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OSCILLATION FOR QUASILINEAR ELLIPTIC EQUATIONS WITH $p(x)$-LAPLACIANS IN GENERAL DOMAINS

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Dedicated to Professor Kusano Takaŝi on his eightieth birthday

Abstract. Oscillation of quasilinear elliptic equations with $p(x)$-Laplacians in general domains are derived by the variational approach as applications of Picone identity. Three examples are given, and generalizations to quasilinear elliptic equations with $p(x)$-Laplacians are shown.

1. Introduction

Recently there has been much interest in establishing Picone identity which plays an important role in Sturmian comparison theorems and oscillation theorems for various differential equations. We refer the reader to Allegretto [1], Allegretto and Huang [3, 4], Bognár and Došlý [5], Došlý and Řehák [9], Dunninger [10], Kusano, Jaroš, and Yoshida [12], Yoshida [17] for p-Laplace equations, and to Allegretto [2], Bognár and Došlý [6], Sahiner and Zafer [15], Yoshida [18, 19, 20, 21] for $p(x)$-Laplace equations.

The operator $\nabla \cdot \big(|\nabla u|^{p(x)-2}\nabla u\big)$ is said to be $p(x)$-Laplacian $(p(x) > 1)$, and becomes p-Laplacian $\nabla\cdot\big(|\nabla u|^{p-2}\nabla u\big)$ if $p(x) = p$ (constant), where the dot $\cdot$ denotes the scalar product,$\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ and $|x|$ denotes the Euclidean length of $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

Various mathematical problems with variable exponent growth condition have been received considerable attention in recent years (see [8, 11]). These problems arise from nonlinear elasticity theory, electrorheological fluids (cf. [14, 22]) and image processing (cf. [7]).

Oscillation results for half-linear elliptic equations have been extensively developed, but most of them pertain to unbounded domains which are exterior or general exterior domains which are not "small" at ∞.

In 1973 Swanson [16] studied the linear elliptic equation

$$\nabla \cdot \big(A(x)\nabla v\big) + C(x)v = 0, \quad \Omega \subset \mathbb{R}^n,$$

where Ω is an unbounded domain in $\mathbb{R}^n$ and it is not required that Ω be quasiconical, quasicylindrical, exterior, general exterior, or even connected.

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The objective of this paper is to provide oscillation criteria for the half-linear elliptic inequality with $p(x)$-Laplacian

$$vQ[v] \le 0 \tag{1.1}$$

in a general domain $\Omega \subset \mathbb{R}^n$, where

$$\begin{aligned} Q[v] := \nabla \cdot \big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ + |\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v + C(x)|v|^{\alpha(x)-1}v. \end{aligned} \tag{1.2}$$

We note that $\log|v|$ in (1.2) has singularities at zeros of v, but $v\log|v|$ becomes continuous at the zeros of v if we define $v\log|v| = 0$ at the zeros, in light of the fact that $\lim_{\varepsilon\to+0}\varepsilon\log\varepsilon = 0$. Therefore, we conclude that $vQ[v]$ has no singularities and is continuous in Ω. From the relation

$$(kv)Q[kv] = |k|^{\alpha(x)+1}vQ[v] \ (k \in \mathbb{R})$$

we see that (1.1) is half-linear in the sense that a constant multiple of a solution v of (1.1) is also a solution of (1.1) (cf. Yoshida [18, Proposition 2.1]).

Our approach is an adaptation of variational method which is based on Picone identity. We refer to Mařík [13] which investigates oscillation of half-linear elliptic equations with p-Laplacian in general exterior domains by the variational approach.

In Section 2 we present oscillation results based on Picone identity. Section 3 is devoted to examples which illustrate a main oscillation theorem in Section 2. Generalizations to more general elliptic inequalities are given in Section 4.

2. Oscillation results

In this section we derive a main oscillation theorem by using Lemmas 2.3 and 2.4 which are deduced from Picone identity.

Let Ω be an unbounded domain in $\mathbb{R}^n$. It is assumed that $A(x) \in C(\Omega;(0,\infty))$, $B(x) \in C(\Omega;\mathbb{R}^n)$, $C(x) \in C(\Omega;\mathbb{R})$, $\alpha(x) \in C^1(\Omega;(0,\infty))$, and $\alpha(x) > 0$.

The domain $\mathcal{D}_Q(\Omega)$ of Q is defined to be the set of all functions v of class $C^1(\Omega;\mathbb{R})$ such that $A(x)|\nabla v|^{\alpha(x)-1}\nabla v \in C^1(\Omega;\mathbb{R}^n)$.

A solution $v \in \mathcal{D}_Q(\Omega)$ of (1.1) is said to be *oscillatory* in Ω if it has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap \{x \in \mathbb{R}^n;\ |x| > r\}.$$

Let G be a bounded domain with piecewise smooth boundary ∂G such that $G \subset \Omega$. We need four lemmas. The following lemma was established by Yoshida [19, Theorem 2.1].

Lemma 2.1 (Picone identity for Q)**.** *If $v \in \mathcal{D}_Q(G)$ and v has no zero in G, then we obtain the following Picone identity for any $u \in C^1(G;\mathbb{R})$:*

$$
\begin{aligned}
&-\nabla\cdot\Big(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)}\Big)\\
&= -A(x)\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1} + C(x)|u|^{\alpha(x)+1}\\
&\quad + A(x)\Big[\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1}\\
&\quad + \alpha(x)\big|\frac{u}{v}\nabla v\big|^{\alpha(x)+1} - (\alpha(x)+1)\big|\frac{u}{v}\nabla v\big|^{\alpha(x)-1}\Big(\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)\\
&\quad - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big)\cdot\big(\frac{u}{v}\nabla v\big)\Big] - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}}\big(vQ[v]\big) \quad \text{in } G,
\end{aligned}
\tag{2.1}
$$

where $\varphi(u) = |u|^{\alpha(x)-1}u$.

The next lemma is stated in Yoshida [19, Lemma 3.1].

Lemma 2.2. *The inequality*

$$
|\xi|^{\alpha(x)+1} + \alpha(x)\,|\eta|^{\alpha(x)+1} - (\alpha(x)+1)|\eta|^{\alpha(x)-1}\xi\cdot\eta \ge 0
$$

is valid for $x \in G$, $\xi, \eta \in \mathbb{R}^n$, where the equality holds if and only if $\xi = \eta$.

For a nonempty piecewise smooth bounded domain $G \subset \Omega$, we define the functional

$$
\begin{aligned}
M[u;G] := \int_G \Big[&A(x)\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1}\\
&- C(x)|u|^{\alpha(x)+1}\Big]\,dx
\end{aligned}
$$

with domain consisting of all real-valued piecewise C^1-functions u on $\overline{G}$.

Lemma 2.3. *Let $\alpha(x) \in C^2(G;(0,\infty))$ and $B(x)/A(x) \in C^1(G;\mathbb{R}^n)$. Assume that there exists a function $u \in C^1(\overline{G};\mathbb{R})$ such that $u = 0$ on ∂G and u has no zero in G. If $M[u;G] \le 0$ and*

(H1) *there is a function $F \in C(\overline{G};\mathbb{R})$ such that $F \in C^1(G;\mathbb{R})$ and*

$$
\nabla F = \frac{\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{B(x)}{(\alpha(x)+1)A(x)} \quad \text{in } G,
$$

then every solution $v \in \mathcal{D}_Q(G)$ of (1.1) must vanish at some point of $\overline{G}$.

Proof. Suppose to the contrary that there exists a solution $v \in \mathcal{D}_Q(G)$ of (1.1) such that v has no zero on $\overline{G}$. Integrating the Picone identity (2.1) over G and using the divergence theorem, we arrive at

$$
0 \ge -M[u;G] + \int_G W(u,v)\,dx, \tag{2.2}
$$

where

$$
\begin{aligned}
&W(u,v)\\
&:= A(x)\Big[\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1}
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha(x)\Big|\frac{u}{v}\nabla v\Big|^{\alpha(x)+1}-(\alpha(x)+1)\Big|\frac{u}{v}\nabla v\Big|^{\alpha(x)-1}\Big(\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)\\
&-\frac{u}{(\alpha(x)+1)A(x)}B(x)\Big)\cdot\big(\frac{u}{v}\nabla v\big)\Big].
\end{aligned}
$$

Lemma 2.2 implies that $W(u,v)\geq 0$ in G, and hence $\int_G W(u,v)\,dx\geq 0$. Since $-M[u;G]\geq 0$, we see that the right-hand side of (2.2) is non-negative. Therefore we conclude that $\int_G W(u,v)\,dx=0$, which yields

$$
\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)A(x)}B(x)\equiv\frac{u}{v}\nabla v\quad\text{in } G
$$

from Lemma 2.2. Using the hypothesis (H1), we obtain

$$
\nabla u+u\nabla F\equiv\frac{u}{v}\nabla v\quad\text{in } G,
$$

from which we get

$$
e^{-F}v\nabla\left(e^F\frac{u}{v}\right)\equiv 0\quad\text{in } G.
$$

Hence, there is a constant c_0 such that $e^F u/v=c_0$ in G, and therefore on $\overline{G}$ by continuity. Since $u=0$ on ∂G, we find that $c_0=0$, that is, $u\equiv 0$ in G, which contradicts the hypothesis that u has no zero in G. This completes the proof. $\square$

Lemma 2.4. *Let $\alpha(x)\in C^2(G;(0,\infty))$ and $B(x)/A(x)\in C^1(G;\mathbb{R}^n)$. Assume that there exists a nontrivial function $u\in C^1(\overline{G};\mathbb{R})$ such that $u=0$ on ∂G. If $M[u;G]<0$, then every solution $v\in\mathcal{D}_Q(G)$ of* (1.1) *must vanish at some point of $\overline{G}$.*

Proof. Suppose that there is a solution $v\in\mathcal{D}_Q(G)$ of (1.1) such that v has no zero on $\overline{G}$. As in the proof of Lemma 2.3, we observe that (2.2) holds. Since $-M[u;G]>0$ and $\int_G W(u,v)\,dx\geq 0$, it can be shown that the right-hand side of (2.2) is positive. The contradiction proves the lemma. $\square$

Remark 2.5. In the hypothesis (H1) of Lemma 2.3, the vector-valued function

$$
\frac{\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{B(x)}{(\alpha(x)+1)A(x)}
$$

must be a C^1-function (cf. [19, Proposition 2.2]), and therefore we suppose that $\alpha(x)\in C^2$, $B(x)/A(x)\in C^1$ in Lemmas 2.3 and 2.4.

Theorem 2.6. *Assume that Ω contains a sequence of domains G_k $(k=1,2,\dots)$ such that G_k are nonempty bounded domains with piecewise smooth boundaries ∂G_k. If for any $r>0$ there exist a natural number $m=m(r)\in\mathbb{N}$ such that $\overline{G_m}\subset\Omega_r$ and a nontrivial piecewise C^1-function u_m on $\overline{G_m}$ with the following properties:*

- (i) *$\alpha(x)=\alpha_m(x)\in C^2(G_m;(0,\infty))$ and*
$$
B(x)/A(x)=B_m(x)/A_m(x)\in C^1(G_m;\mathbb{R}^n)\quad\text{on } G_m;
$$
- (ii) *$u_m=0$ on ∂G_m;*
- (iii) *u_m has no zero in G_m;*
- (iv) *$M[u_m;G_m]\leq 0$;*
- (v) *there exists a function $F_m\in C(\overline{G_m};\mathbb{R})$ such that $F_m\in C^1(G_m;\mathbb{R})$ and*

$$
\nabla F_m=\frac{\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x)-\frac{B_m(x)}{(\alpha_m(x)+1)A_m(x)}\quad\text{in } G_m,\tag{2.3}
$$

then every solution $v \in \mathcal{D}_Q(\Omega)$ of (1.1) *is oscillatory in Ω. Furthermore, if $M[u_m; G_m] < 0$ in the hypothesis* (iv)*, then the conclusion is valid without the hypotheses* (iii), (v).

Proof. For any $r > 0$ there exist a natural number $m = m(r) \in \mathbb{N}$ such that $\overline{G_m} \subset \Omega_r$ and a nontrivial piecewise C^1-function u_m on $\overline{G_m}$ satisfying (i)–(v). It follows from Lemma 2.3 that every solution $v \in \mathcal{D}_Q(G_m)$ of (1.1) has a zero on $\overline{G_m} \subset \Omega_r$, that is, v has a zero in Ω_r for any $r > 0$, which implies that v is oscillatory in Ω. This completes the proof of the first statement of the theorem. If $M[u_m; G_m] < 0$, the second statement follows from Lemma 2.4 by the same arguments as were used in the first statement. □

Remark 2.7. Let $p \geq 1$. It follows from Jensen's inequality that

$$\Big(\frac{a+b+c}{3}\Big)^p \leq \frac{a^p + b^p + c^p}{3}$$

for any $a, b, c \geq 0$. We let $\alpha(x) > 0$, and obtain

$$\begin{aligned} |a+b+c|^{\alpha(x)+1} &\leq (|a|+|b|+|c|)^{\alpha(x)+1} \\ &\leq 3^{\alpha(x)}\Big(|a|^{\alpha(x)+1} + |b|^{\alpha(x)+1} + |c|^{\alpha(x)+1}\Big) \end{aligned}$$

for any $a, b, c \in \mathbb{R}$. Hence, we obtain

$$\begin{aligned} &\Big|\nabla u + \frac{u \log |u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1} \\ &\leq 3^{\alpha(x)}\Big(|\nabla u|^{\alpha(x)+1} + \Big|\frac{u \log |u|}{\alpha(x)+1}\nabla\alpha(x)\Big|^{\alpha(x)+1} + \Big|\frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1}\Big). \end{aligned}$$

Defining

$$\begin{aligned} \tilde{M}[u; G] := \int_G \Big[A(x)3^{\alpha(x)}\Big(|\nabla u|^{\alpha(x)+1} + \Big|\frac{u \log |u|}{\alpha(x)+1}\nabla\alpha(x)\Big|^{\alpha(x)+1} \\ + \Big|\frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1}\Big) - C(x)|u|^{\alpha(x)+1}\Big]dx, \end{aligned}$$

we observe that $M[u; G] \leq \tilde{M}[u; G]$. Consequently, Lemmas 2.3, 2.4 and Theorem 2.6 remain true if $M[u; G], M[u_m; G_m]$ are replaced by $\tilde{M}[u; G], \tilde{M}[u_m; G_m]$, respectively.

3. Examples

Three examples which illustrate Theorem 2.6 are given in this section. We consider the cases where Ω contains a sequence of rectangular domains, annular domains, or balls. Using the similar arguments as in [19, Remark 2.4], we give the following example.

Example 3.1. We consider the case where $n = 2$ and Ω contains a sequence of rectangular domains

$$G_k = \big(a_k, a_k + \pi\big) \times \big(b_k, b_k + \pi\big) \quad (k = 1, 2, \dots),$$

where $\lim_{k\to\infty} a_k = \infty$ or $\lim_{k\to\infty} b_k = \infty$. Assume that

$$A(x) \leq A_k, \quad C(x) \geq C_k \quad \text{on } G_k$$

for some constants $A_k > 0$, $C_k > 0$, and let

$$\begin{aligned}\alpha(x) = \alpha_k(x) &= e^{\sin(x_1-a_k)\sin(x_2-b_k)+1} - 1,\\ \frac{B(x)}{A(x)} = \frac{B_k(x)}{A_k(x)} &\\ = e^{\sin(x_1-a_k)\sin(x_2-b_k)+1}\Big(&-\cos(x_1-a_k)\sin(x_2-b_k),\\ &-\sin(x_1-a_k)\cos(x_2-b_k)\Big)\end{aligned}$$

on each G_k. For any $r > 0$ there exists an integer $m = m(r) \in \mathbb{N}$ such that $\overline{G_m} \subset \Omega_r$, because $\lim_{k\to\infty} a_k = \infty$ or $\lim_{k\to\infty} b_k = \infty$. Letting

$$u_m = \sin\big(x_1 - a_m\big)\sin\big(x_2 - b_m\big),$$

we see that $u_m \in C^1(\overline{G_m};\mathbb{R})$, $u_m = 0$ on ∂G_m, $u_m > 0$ in G_m, and that there exists a function $F_m \in C(\overline{G_m};\mathbb{R}) \cap C^1(G_m;\mathbb{R})$ satisfying (2.3). Since

$$\frac{B_m(x)}{A_m(x)} = -(\alpha_m(x)+1)\nabla u_m, \quad \frac{\nabla\alpha_m(x)}{\alpha_m(x)+1} = \nabla u_m,$$

we obtain

$$\begin{aligned}\frac{\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x) - \frac{B_m(x)}{(\alpha_m(x)+1)A_m(x)} &= (\log u_m)\nabla u_m + \nabla u_m\\ &= \nabla F_m \quad \text{in } G_m,\end{aligned}$$

where

$$F_m = u_m \log u_m.$$

Moreover, we observe that $F_m \in C(\overline{G_m};\mathbb{R})$. It is easy to see that

$$\begin{aligned}&M[u_m; G_m]\\ &\le \int_{a_m}^{a_m+\pi}\int_{b_m}^{b_m+\pi}\Big[A_m\Big|\nabla u_m + \frac{u_m\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x)\\ &\quad - \frac{u_m}{(\alpha_m(x)+1)A_m(x)}B_m(x)\Big|^{\alpha_m(x)+1} - C_m|u_m|^{\alpha_m(x)+1}\Big]dx_1dx_2.\end{aligned} \tag{3.1}$$

We easily obtain

$$\begin{aligned}&\nabla u_m + \frac{u_m\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x) - \frac{u_m}{(\alpha_m(x)+1)A_m(x)}B_m(x)\\ &= (1 + u_m\log u_m + u_m)\nabla u_m\end{aligned}$$

and hence

$$\begin{aligned}&\Big|\nabla u_m + \frac{u_m\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x) - \frac{u_m}{(\alpha_m(x)+1)A_m(x)}B_m(x)\Big|\\ &\le \sqrt{2}|1 + u_m\log u_m + u_m|\end{aligned} \tag{3.2}$$

in view of the inequality $|\nabla u_m| \le \sqrt{2}$. Since

$$\begin{aligned}u_m\log u_m = \sin(x_2-b_m)&\Big[\sin(x_1-a_m)\log\sin(x_1-a_m)\Big]\\ &+ \sin(x_1-a_m)\Big[\sin(x_2-b_m)\log\sin(x_2-b_m)\Big],\end{aligned}$$

we see that

$$|u_m \log u_m| \le \frac{2}{e} \tag{3.3}$$

in light of the inequality

$$|x \log x| \le \frac{1}{e} \quad (0 \le x \le 1).$$

Since $\alpha_m(x) + 1 \le e^2$, it follows from (3.2) and (3.3) that

$$\begin{aligned} &\Big|\nabla u_m + \frac{u_m \log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x) - \frac{u_m}{(\alpha_m(x)+1)A_m(x)}B_m(x)\Big|^{\alpha_m(x)+1} \\ &\le \Big(\sqrt{2}\Big(1+\frac{2}{e}+1\Big)\Big)^{e^2} = \Big(2\sqrt{2}\Big(1+\frac{1}{e}\Big)\Big)^{e^2}. \end{aligned} \tag{3.4}$$

Since $\alpha_m(x) + 1 \le e^2 \le 8$, it can be shown that

$$\begin{aligned} &\int_{a_m}^{a_m+\pi}\int_{b_m}^{b_m+\pi} |u_m|^{\alpha_m(x)+1} dx_1 dx_2 \\ &= \int_{a_m}^{a_m+\pi}\int_{b_m}^{b_m+\pi} |\sin(x_1-a_m)\sin(x_2-b_m)|^{\alpha_m(x)+1}\, dx_1 dx_2 \\ &\ge \int_{a_m}^{a_m+\pi}\int_{b_m}^{b_m+\pi} \sin^8(x_1-a_m)\sin^8(x_2-b_m) dx_1 dx_2 \\ &= \Big(\int_0^\pi \sin^8 x\, dx\Big)^2. \end{aligned}$$

It is known that

$$\begin{aligned} \int_0^\pi \sin^8 x\, dx &= 2\int_0^{\pi/2} \sin^8 x\, dx \\ &= 2\cdot\frac{\pi}{2}\frac{7!!}{8!!} = \pi\cdot\frac{7\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2} = \frac{105}{384}\pi, \end{aligned}$$

and therefore we have

$$\int_{a_m}^{a_m+\pi}\int_{b_m}^{b_m+\pi} |u_m|^{\alpha_m(x)+1} dx_1 dx_2 \ge \Big(\frac{105}{384}\Big)^2 \pi^2. \tag{3.5}$$

Combining (3.1), (3.4) and (3.5) yields

$$M[u_m; G_m] \le A_m\Big(2\sqrt{2}\Big(1+\frac{1}{e}\Big)\Big)^{e^2}\pi^2 - C_m\Big(\frac{105}{384}\Big)^2\pi^2.$$

If

$$C_m \ge \Big(\frac{384}{105}\Big)^2\Big(2\sqrt{2}\Big(1+\frac{1}{e}\Big)\Big)^{e^2} A_m,$$

then we observe that $M[u_m; G_m] \le 0$, and consequently it follows from Theorem 2.6 that every solution $v \in \mathcal{D}_Q(\Omega)$ of (1.1) is oscillatory in Ω.

Example 3.2. We consider the case where Ω contains a sequence of annular domains

$$G_k = \{x \in \mathbb{R}^n;\ a_k < |x| < b_k\} \quad (k = 1, 2, \dots),$$

where $0 < b_k - a_k \le 1$ and $\lim_{k\to\infty} a_k = \infty$. It is assumed that

$$A(x) \le A_k, \quad C(x) \ge C_k \quad \text{on } G_k$$

for some constants $A_k > 0$, $C_k > 0$, and let

$$\begin{aligned}
\alpha(x) &= \alpha_k(|x|) = e^{(b_k-|x|)(|x|-a_k)+1} - 1,\\
\frac{B(x)}{A(x)} &= \frac{B_k(x)}{A_k(x)} = \tilde{B}_k(|x|)\frac{x}{|x|}\\
&= -e^{(b_k-|x|)(|x|-a_k)+1}\big(a_k + b_k - 2|x|\big)\frac{x}{|x|}
\end{aligned}$$

on each G_k. For any $r > 0$ there exists an integer $m = m(r) \in \mathbb{N}$ such that $\overline{G_m} \subset \Omega_r$, because $\lim_{k\to\infty} a_k = \infty$. Let

$$u_m = \big(b_m - |x|\big)\big(|x| - a_m\big),$$

we find that $u_m \in C^1(\overline{G_m};\mathbb{R})$, $u_m = 0$ on ∂G_m, u_m has no zero in G_m, and that there exists a function $F_m \in C(\overline{G_m};\mathbb{R}) \cap C^1(G_m;\mathbb{R})$ satisfying (2.3). In fact, we derive

$$\begin{aligned}
&\frac{\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x) - \frac{B_m(x)}{(\alpha_m(x)+1)A_m(x)}\\
&= \frac{\log\big((b_m-|x|)(|x|-a_m)\big)}{e^{(b_m-|x|)(|x|-a_m)+1}}\Big(\alpha_m'(r)\Big|_{r=|x|}\Big)\frac{x}{|x|} + (a_m + b_m - 2|x|)\frac{x}{|x|}\\
&= \Big[(a_m + b_m - 2|x|)\log\big((b_m-|x|)(|x|-a_m)\big) + a_m + b_m - 2|x|\Big]\frac{x}{|x|}\\
&= \nabla F_m \quad \text{in } G_m,
\end{aligned}$$

where

$$F_m = f_m(|x|),$$

$f_m(r)$ being the function defined by

$$f_m(r) = (b_m - r)(r - a_m)\log\big((b_m - r)(r - a_m)\big).$$

Moreover, we see that $F_m \in C(\overline{G_m};\mathbb{R})$. Introducing hyperspherical coordinates (r,θ), and letting $u_m(x) = z_m(|x|)$, we have

$$\begin{aligned}
&M[u_m;G_m]\\
&\le \int_{a_m}^{b_m}\int_{S_1}\Big[A_m\Big|z_m'(r) + \frac{z_m(r)\log|z_m(r)|}{\alpha_m(r)+1}\alpha_m'(r)\\
&\quad - \frac{z_m(r)}{(\alpha_m(r)+1)}\tilde{B}_m(r)\Big|^{\alpha_m(r)+1} - C_m|z_m(r)|^{\alpha_m(r)+1}\Big]r^{n-1}\,dr\,d\omega,
\end{aligned} \tag{3.6}$$

where ω denotes the measure on the unit sphere S_1. It is readily seen that

$$\begin{aligned}
&z_m'(r) + \frac{z_m(r)\log|z_m(r)|}{\alpha_m(r)+1}\alpha_m'(r) - \frac{z_m(r)}{(\alpha_m(r)+1)}\tilde{B}_m(r)\\
&= z_m'(r) + z_m(r)f_m'(r)\\
&= \big(a_m + b_m - 2r\big)\Big[1 + (b_m - r)(r - a_m)\log\big((b_m - r)(r - a_m)\big)\\
&\quad + (b_m - r)(r - a_m)\Big].
\end{aligned}$$

Simple computations show that

$$|a_m + b_m - 2r| \le b_m - a_m,$$

$$|(b_m - r)(r - a_m)\log\big((b_m - r)(r - a_m)\big)| \le \frac{2}{e}(b_m - a_m),$$

$$
\begin{gathered}
|(b_m - r)(r - a_m)| \le \frac{(b_m - a_m)^2}{4} \\
e \le \alpha_m(r) + 1 = e^{(b_m - r)(r - a_m) + 1} \le e^{(b_m - a_m)^2/4 + 1},
\end{gathered}
$$

on $[a_m, b_m]$, and therefore we arrive at

$$
\begin{aligned}
&\Big| z_m'(r) + \tfrac{z_m(r)\log|z_m(r)|}{\alpha_m(r)+1}\alpha_m'(r) - \tfrac{z_m(r)}{(\alpha_m(r)+1)}\tilde{B}_m(r)\Big|^{\alpha_m(r)+1} \qquad (3.7)\\
&\le \big(b_m - a_m\big)^e \Big[1 + \tfrac{2}{e}(b_m - a_m) + \tfrac{(b_m - a_m)^2}{4}\Big]^{\tilde{\alpha}_m}, \qquad (3.8)
\end{aligned}
$$

where $\tilde{\alpha}_m = e^{(b_m - a_m)^2/4 + 1}$. It is easily verified that

$$
\begin{aligned}
&\int_{a_m}^{b_m} |z_m(r)|^{\alpha_m(r)+1} r^{n-1} dr \\
&= \int_{a_m}^{b_m} |(b_m - r)(r - a_m)|^{\alpha_m(r)+1}\, r^{n-1} dr \qquad (3.9)\\
&\ge a_m^{n-1} \int_{a_m}^{b_m} (b_m - r)^{\alpha_m(r)+1}(r - a_m)^{\alpha_m(r)+1} dr.
\end{aligned}
$$

By making a change of variable $r = a_m + (b_m - a_m)t$, we obtain

$$
\begin{aligned}
&\int_{a_m}^{b_m} (b_m - r)^{\alpha_m(r)+1}(r - a_m)^{\alpha_m(r)+1} dr \\
&\ge (b_m - a_m)^{2\tilde{\alpha}_m + 3} \int_0^1 (1 - t)^{\tilde{\alpha}_m + 1} t^{\tilde{\alpha}_m + 1} dt \qquad (3.10)\\
&= (b_m - a_m)^{2\tilde{\alpha}_m + 3} B(\tilde{\alpha}_m + 2, \tilde{\alpha}_m + 2),
\end{aligned}
$$

where $B(s, t)$ denotes the Bessel function. Combining (3.6)–(3.10), we observe that

$$
\begin{aligned}
&M[u_m; G_m] \\
&\le \omega_n A_m b_m^{n-1}(b_m - a_m)\big(b_m - a_m\big)^e \Big[1 + \frac{2}{e}(b_m - a_m) + \frac{(b_m - a_m)^2}{4}\Big]^{\tilde{\alpha}_m} \\
&\quad - \omega_n C_m a_m^{n-1}(b_m - a_m)^{2\tilde{\alpha}_m + 3} B(\tilde{\alpha}_m + 2, \tilde{\alpha}_m + 2),
\end{aligned}
$$

and hence $M[u_m; G_m] \le 0$ if

$$
\begin{aligned}
&C_m a_m^{n-1}(b_m - a_m)^{2\tilde{\alpha}_m + 2} B(\tilde{\alpha}_m + 2, \tilde{\alpha}_m + 2) \\
&\ge A_m b_m^{n-1}\big(b_m - a_m\big)^e \Big[1 + \frac{2}{e}(b_m - a_m) + \frac{(b_m - a_m)^2}{4}\Big]^{\tilde{\alpha}_m}.
\end{aligned}
$$

Then, Theorem 2.6 implies that every solution $v \in \mathcal{D}_Q(\Omega)$ of (1.1) is oscillatory in Ω.

Example 3.3. Suppose that Ω contains a sequence of balls

$$
G_k = \{x \in \mathbb{R}^n;\ |x - a_k| < b_k\} \quad (k = 1, 2, \dots),
$$

where $0 < b_k \le 1$ and $\lim_{k\to\infty} |a_k| = \infty$. We assume that

$$
A(x) \le A_k, \quad C(x) \ge C_k \quad \text{on } G_k
$$

for some constants $A_k > 0$, $C_k > 0$, and let

$$
\alpha(x) = \alpha_k(|x - a_k|) = e^{|x - a_k| + 1} - 1,
$$

$$\frac{B(x)}{A(x)} = \frac{B_k(x)}{A_k(x)} = \tilde{B}_k(|x-a_k|)\frac{x-a_k}{|x-a_k|} = -e^{|x-a_k|+1}\frac{x-a_k}{|x-a_k|}$$

on each G_k. For any $r>0$ there is an integer $m=m(r)\in\mathbb{N}$ such that $\overline{G_m}\subset\Omega_r$, because $\lim_{k\to\infty}|a_k|=\infty$. Letting

$$u_m = b_m - |x-a_m|,$$

we observe that $u_m\in C^1(\overline{G_m};\mathbb{R})$, $u_m=0$ on ∂G_m, u_m has no zero in G_m. A simple calculation yields

$$\begin{aligned}
&\nabla u_m + \frac{u_m\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x) - \frac{u_m}{(\alpha_m(x)+1)A_m(x)}B_m(x)\\
&= -\frac{x-a_m}{|x-a_m|} + \big(b_m-|x-a_m|\big)\Big(\frac{x-a_m}{|x-a_m|}\Big)\log\big|b_m-|x-a_m|\big|\\
&\quad + \big(b_m-|x-a_m|\big)\Big(\frac{x-a_m}{|x-a_m|}\Big),
\end{aligned}$$

and therefore

$$\begin{aligned}
&\Big|\nabla u_m + \frac{u_m\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x) - \frac{u_m}{(\alpha_m(x)+1)A_m(x)}B_m(x)\Big|\\
&\le 1 + \big(b_m-|x-a_m|\big)\log\big(b_m-|x-a_m|\big) + b_m\\
&\le 1 + \frac{1}{e} + b_m,
\end{aligned}$$

in view of the inequality $0\le b_m-|x-a_m|\le b_m$ and

$$\big(b_m-|x-a_m|\big)\log\big(b_m-|x-a_m|\big)\le\frac{1}{e}.$$

Since $\alpha_m(x)+1\le e^{b_m+1}$, we obtain

$$\begin{aligned}
&\Big|\nabla u_m + \frac{u_m\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x) - \frac{u_m}{(\alpha_m(x)+1)A_m(x)}B_m(x)\Big|^{\alpha_m(x)+1}\\
&\le \Big(1+\frac{1}{e}+b_m\Big)^{e^{b_m+1}}.
\end{aligned}\tag{3.11}$$

Introducing hyperspherical coordinates in the ball G_m, we find that

$$\begin{aligned}
\int_{G_m}|u_m|^{\alpha_m(x)+1}dx &= \int_{G_m}\big(b_m-|x-a_m|\big)^{\alpha_m(x)+1}dx\\
&= \int_0^{b_m}\int_{S_1}(b_m-r)^{e^r+1}r^{n-1}dr\,d\omega\\
&= \omega_n\int_0^{b_m}(b_m-r)^{e^r+1}r^{n-1}dr.
\end{aligned}\tag{3.12}$$

By making a change of variable $r/b_m=s$, we obtain

$$\begin{aligned}
\int_0^{b_m}(b_m-r)^{e^r+1}r^{n-1}dr &= \int_0^1\big(b_m-b_ms\big)^{e^{b_ms+1}}(b_ms)^{n-1}b_m\,ds\\
&\ge b_m^{e^{b_m+1}+n}\int_0^1(1-s)^{e^{b_m+1}}s^{n-1}ds\\
&= b_m^{e^{b_m+1}+n}B(e^{b_m+1}+1,n),
\end{aligned}\tag{3.13}$$

in light of $0 < b_m \le 1$. Combining (3.11)–(3.13), we have

$$\begin{aligned} M[u_m; G_m] &\le \int_{G_m} \Big[A_m \Big| \nabla u_m + \frac{u_m \log |u_m|}{\alpha_m(x)+1} \nabla \alpha_m(x) \\ &\quad - \frac{u_m}{(\alpha_m(x)+1)A_m(x)} B_m(x) \Big|^{\alpha_m(x)+1} - C_m |u_m|^{\alpha_m(x)+1} \Big] dx \\ &\le A_m \Big(1 + \frac{1}{e} + b_m\Big)^{e^{b_m+1}} \frac{\omega_n}{n} b_m^n - C_m \omega_n b_m^{e^{b_m+1}+n} B(e^{b_m+1}+1, n), \end{aligned}$$

and hence

$$M[u_m; G_m] < 0$$

if

$$C_m b_m^{e^{b_m+1}} n B(e^{b_m+1}+1, n) > A_m \Big(1 + \frac{1}{e} + b_m\Big)^{e^{b_m+1}}.$$

It follows from Theorem 2.6 that every solution $v \in \mathcal{D}_Q(\Omega)$ of (1.1) is oscillatory in Ω.

4. Generalizations

In this section we treat the more general elliptic operator $\hat{Q}$ defined by

$$\begin{aligned} \hat{Q}[v] &:= \nabla \cdot \big(A(x)|\nabla v|^{\alpha(x)-1} \nabla v\big) - A(x)(\log |v|)|\nabla v|^{\alpha(x)-1} \nabla \alpha(x) \cdot \nabla v \\ &\quad + |\nabla v|^{\alpha(x)-1} B(x) \cdot \nabla v + C(x)|v|^{\alpha(x)-1} v \\ &\quad + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i(x)-1} v + \sum_{j=1}^{L} E_j(x)|v|^{\gamma_j(x)-1} v, \end{aligned}$$

where $\beta_i(x) > \alpha(x) > \gamma_j(x) > 0$, and $D_i(x), E_j(x) \in C(\overline{G}; [0,\infty))$ ($i = 1, 2, \dots, \ell$; $j = 1, 2, \dots, L$). The domain $\mathcal{D}_{\hat{Q}}(G)$ of $\hat{Q}$ is defined as the same as $\mathcal{D}_Q(G)$. Let $N = \min\{\ell, L\}$ and we define

$$\hat{C}(x) = \sum_{i=1}^{N} H(\beta_i(x), \alpha(x), \gamma_i(x); D_i(x), E_i(x)),$$

where

$$\begin{aligned} &H(\beta(x), \alpha(x), \gamma(x); D(x), E(x)) \\ &= \Big(\frac{\beta(x)-\gamma(x)}{\alpha(x)-\gamma(x)}\Big)\Big(\frac{\beta(x)-\alpha(x)}{\alpha(x)-\gamma(x)}\Big)^{\frac{\alpha(x)-\beta(x)}{\beta(x)-\gamma(x)}} D(x)^{\frac{\alpha(x)-\gamma(x)}{\beta(x)-\gamma(x)}} E(x)^{\frac{\beta(x)-\alpha(x)}{\beta(x)-\gamma(x)}}. \end{aligned}$$

Then we have the following Lemma which is analogous to Lemma 2.1 (see [21]).

Lemma 4.1 (Picone-type inequality for $\hat{Q}$)**.** *If $v \in \mathcal{D}_{\hat{Q}}(G)$ and v has no zero in G, then we obtain the following Picone-type inequality for any $u \in C^1(G; \mathbb{R})$:*

$$\begin{aligned} &- \nabla \cdot \Big(u\varphi(u) \frac{A(x)|\nabla v|^{\alpha(x)-1} \nabla v}{\varphi(v)}\Big) \\ &\ge -A(x)\Big|\nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x)\Big|^{\alpha(x)+1} \\ &\quad + \big(C(x) + \tilde{C}(x)\big)|u|^{\alpha(x)+1} \\ &\quad + A(x)\Big[\Big|\nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x)\Big|^{\alpha(x)+1} \end{aligned}$$

$$
\begin{aligned}
&+ \alpha(x)\Big|\frac{u}{v}\nabla v\Big|^{\alpha(x)+1} - (\alpha(x)+1)\Big|\frac{u}{v}\nabla v\Big|^{\alpha(x)-1}\Big(\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) \\
&- \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big)\cdot\big(\frac{u}{v}\nabla v\big)\Big] - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}}\big(v\hat{Q}[v]\big) \quad \text{in } G.
\end{aligned}
$$

For a nonempty piecewise smooth bounded domain $G \subset \Omega$, we define the functional

$$
\begin{aligned}
\hat{M}[u;G] := \int_G \Big[A(x)\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1} \\
- \big(C(x)+\hat{C}(x)\big)|u|^{\alpha(x)+1}\Big]dx
\end{aligned}
$$

with domain consisting of all real-valued piecewise C^1-functions u on $\overline{G}$.

By the same arguments as were used in Lemmas 2.3, 2.4 and Theorem 2.6 we obtain the following Lemma and Theorem.

Lemma 4.2. *Let $\alpha(x) \in C^2(G;(0,\infty))$ and $B(x)/A(x) \in C^1(G;\mathbb{R}^n)$. If there exists a function $u \in C^1(\overline{G};\mathbb{R})$ such that $u = 0$ on ∂G and u has no zero in G, $\hat{M}[u;G] \le 0$ and the hypothesis* (H1) *of Lemma* 2.3 *holds, then every solution $v \in \mathcal{D}_{\hat{Q}}(G)$ of $v\hat{Q}[v] \le 0$ must vanish at some point of $\overline{G}$. Moreover, If there exists a nontrivial function $u \in C^1(\overline{G};\mathbb{R})$ such that $u = 0$ on ∂G and $\hat{M}[u;G] < 0$, then every solution $v \in \mathcal{D}_{\hat{Q}}(G)$ of $v\hat{Q}[v] \le 0$ must vanish at some point of $\overline{G}$.*

Theorem 4.3. *Assume that Ω contains a sequence of domains G_k $(k = 1,2,\dots)$ such that G_k are nonempty bounded domains with piecewise smooth boundaries ∂G_k. If for any $r > 0$ there exist a natural number $m = m(r) \in \mathbb{N}$ such that $\overline{G_m} \subset \Omega_r$ and a nontrivial piecewise C^1-function u_m on $\overline{G_m}$ with the following properties:*

(i) *$\alpha(x) = \alpha_m(x) \in C^2(G_m;(0,\infty))$ and*

$$
B(x)/A(x) = B_m(x)/A_m(x) \in C^1(G_m;\mathbb{R}^n) \quad \text{on } G_m;
$$

(ii) *$u_m = 0$ on ∂G_m;*

(iii) *u_m has no zero in G_m;*

(iv) *$\hat{M}[u_m;G_m] \le 0$;*

(v) *there exists a function $F_m \in C(\overline{G_m};\mathbb{R})$ such that $F_m \in C^1(G_m;\mathbb{R})$ and*

$$
\nabla F_m = \frac{\log|u_m|}{\alpha_m(x)+1}\nabla\alpha_m(x) - \frac{B_m(x)}{(\alpha_m(x)+1)A_m(x)} \quad \text{in } G_m,
$$

then every solution $v \in \mathcal{D}_{\hat{Q}}(\Omega)$ of $v\hat{Q}[v] \le 0$ is oscillatory in Ω. Furthermore, if $\hat{M}[u_m;G_m] < 0$ in the hypothesis (iv)*, then the conclusion is valid without the hypotheses* (iii), (v).

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