z) < 0 \tag{2.2}$$

holds. This implies the first part of the proposition. Now, we prove the claim. Indeed, for $(t, z) \ne (0, 0)$, since

$$\langle \nabla J(u), u\rangle = \langle \nabla J(u), z\rangle = 0,$$

we deduce that

$$\begin{aligned} &(DF(u)(tu + z)) \cdot (t, z) \\ &= \Big(\int_\Omega |\nabla z|^2 \, dx - \lambda \int_\Omega |z|^2 \, dx\Big) \\ &\quad - \int_\Omega \Big((2_s^* - 2)t^2|u|^2 + 2(2_s^* - 2)tzu + (2_s^* - 1)|z|^2\Big)\frac{|u|^{2_s^*-2}}{|y|^s} dx. \end{aligned} \tag{2.3}$$

For $\lambda_m \le \lambda < \lambda_{m+1}$, it is readily verified that (2.2) holds.

Next, we verify as in [15] that $w \in E$ is a critical point of J if and only if $u \in \mathcal{N}$ and $DJ(u)|_{T_u\mathcal{N}} = 0$. The proof is complete. □

We recall that a ground state solution u to (1.1) is any element of $\mathcal{N}$ such that $DJ(u)$ vanishes on $T_u\mathcal{N}$ and $J(u) = c$, where

$$c = \inf_{\mathcal{N}} J. \tag{2.4}$$

By the argument in [14], for every $v \in E^+ \setminus \{0\}$, there is a unique continuous map pair $(f(v), g(v)) \in (0, \infty) \times E^-$ such that $F(f(v)v + g(v)) = 0$ and

$$J(f(v)v + g(v)) = \max_{t>0, z\in E^-} J(tv + z).$$

Hence,

$$c = \inf_{\mathcal{N}} J = \inf_{v\neq 0, v\in E^+} J(f(v)v + g(v)) = \inf_{\{v\neq 0, v\in E^+\}} \max_{\{t>0, z\in E^-\}} J(tv + z). \tag{2.5}$$

3. Existence results

In this section, we show that problem (2.4) is achieved. The minimizer of problem (2.4) is actually a ground state solution of (1.1). Let

$$S = \inf_{u\in E, u\neq 0} \Big\{ \frac{\int_\Omega |\nabla u|^2\, dx}{(\int_\Omega \frac{u^{2_s^*}}{|y|^s}\, dx)^{2/2_s^*}} \Big\}. \tag{3.1}$$

We know from [3] that S can be achieved, which is independent of Ω and depends only by N, k, s, moreover the infimum S is never achieved when Ω is a bounded domain, we denote the minimizer by $U(x) > 0$. By (1.7),

$$\frac{C_1}{1 + |x|^{N-2}} \le U(x) \le \frac{C_2}{1 + |x|^{N-2}}.$$

The following elementary lemma is readily verified.

Lemma 3.1. *Suppose $A > 0$, $B > 0$. Then*

$$\max_{t>0}(A\frac{t^2}{2} - B\frac{t^{2_s^*}}{2_s^*}) = \frac{2-s}{2(N-s)}\Big(\frac{A}{B^{2/2_s^*}}\Big)^{\frac{N-s}{2-s}}.$$

Lemma 3.2. *Suppose that*

$$c < \frac{2-s}{2(N-s)} S^{\frac{N-s}{2-s}}, \tag{3.2}$$

then there exists $v \in E^+ \setminus \{0\}$ such that

$$\max_{t>0, w\in E^-} J(tv + w) = J(f(v)v + g(v)) = c.$$

Proof. Take any sequence $\{v_n\}$ in $E^+ \setminus \{0\}$ such that $\|v_n\| = 1$ and

$$\max_{t>0, w\in E^-} J(tv_n + w) \to c. \tag{3.3}$$

Without loss of generality, we can assume that

$$v_n \rightharpoonup v \quad \text{in } E^+,$$
$$v_n \to v \quad \text{in } L^2(\Omega),$$
$$v_n \to v \quad \text{a.e. } \Omega.$$

Suppose

$$A=\lim_{n\to\infty}\int_\Omega |\nabla(v_n-v)|^2dx\,,\quad B=\lim_{n\to\infty}\int_\Omega \frac{|v_n-v|^{2_s^*}}{|y|^s}dx\,.$$

Using the Brezis-Lieb's Lemma, from (3.3) we obtain

$$J(tv+w)+\frac12 At^2-\frac{1}{2_s^*}Bt^{2_s^*}\le c,\quad \forall t>0,\ \forall w\in E^-. \tag{3.4}$$

If $v=0$ and $B=0$, from the assumption $\|v_n\|=1$, we deduce that $A=1$. Hence $t^2\le 2c-2J(w)$ for every $t>0$ and every $w\in E^-$, a contradiction.

Assume now $B\ne 0$. From Lemma 3.1, we obtain that

$$\frac{2-s}{2(N-s)}S^{\frac{N-s}{2-s}}\le \frac{2-s}{2(N-s)}\Big(\frac{A}{B^{2/2_s^*}}\Big)^{\frac{N-s}{2-s}}=\max_{t>0}\Big(\frac12 At^2-\frac{1}{2_s^*}Bt^{2_s^*}\Big). \tag{3.5}$$

If $v=0$, we obtain from (3.2), (3.4) and (3.5) that

$$\frac{2-s}{2(N-s)}S^{\frac{N-s}{2-s}}\le c<\frac{2-s}{2(N-s)}S^{\frac{N-s}{2-s}},$$

a contradiction. Thus $v\ne 0$.

Denote $h=g(v)/f(v)$. It follows from the definition of c that

$$\begin{aligned} c&\le J(f(v)(v+h))=\max_{t>0}J(t(v+h))\\ &=\frac{2-s}{2(N-s)}\Big\{\frac{\int_\Omega|\nabla(v+h)|^2\,dx-\lambda\int_\Omega|v+h|^2dx}{\big(\int_\Omega\frac{|v+h|^{2_s^*}}{|y|^s}dx\big)^{2/2_s^*}}\Big\}^{\frac{N-s}{2-s}}. \end{aligned}\tag{3.6}$$

By (3.4) and Lemma 3.1,

$$\begin{aligned} c&\ge\max_{t>0}\Big(J(t(v+h))+\frac12At^2-\frac{1}{2_s^*}Bt^{2_s^*}\Big)\\ &=\frac{2-s}{2(N-s)}\Big\{\frac{A+\int_\Omega|\nabla(v+h)|^2\,dx-\lambda\int_\Omega|v+h|^2\,dx}{\big(B+\int_\Omega\frac{|v+h|^{2_s^*}}{|y|^s}\,dx\big)^{2/2_s^*}}\Big\}^{\frac{N-s}{2-s}}. \end{aligned}\tag{3.7}$$

Putting together (3.2), (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned} &\Big(\frac{2(N-s)}{2-s}c\Big)^{\frac{2-s}{N-s}}\Big(B+\int_\Omega\frac{|v+h|^{2_s^*}}{|y|^s}\,dx\Big)^{2/2_s^*}\\ &<\Big(\frac{2(N-s)}{2-s}c\Big)^{\frac{2-s}{N-s}}\Big(B^{2/2_s^*}+\Big(\int_\Omega\frac{|v+h|^{2_s^*}}{|y|^s}\,dx\Big)^{2/2_s^*}\Big)\\ &<A+\int_\Omega|\nabla(v+h)|^2\,dx-\lambda\int_\Omega|v+h|^2dx\\ &\le\Big(\frac{2(N-s)}{2-s}c\Big)^{\frac{2-s}{N-s}}\Big(B+\int_\Omega\frac{|v+h|^{2_s^*}}{|y|^s}\,dx\Big)^{2/2_s^*}, \end{aligned}\tag{3.8}$$

a contradiction. Therefore, $B=0$ and (3.4) yield

$$c\le J(f(v)v+g(v))\le c.$$

The assertion follows. $\square$

From Lemma 3.2, we know that there exists a minimizer of problem (2.4) provided that (3.2) holds. By Proposition 2.1, such a minimizer is actually a solution of problem (1.1). Therefore, to prove Theorem 1.1, it is sufficient to verify condition (3.2). Choosing $B_\rho(0,z_0)\subset\Omega\subset B_R(0,z_0)$. Let $\varphi\in C_0^\infty(\Omega)$ be a cut-off function satisfying

$$\varphi(x)=\begin{cases}1, & x\in B_{\frac{\rho}{2}}(0,z_0)\\ 0, & x\notin B_\rho(0,z_0).\end{cases}$$

For $\varepsilon>0$, we define $U_\varepsilon(x)=\varepsilon^{\frac{2-N}{2}}U(\frac{x-(0,z_0)}{\varepsilon})$, $u_\varepsilon=\varphi(x)U_\varepsilon(x)$, Then $u_\varepsilon\in E$ for $\varepsilon>0$ small. We have following estimates for u_ε.

Lemma 3.3. *Suppose $N\geq 3$, we have*

$$\|u_\varepsilon\|^2=\|U\|^2+O(\varepsilon^{N-2})+O(\varepsilon^{N-s}), \tag{3.9}$$

$$\int_\Omega\frac{|u_\varepsilon|^{2_s^*}}{|y|^s}dx=\int_{R^N}\frac{|U|^{2_s^*}}{|y|^s}dx+O(\varepsilon^{N-s}), \tag{3.10}$$

$$\int_\Omega|u_\varepsilon(x)|^2dx\geq\begin{cases}C\varepsilon^2+O(\varepsilon^{N-2}), & N\geq 5,\\ C\varepsilon^2|\ln\varepsilon|+O(\varepsilon^2), & N=4,\\ C\varepsilon+O(\varepsilon^2), & N=3,\end{cases} \tag{3.11}$$

$$\int_\Omega u_\varepsilon(x)dx\leq C\varepsilon^{(N-2)/2}, \tag{3.12}$$

$$\int_\Omega\frac{|u_\varepsilon|^{2_s^*-1}}{|y|^s}dx\leq C\varepsilon^{(N-2)/2}. \tag{3.13}$$

Proof. First, we estimate (3.10). There holds

$$\begin{aligned}\int_\Omega\frac{|u_\varepsilon|^{2_s^*}}{|y|^s}dx&=\int_\Omega\frac{|\varphi U_\varepsilon|^{2_s^*}}{|y|^s}dx=\int_\Omega\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx-\int_\Omega(1-\varphi^{2_s^*})\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx\\&=\int_{R^N}\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx-\int_{R^N\setminus\Omega}\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx-\int_\Omega(1-\varphi^{2_s^*})\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx\\&=\int_{R^N}\frac{U^{2_s^*}}{|y|^s}dx-\int_{R^N\setminus\Omega}\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx-\int_{\Omega\setminus B_{\frac{\rho}{2}}(0,z_0)}(1-\varphi^{2_s^*})\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx.\end{aligned}$$

Since

$$\int_{R^N\setminus B_R(0,z_0)}\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx\leq\int_{R^N\setminus\Omega}\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx\leq\int_{R^N\setminus B_\rho(0,z_0)}\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx,$$

while

$$\begin{aligned}\int_{R^N\setminus B_R(0,z_0)}\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx&=\int_{R^N\setminus B_R(0,z_0)}\varepsilon^{s-N}\frac{U(\frac{x-(0,z_0)}{\varepsilon})^{2_s^*}}{|y|^s}dx\\&=\int_{R^N\setminus B_R(0)}\varepsilon^{s-N}\frac{U(\frac{x}{\varepsilon})^{2_s^*}}{|y|^s}dx\\&\leq C\varepsilon^{s-N}\int_{R^N\setminus B_R(0)}\Big(\frac{1}{1+|\frac{x}{\varepsilon}|^{N-2}}\Big)^{2_s^*}\frac{1}{|y|^s}dx\end{aligned}$$

$$
\begin{aligned}
&= C\varepsilon^{N-s}\int_{R^N\setminus B_R(0)}\Big(\frac{1}{\varepsilon^{N-2}+|x|^{N-2}}\Big)^{2_s^*}\frac{1}{|y|^s}dx\\
&= O(\varepsilon^{N-s}),
\end{aligned}
$$

and similarly,

$$\int_{R^N\setminus B_\rho(0,z_0)}\frac{U_\varepsilon^{2_s^*}}{|y|^s}dx = O(\varepsilon^{N-s}).$$

Thus, we obtain

$$\int_\Omega\frac{|u_\varepsilon|^{2_s^*}}{|y|^s}dx = \int_{R^N}\frac{U^{2_s^*}}{|y|^s}dx + O(\varepsilon^{N-s}).$$

That is, (3.10) holds.

Next, we estimate (3.11). In fact,

$$
\begin{aligned}
\int_\Omega|u_\varepsilon|^2dx &= \int_\Omega\varphi^2|U_\varepsilon|^2dx \le \int_{B_\rho(0,z_0)}|U_\varepsilon|^2\,dx\\
&= \varepsilon^{2-N}\int_{B_\rho(0)}U(\frac{x}{\varepsilon})^2dx\\
&\le \varepsilon^{2-N}\int_{B_\rho(0)}\frac{C}{\big(1+|\frac{x}{\varepsilon}|^{N-2}\big)^2}dx\\
&= \varepsilon^{N-2}\int_{B_\rho(0)}\frac{C}{\big(\varepsilon^{N-2}+|x|^{N-2}\big)^2}dx\\
&\le \varepsilon^{N-2}\int_{B_\varepsilon(0)}\frac{C}{\varepsilon^{2(N-2)}}dx + \varepsilon^{N-2}\int_{B_\rho(0)\setminus B_\varepsilon(0)}\frac{C}{|x|^{2(N-2)}}dx\\
&= \begin{cases} C\varepsilon^2 + O(\varepsilon^{N-2}), & N\ge 5,\\ C\varepsilon^2|\ln\varepsilon| + O(\varepsilon^2), & N=4,\\ C\varepsilon + O(\varepsilon^2), & N=3.\end{cases}
\end{aligned}
$$

Now, we estimate (3.9). Observe that

$$\int_\Omega|\nabla u_\varepsilon|^2\,dx = \int_\Omega U_\varepsilon^2|\nabla\varphi|^2dx + \int_\Omega\nabla U_\varepsilon\nabla(\varphi^2U_\varepsilon)\,dx$$

and $-\Delta U_\varepsilon = U_\varepsilon^{2*-1}/|y|^s$, we find

$$\int_\Omega\nabla U_\varepsilon\nabla(\varphi^2U_\varepsilon)\,dx = \int_\Omega\varphi^2\frac{U_\varepsilon^{2*}}{|y|^s}\,dx$$

and

$$\int_\Omega|\nabla u_\varepsilon|^2\,dx = \int_\Omega|\nabla\varphi|^2U_\varepsilon^2\,dx + \int_\Omega\varphi^2\frac{U_\varepsilon^{2*}}{|y|^s}.$$

Since $\nabla\varphi = 0$ in $B_\rho(0,z_0)$, we have

$$
\begin{aligned}
\int_\Omega|\nabla\varphi|^2U_\varepsilon^2\,dx &= \int_{\Omega\setminus B_\rho(0,z_0)}|\nabla\varphi|^2U_\varepsilon^2\,dx\\
&\le \int_{B_R(0,z_0)\setminus B_\rho(0,z_0)}|\nabla\varphi|^2U_\varepsilon^2\,dx\\
&\le C\int_{B_R(0,z_0)\setminus B_\rho(0,z_0)}U_\varepsilon^2\,dx
\end{aligned}
$$

$$\le C\int_{B_R(0)\setminus B_\rho(0)}\varepsilon^{2-N}\frac{1}{(1+|\frac{x}{\varepsilon}|^{N-2})^2}\,dx=O(\varepsilon^{N-2}).$$

On the other hand, we can show that

$$\int_\Omega \varphi^2\frac{|U_\varepsilon|^{2_s^*}}{|y|^s}dx=\int_{R^N}\frac{U^{2_s^*}}{|y|^s}dx+O(\varepsilon^{N-s})=\int_{R^N}|\nabla U|^2\,dx+O(\varepsilon^{N-s}).$$

Therefore,

$$\int_\Omega|\nabla u_\varepsilon|^2\,dx=\|\nabla U\|^2+O(\varepsilon^{N-2})+O(\varepsilon^{N-s}).$$

Now, we estimate (3.12).

$$\begin{aligned}\int_\Omega u_\varepsilon dx&=\int_{B_\rho(0,z_0)}\varepsilon^{\frac{2-N}{2}}U(\frac{x-(0,z_0)}{\varepsilon})\\&\le C\int_{B_\rho(0)}\varepsilon^{\frac{2-N}{2}}\frac{1}{1+|\frac{x}{\varepsilon}|^{N-2}}dx\\&=\varepsilon^{\frac{N-2}{2}}\int_{B_\rho(0)}\frac{1}{\varepsilon^{N-2}+|x|^{N-2}}dx\\&\le C\varepsilon^{\frac{N-2}{2}}\int_{B_\varepsilon(0)}\frac{1}{\varepsilon^{N-2}}dx+C\varepsilon^{\frac{N-2}{2}}\int_{B_\rho(0)\setminus B_\varepsilon(0)}\frac{1}{|x|^{N-2}}dx\\&\le C\varepsilon^{\frac{N-2}{2}}.\end{aligned}$$

Finally, there holds

$$\begin{aligned}\int_\Omega\frac{|u_\varepsilon|^{2_s^*-1}}{|y|^s}\,dx&\le\int_{B_\rho(0,z_0)}\frac{|U_\varepsilon|^{2_s^*-1}}{|y|^s}\,dx\\&\le\varepsilon^{\frac{N+2-2s}{2}}\int_{B_\rho(0)}\Big(\frac{1}{\varepsilon^{N-2}+|x|^{N-2}}\Big)^{2_s^*-1}\frac{dx}{|y|^s}\\&=\varepsilon^{\frac{N+2-2s}{2}}\int_{B_\varepsilon(0)}\Big(\frac{1}{\varepsilon^{N-2}+|x|^{N-2}}\Big)^{2_s^*-1}\frac{dx}{|y|^s}\\&\quad+\varepsilon^{\frac{N+2-2s}{2}}\int_{B_\rho(0)\setminus B_\varepsilon(0)}\Big(\frac{1}{\varepsilon^{N-2}+|x|^{N-2}}\Big)^{2_s^*-1}\frac{dx}{|y|^s}\\&\le C\varepsilon^{(N-2)/2}+\varepsilon^{\frac{N+2-2s}{2}}\int_{B_\rho(0)\setminus B_\varepsilon(0)}\frac{1}{(|y|^2+|z|^2)^{\frac{N+2-2s}{2}}}\frac{dx}{|y|^s}\end{aligned}$$

and

$$\begin{aligned}&\varepsilon^{\frac{N+2-2s}{2}}\int_{B_\rho(0)\setminus B_\varepsilon(0)}\frac{1}{(|y|^2+|z|^2)^{\frac{N+2-2s}{2}}}\frac{dx}{|y|^s}\\&=\varepsilon^{\frac{N+2-2s}{2}}\int_{(B_\rho(0)\setminus B_\varepsilon(0))\cap\{x=(y,z):|y|\ge|z|\}}\frac{1}{(|y|^2+|z|^2)^{\frac{N+2-2s}{2}}}\frac{dx}{|y|^s}\\&\quad+\varepsilon^{\frac{N+2-2s}{2}}\int_{(B_\rho(0)\setminus B_\varepsilon(0))\cap\{x=(y,z):|y|<|z|\}}\frac{1}{(|y|^2+|z|^2)^{\frac{N+2-2s}{2}}}\frac{dx}{|y|^s}\\&\le C\varepsilon^{\frac{N+2-2s}{2}}\int_{\{x=(y,z):\frac{\varepsilon}{\sqrt2}<|y|,|z|<\rho\}}\frac{1}{|z|^{N+2-2s}}\frac{dx}{|y|^s}\\&\quad+C\varepsilon^{\frac{N+2-2s}{2}}\int_{\{x=(y,z):\frac{\varepsilon}{\sqrt2}<|y|,|z|<\rho\}}\frac{1}{|y|^{N+2-2s}}\frac{dx}{|y|^s}\end{aligned}$$

$$\le C\varepsilon^{(N-2)/2},$$

which implies (3.13). □

Proposition 3.4. *There holds*

$$c < \frac{2-s}{2(N-s)} S^{\frac{N-s}{2-s}}.$$

Proof. We will check that

$$\max_{t>0, v\in E^-} J(tu_\varepsilon + v) < \frac{2-s}{2(N-s)} S^{\frac{N-s}{2-s}}. \tag{3.14}$$

Let $\omega = \Omega \setminus \operatorname{supp}\varphi$. By [15, Lemma 3.3], $v \mapsto \|v\|_{L^{2^*_s}(\omega)}$ defines a norm on E^-. Since $\dim E^- = m < +\infty$, all the norms are equivalent on E^-. For every $t > 0$ and every $v \in E^-$, by convexity we deduce

$$\begin{aligned} &\int_\Omega \frac{|tu_\varepsilon(x) + v(x)|^{2^*_s}}{|y|^s}\,dx \\ &= \int_{\Omega\setminus\omega} \frac{|tu_\varepsilon(x) + v(x)|^{2^*_s}}{|y|^s}\,dx + \int_\omega \frac{|v(x)|^{2^*_s}}{|y|^s}\,dx \\ &\ge t^{2^*_s} \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_s}}{|y|^s}\,dx + 2^*_s t^{2^*_s-1} \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_s-1} v(x)}{|y|^s}\,dx + 2^*_s C_1 \|v\|^{2^*_s}. \end{aligned} \tag{3.15}$$

It follows that

$$\begin{aligned} J(tu_\varepsilon + v) \le J(tu_\varepsilon) &+ t\int_\Omega \nabla u_\varepsilon \nabla v + \frac{1}{2}\int_\Omega |\nabla v|^2\,dx \\ &- \lambda t \int_\Omega u_\varepsilon(x) v(x)\,dx - \frac{\lambda}{2}\int_\Omega |v(x)|^2\,dx \\ &- t^{2^*_s-1} \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_s-1} v(x)}{|y|^s} dx - C_1 \|v\|^{2^*_s}. \end{aligned} \tag{3.16}$$

By the assumption $\lambda_m \le \lambda < \lambda_{m+1}$,

$$\int_\Omega |\nabla v|^2\,dx - \lambda \int_\Omega |v(x)|^2\,dx \le (\lambda_m - \lambda)\|v\|^2 \le 0. \tag{3.17}$$

In particular, we can write

$$J(tu_\varepsilon + z) \le A(t^2 + t\|v\| + t^{2^*_s-1}\|v\|) - B(t^{2^*_s} + \|v\|^{2^*_s})$$

for suitable constants $A > 0$ and $B > 0$. Hence there exists $R > 0$ such that, for ε small, $t > R$ and $v \in E^-$ there holds $J(tu_\varepsilon + v) \le 0$. On the other hand, whenever $t \le R$,

$$J(tu_\varepsilon + v) \le J(tu_\varepsilon) + O\big(\varepsilon^{(N-2)/2}\big)\|v\| - C_1\|v\|^{2^*_s} \le J(tu_\varepsilon) + O\Big(\varepsilon^{\frac{(N-2)(N-s)}{N+2-2s}}\Big). \tag{3.18}$$

Indeed, integrating by parts and using the definition of E^-, we obtain

$$
\begin{aligned}
&\int_\Omega \nabla u_\varepsilon \nabla v\,dx - \lambda \int_\Omega u_\varepsilon(x)v(x)\,dx \\
&= \int_\Omega (-\Delta v)u_\varepsilon\,dx - \lambda \int_\Omega u_\varepsilon(x)v(x)\,dx \\
&\le |\lambda_m - \lambda| \int_\Omega |u_\varepsilon(x)v(x)|\,dx \le |\lambda_m-\lambda|\|v(\cdot)\|_{L^\infty} \int_\Omega |u_\varepsilon(x)|\,dx \\
&\le C|\lambda_m - \lambda|\|v\| \int_\Omega |u_\varepsilon(x)|\,dx
\end{aligned} \tag{3.19}
$$

and

$$
\Big|\int_\Omega \frac{|u_\varepsilon(x)|^{2_s^*-1}v}{|y|^s}\,dx\Big| \le C\|v\| \int_\Omega \frac{|u_\varepsilon(x)|^{2_s^*-1}}{|y|^s}\,dx.
$$

By (3.12) and (3.13) , we get

$$
\int_\Omega |u_\varepsilon(x)|\,dx \le C\varepsilon^{(N-2)/2}, \quad \int_\Omega \frac{|u_\varepsilon(x)|^{2_s^*-1}}{|y|^s}\,dx \le C\varepsilon^{(N-2)/2}.
$$

By the Young inequality,

$$
O\Big(\varepsilon^{(N-2)/2}\Big)\|v\| \le O\Big(\varepsilon^{\frac{(N-2)(N-s)}{N+2-2s}}\Big) + C_1\|v\|^{2_s^*}.
$$

Therefore, together with (3.17), we see that (3.18) holds.

Since $N \ge 5$ implies $\frac{(N-2)(N-s)}{N+2-2s} > 2$. By Lemma 3.2, for $\varepsilon > 0$ small enough,

$$
\begin{aligned}
&\max_{t>0, v\in E^-} J(tu_\varepsilon + v) \\
&\le \max_{t>0} J(tu_\varepsilon) + O\Big(\varepsilon^{\frac{(N-2)(N-s)}{N+2-2s}}\Big) \\
&= \frac{2-s}{2(N-s)}\Big(\frac{\|u_\varepsilon\|^2 - \lambda\|u_\varepsilon(x)\|^2_{L^2(\Omega)}}{(\int_\Omega \frac{|u_\varepsilon(x)|^{2_s^*}}{|y|^s}\,dx)^{2/2_s^*}}\Big)^{\frac{N-s}{2-s}} + O\Big(\varepsilon^{\frac{(N-2)(N-s)}{N+2-2s}}\Big) \\
&\le \frac{2-s}{2(N-s)}\Big(S - C\lambda\varepsilon^2 + O(\varepsilon^{N-2})\Big)^{\frac{N-s}{2-s}} + O\Big(\varepsilon^{\frac{(N-2)(N-s)}{N+2-2s}}\Big) \\
&< \frac{2-s}{2(N-s)} S^{\frac{N-s}{2-s}}.
\end{aligned}
$$

Assume now that $N = 4$. From (3.12) and (3.13) , we obtain

$$
\int_\Omega |u_\varepsilon(x)|\,dx \le C\varepsilon, \quad \int_\Omega \frac{|u_\varepsilon(x)|^{2_s^*-1}}{|y|^s}\,dx \le C\varepsilon.
$$

By the assumption $\lambda_m < \lambda < \lambda_{m+1}$,

$$
\int_\Omega |\nabla v|^2\,dx - \lambda \int_\Omega |v(x)|^2\,dx \le (\lambda_m - \lambda)\|v\|^2 = -C_2\|v\|^2. \tag{3.20}
$$

Inequality (3.16), (3.19) and (3.20) imply that, for $t \le R$,

$$
J(tu_\varepsilon + v) \le J(tu_\varepsilon) + O(\varepsilon)\|v\| - C_2\|v\|^2 \le J(tu_\varepsilon) + O(\varepsilon^2).
$$

From Lemma 3.2, for $\varepsilon > 0$ small enough, we obtain

$$
\max_{t>0, v\in E^-} J(tu_\varepsilon + v)
$$

$$
\begin{aligned}
&\le \frac{2-s}{2(4-s)}\Big(\frac{\|u_\varepsilon\|^2-\lambda\|u_\varepsilon\|^2_{L^2(\Omega)}}{(\int_\Omega \frac{|u_\varepsilon|^{2^*_s}}{|y|^s}\,dx)^{2/2^*_s}}\Big)^{\frac{4-s}{2-s}}+O(\varepsilon^2)\\
&\le \frac{2-s}{2(4-s)}\Big(\frac{\|U\|^2+O\left(\varepsilon^2\right)-\lambda\left(C\varepsilon^2\left|\ln\varepsilon\right|+O(\varepsilon^2)\right)}{\left(\int_{R^N}\frac{|U|^{2^*_s}}{|y|^s}\,dx+O\left(\varepsilon^{4-s}\right)\right)^{2/(4-s)}}\Big)^{\frac{4-s}{2-s}}+O(\varepsilon^2)\\
&\le \frac{2-s}{2(4-s)}\left(S-C\lambda\varepsilon^2\left|\ln\varepsilon\right|+O\left(\varepsilon^2\right)\right)^{\frac{4-s}{2-s}}+O(\varepsilon^2)\\
&< \frac{2-s}{2(4-s)}S^{\frac{4-s}{2-s}}.
\end{aligned}
$$

□

Proof of Theorem 1.1. By Lemma 3.1 and Proposition 3.4, there exists $u\in\mathcal{N}$ such that $J(u)=c$ and $DJ(u)|_{T_u\mathcal{N}}=0$. It follows from Proposition 2.1 that $DJ(u)=0$ on X. □

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