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PERIODIC SOLUTIONS FOR FOURTH-ORDER p-LAPLACIAN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH SIGN-VARIABLE COEFFICIENT

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Abstract. Using the theory of coincidence degree, we show the existence of periodic solutions to the fourth-order p-Laplacian differential equations of Liénard-type

$$\begin{aligned}&\phi_p(x''))'' + f(x(t))x'(t) + \alpha(t)g_1(x(t-\tau_1(t,x(t))))\\&+ \beta(t)g_2(x(t-\tau_1(t,x(t)))) = p(t).\end{aligned}$$

The rate of growth of $g_1(u)$ with respect to the variable u is allowed to be greater than $p-1$, and the coefficient $\beta(t)$ is allowed to change sign.

1. Introduction

The study of the fourth-order differential equations is of great practical significance, whose classical application is to describe the equilibrium of elastic beams. The study on periodic oscillations of the fourth-order differential equations has gained more and more attention by many researchers, and some profound results have been obtained (see [3, 7, 8, 10]). However, the results of periodic solutions to a fourth order p-Laplacian delay differential equation are relatively rare.

In this article, we consider the existence of periodic solutions to the fourth-order p-Laplacian differential equations with multiple deviating arguments:

$$\begin{aligned}&\phi_p(x''))'' + f(x(t))x'(t) + \alpha(t)g_1\big(x(t-\tau_1(t,x(t)))\big)\\&+ \beta(t)g_2\big(x(t-\tau_2(t,x(t)))\big) = p(t)\end{aligned}\tag{1.1}$$

where $p > 1$, $\phi_p(s) = |s|^{p-2}s$ $(s \neq 0)$, $\phi_p(0) = 0$, $\alpha(t), \beta(t), p(t) \in C(\mathbb{R},\mathbb{R})$, $\int_0^T p(t)dt = 0$, $\int_0^T \beta(t)dt \neq 0$, $\alpha(t) \geq 0$ (≤ 0) for $t \in \mathbb{R}$, $\int_0^T \alpha(t)dt > 0$ (< 0), $\alpha(t+T) = \alpha(t)$, $\beta(t+T) = \beta(t)$, $p(t+T) = p(t)$ $\tau_i \in C(\mathbb{R}^2,\mathbb{R})$, $\tau_i(t+T,x) = \tau_i(t,x)$, $g_i \in C(\mathbb{R},\mathbb{R})$, $i = 1, 2$, $T > 0$.

In recent years, there have been a number of results on the existence of periodic solutions of the second order p-Laplacian differential equations; see [1, 2, 5, 6, 9, 11]

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and the references therein. Cheung and Ren [9] studied the existence of periodic solutions for the p-Laplacian delay equation

$$\phi_p(x'))' + f(x'(t)) + \beta g(x(t-\tau(t))) = e(t)$$

where $\beta > 0$ is a constant. Gao and Lu [2] studied the periodic solutions for the p-Laplacian Rayleigh differential equation with a delay,

$$\phi_p(x'))' + f(x'(t)) + \beta(t) g(x(t-\tau(t))) = e(t)\,.$$

In 2007, Cheung and Ren [1] discussed the solvability of periodic problems for the Lienard-type p-Laplacian delay differential equation

$$\phi_p(x'))' + f(x(t))x'(t) + g(t, x(t-\tau(t))) = e(t)\,,$$

under the assumption

$$\lim_{|x|\to\infty} \frac{|g(x)|}{|x|^{p-1}} = r \geq 0.$$

Motivated by the above works, we will present the existence of periodic solutions for (1.1) by using Mawhin's continuation theorem. Our main results are different from those results in the literature. For instance, in our study we allow the growth rate of $g_1(u)$, with respect to u, to be greater than $p-1$. Also we allow the coefficient $\beta(t)$ to change sign $\mathbb{R}$.

2. Preliminaries

For simplicity, we use the following symbols in this article

$$C_T = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\}, \quad |x|_\infty = \max_{t\in[0,T]} |x(t)|,$$
$$C_T^1 = \{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\}, \quad \|x\| = \max\{|x|_\infty, |x'|_\infty\},$$
$$|x|_p = \Big(\int_0^T |x(t)|^p dt\Big)^{1/p}, \quad D_p = \begin{cases} 1, & 0 < p \leq 1, \\ 2^{p-1}, & p > 1. \end{cases}$$

To state our main results, we introduce several technical lemmas.

Lemma 2.1 ([6]). *Assume that Ω is an open bounded set in C_T^1 such that the following three conditions hold:*

(1) *For each $\lambda \in (0, 1)$, the equation*
$$(\phi_p(x''))'' = \lambda f(t, x(t), x(t-\mu(t)), x'(t)), \tag{2.1}$$
has no T-periodic solution on $\partial\Omega$, where $f(t, x, y, z) \in C(\mathbb{R}^4, \mathbb{R})$ and $f(t + T, \cdot, \cdot, \cdot)) = f(t, \cdot, \cdot, \cdot))$.

(2) *The equation*
$$F(a) = \frac{1}{2\pi} \int_0^T f(t, a, a, 0) dt = 0,$$
has no solution on $\partial\Omega \cap \mathbb{R}$.

(3) *The Brouwer degree satisfies* $\deg_B(F, \Omega \cap \mathbb{R}, 0) \neq 0$.

Then (2.1) *has a T-periodic solution in $\bar{\Omega}$ when $\lambda = 1$.*

Lemma 2.2 ([11]). *If $\omega(t) \in c^1(\mathbb{R},\mathbb{R})$ and $\omega(0)=\omega(T)=0$, then there holds*

$$\int_0^T |\omega(t)|^p dt \le \Big(\frac{T}{\pi_p}\Big)^p \int_0^T |\omega'(t)|^p dt,$$

where

$$\pi_p = \int_0^{(p-1)/p} \frac{ds}{(1-(p-1)^{-1}s^p)^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}.$$

Lemma 2.3 ([4]). *Let $a, b, p > 0$, then there holds*

$$(a+b)^p \le D_p(a^p + b^p).$$

For the sake of convenience, we list the following assumptions which will be used frequently in Section 3.

(H1) For $i = 1, 2$, there are positive constants r_i, r_i^*, m_i with $m_2 \le p-1$ and $m_1 > p-1$ such that for $|u| > 1$ there hold
  (i) $r_1|u|^{m_1} \le |g_1(u)| \le r_2|u|^{m_1}$ and $r_1^*|u|^{m_2} \le |g_2(u)| \le r_2^*|u|^{m_2}$.
  (ii) $ug_i(u) > 0$.

(H2) $A = D_{\frac{1}{m_1}}(\frac{r_2^*\bar\beta}{\bar\alpha r_1})^{1/m_1} < 1$.

(H3) There are constants $\gamma, r_3 > 0$ and $k_0 \in Z$ such that $m_1 = r_3 + p - 1$ and

$$0 \le \tau_1(t, x(t)) - k_0T \le \max\{\frac{\gamma^q}{1+|x|_\infty^{r_3 q}}, T\}, \quad \forall t \in [0,T],\ x(t) \in C[0,T].$$

where $q > 1 : \frac{1}{p} + \frac{1}{q} = 1$

3. Main Results

Theorem 3.1. *Suppose that* (H1)–(H3). *Then* (1.1) *has at least one T-periodic solution if one of the following two conditions holds*

(1) $m_2 = p-1$, $\Delta_1 + \Delta_2 < 1$,
(2) $m_2 < p-1$, $\Delta_1 < 1$,

where

$$\Delta_1 = \frac{D_{p-1}\overline{\alpha} r_2 T^{\frac{p-1}{q}}\gamma}{2^{p-1}(1-A)^{p-1}}\Big(\frac{T}{\pi_p}\Big)^p, \quad \Delta_2 = \frac{r_2^*\overline{\beta}D_{m_2+1}T^{\frac{m_2+1}{q}}}{2^{m_2+1}(1-A)^{m_2+1}}\Big(\frac{T}{\pi_p}\Big)^{m_2+1}.$$

Proof. Without loss of generality, we assume $\alpha(t) \ge 0$, $t \in \mathbb{R}$, $\int_0^T \alpha(t)dt > 0$, and $\int_0^T \beta(t)dt > 0$. Consider the homotopy equation

$$\begin{aligned} &\phi_p(x''))'' + \lambda f(x(t))x'(t) + \lambda\alpha(t)g_1(x(t-\tau_1(t,x(t)))) \\ &+ \lambda\beta(t)g_2(x(t-\tau_2(t,x(t)))) = \lambda p(t). \end{aligned} \tag{3.1}$$

Suppose that $x(t)$ is an arbitrary T-periodic solution of (3.1). Integrating both sides of equation (3.1) on $[0,T]$ we obtain

$$\int_0^T \alpha(t)g_1(x(t-\tau_1(t,x(t))))dt = -\int_0^T \beta(t)g_2(x(t-\tau_2(t,x(t))))dt.$$

Applying the mean value theorem, then there exists a constant $\xi \in [0,T]$ such that

$$g_1(x(\xi-\tau_1(\xi,x(\xi)))) \int_0^T \alpha(t)dt = -\int_0^T \beta(t)g_2(x(t-\tau_2(t,x(t))))dt. \tag{3.2}$$

Now, we claim that the inequality

$$|x(\xi - \tau_1(\xi, x(\xi)))| \le A|x|_\infty + B \tag{3.3}$$

holds, where

$$\begin{gathered} A = D_{\frac{1}{m_1}}(\frac{r_2^*\bar{\beta}}{\bar{\alpha}r_1})^{1/m_1}, \quad B = D_{\frac{1}{m_1}}(\frac{M_{g_2}\bar{\beta}}{\bar{\alpha}r_1})^{1/m_1} + 1, \\ \bar{\alpha} = \int_0^T \alpha(t)dt, \quad \bar{\beta} = \int_0^T |\beta(t)|dt, \quad M_{g_2} = \max_{|u|\le 1} |g_2(u)|. \end{gathered}$$

In fact, if $|x(\xi-\tau_1(\xi, x(\xi)))| \le 1$, then inequality (3.3) holds. If $|x(\xi-\tau_1(\xi, x(\xi)))| > 1$, we define

$$\begin{gathered} E_1 = \{t \in [0, T] : |x(t - \tau_1(t, x(t)))| \le 1\}, \\ E_2 = \{t \in [0, T] : |x(t - \tau_1(t, x(t)))| > 1\}. \end{gathered}$$

It follows from (H1)(i) that

$$\begin{aligned} \bar{\alpha}r_1|x(\xi - \tau_1(\xi, x(\xi)))|^{m_1} &\le \int_0^T \beta(t)g_2(x(t - \tau_2(t, x(t))))dt \\ &= \int_{E_1} + \int_{E_2} \beta(t)g_2(x(t - \tau_2(t, x(t))))dt \\ &\le r_2^*\bar{\beta}|x|_\infty^{m_2} + M_{g_2}\bar{\beta}. \end{aligned}$$

This implies that

$$\begin{aligned} |x(\xi - \tau_1(\xi, x(\xi)))| &\le [\frac{1}{\bar{\alpha}r_1}(r_2^*\bar{\beta}|x|_\infty^{m_2} + M_{g_2}\bar{\beta})]^{1/m_1} \\ &\le D_{\frac{1}{m_1}}[(\frac{r_2^*\bar{\beta}}{\bar{\alpha}r_1})^{1/m_1}|x|_\infty^{\frac{m_2}{m_1}} + (\frac{M_{g_2}\bar{\beta}}{\bar{\alpha}r_1})^{1/m_1}] \\ &\le D_{\frac{1}{m_1}}(\frac{r_2^*\bar{\beta}}{\bar{\alpha}r_1})^{1/m_1}|x|_\infty + D_{\frac{1}{m_1}}(\frac{M_{g_2}\bar{\beta}}{\bar{\alpha}r_1})^{1/m_1}. \end{aligned}$$

Thus, it can be easily seen that (3.3) holds. Let

$$\xi - \tau_1(\xi, x(\xi)) = kT + \bar{\xi}, \tag{3.4}$$

where k is an integer and $\bar{\xi} \in [0, T]$, thus we have

$$x(\xi - \tau_1(\xi, x(\xi))) = x(kT + \bar{\xi}) = x(\bar{\xi}).$$

Noting that

$$|x(t)| \le |x(\bar{\xi})| + \frac{1}{2}\int_0^T |x'(s)|ds,$$

we have

$$|x|_\infty = \max_{t\in[0,T]} |x(t)| \le A|x|_\infty + B + \frac{1}{2}\int_0^T |x'(s)|ds,$$

which yields

$$|x|_\infty \le \frac{\int_0^T |x'(s)|ds}{2(1 - A)} + \frac{B}{1 - A}. \tag{3.5}$$

On the other hand, multiplying both sides of (3.1) by $x(t)$, and integrating on $[0,T]$, we obtain

$$\begin{aligned}
\int_0^T |x''(t)|^p dt &= -\lambda\int_0^T f(x(t))x'(t)x(t)dt - \lambda\int_0^T \alpha(t)g_1(x(t-\tau_1(t,x(t))))x(t)dt \\
&\quad - \lambda\int_0^T \beta(t)g_2(x(t-\tau_2(t,x(t))))x(t)dt + \lambda\int_0^T p(t)x(t)dt \\
&\le \lambda\int_0^T \alpha(t)|g_1(x(t-\tau_1(t,x(t))))|\,|x(t)-x(t-\tau_1(t,x(t)))|dt \\
&\quad - \lambda\int_0^T \alpha(t)g_1(x(t-\tau_1(t,x(t))))x(t-\tau_1(t,x(t)))dt \\
&\quad + \int_0^T |\beta(t)g_2(x(t-\tau_2(t,x(t))))x(t)|dt + \bar{p}|x|_\infty,
\end{aligned} \tag{3.6}$$

where $\bar{p} = \int_0^T |p(t)|dt$.

By the condition (H1)(ii), we have

$$\begin{aligned}
&-\lambda\int_0^T \alpha(t)g_1(x(t-\tau_1(t,x(t))))x(t-\tau_1(t,x(t)))dt \\
&= -\lambda\int_{E_1} -\lambda\int_{E_2} \alpha(t)g_1(x(t-\tau_1(t,x(t))))x(t-\tau_1(t,x(t)))dt \\
&\le \int_{E_1} \alpha(t)|g_1(x(t-\tau_1(t,x(t))))x(t-\tau_1(t,x(t)))|dt \\
&\le \overline{\alpha}M_{g_1},
\end{aligned} \tag{3.7}$$

where $M_{g_1} = \max_{|u|\le 1}|g_1(u)|$. Using the condition (H1) again, we obtain

$$\begin{aligned}
&\int_0^T \alpha(t)|g_1(x(t-\tau_1(t,x(t))))||x(t)-x(t-\tau_1(t,x(t)))|dt \\
&= \int_{E_1} + \int_{E_2} \alpha(t)|g_1(x(t-\tau_1(t,x(t))))||x(t)-x(t-\tau_1(t,x(t)))|dt \\
&\le \overline{\alpha}M_{g_1} + \overline{\alpha}M_{g_1}|x|_\infty + \overline{\alpha}r_2\max_{t\in[0,T]}|x(t)-x(t-\tau_1(t,x(t)))| \times |x|_\infty^{m_1},
\end{aligned}$$

and

$$\int_0^T |\beta(t)||g_2(x(t-\tau_2(t,x(t))))x(t)|dt \le \overline{\beta}r_2^*|x|_\infty^{m_2+1} + \overline{\beta}M_{g_2}|x|_\infty,$$

where $M_{g_2} = \max_{|u|\le 1}|g_2(u)|$ and $\overline{\beta} = \int_0^T |\beta(t)|dt$. So (3.6) yields

$$\begin{aligned}
\int_0^T |x''(t)|^p dt &\le \overline{\alpha}r_2\max_{t\in[0,T]}|x(t)-x(t-\tau_1(t,x(t)))| \times |x|_\infty^{m_1} + \overline{\beta}r_2^*|x|_\infty^{m_2+1} \\
&\quad + (\overline{\alpha}M_{g_1} + \overline{\beta}M_{g_2} + \overline{p})|x|_\infty + 2\overline{\alpha}M_{g_1} \\
&= \overline{\alpha}r_2\max_{t\in[0,T]}|x(t)-x(t-\tau_1(t,x(t)))| \times |x|_\infty^{m_1} + \overline{\beta}r_2^*|x|_\infty^{m_2+1} \\
&\quad + \theta|x|_\infty + K,
\end{aligned} \tag{3.8}$$

where $\theta = \overline{\alpha}M_{g_1} + \overline{\beta}M_{g_2} + \overline{p}$ and $K = 2\overline{\alpha}M_{g_1}$.

Since $x(0) = x(T)$, there exists a constant $\zeta \in [0, T]$ such that $x'(\zeta) = 0$. Let $\omega(t) = x'(t + \zeta)$, then $\omega(0) = \omega(T) = 0$. By Lemma 2.2, we have

$$\int_0^T |x'(t)|^p dt \le \big(\frac{T}{\pi_p}\big)^p \Big(\int_0^T |x''(t)|^p dt\Big).$$

From (H3) and Hölder's inequality, we have

$$\begin{aligned}
&\max_{t\in[0,T]} |x(t) - x(t - \tau_1(t, x(t)))| \\
&= \max_{t\in[0,T]} |x(t) - x(t - \tau_1(t, x(t)) + k_0T)| \\
&= \max_{t\in[0,T]} \Big|\int_{t-\tau_1(t,x(t))+k_0T}^{t} x'(s)ds\Big| \\
&\le \max_{t\in[0,T]} |\tau_1(t, x(t)) - k_0T|^{1/q} \Big(\int_{t-\tau_1(t,x(t))+k_0T}^{t} |x'(s)|^p ds\Big)^{1/p} \\
&\le \max_{t\in[0,T]} |\tau_1(t, x(t)) - k_0T|_\infty^{1/q} \Big(\int_0^T |x'(s)|^p ds\Big)^{1/p}.
\end{aligned} \tag{3.9}$$

Moreover, from (3.5) and by Hölder's inequality, we have

$$\begin{aligned}
r_2^*\overline{\beta}|x|_\infty^{m_2+1} &\le r_2^*\overline{\beta}\Big[\frac{\int_0^T |x'(s)|ds}{2(1-A)} + \frac{B}{1-A}\Big]^{m_2+1} \\
&\le \frac{r_2^*\overline{\beta}D_{m_2+1}}{2^{m_2+1}(1-A)^{m_2+1}}\Big(\int_0^T |x'(s)|ds\Big)^{m_2+1} + \frac{r_2^*\overline{\beta}D_{m_2+1}B^{m_2+1}}{(1-A)^{m_2+1}} \\
&\le \frac{r_2^*\overline{\beta}D_{m_2+1}T^{\frac{m_2+1}{q}}}{2^{m_2+1}(1-A)^{m_2+1}}\big(\frac{T}{\pi_p}\big)^{m_2+1}\Big(\int_0^T |x''(s)|^p ds\Big)^{\frac{m_2+1}{p}} \\
&\quad + \frac{r_2^*\overline{\beta}D_{m_2+1}B^{m_2+1}}{(1-A)^{m_2+1}},
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\theta|x|_\infty &\le \theta\Big[\frac{\int_0^T |x'(s)|ds}{2(1-A)} + \frac{B}{1-A}\Big] \\
&\le \frac{\theta T^{1/q}}{2(1-A)}\big(\frac{T}{\pi_p}\big)\Big(\int_0^T |x''(s)|^p ds\Big)^{1/p} + \frac{\theta B}{1-A}.
\end{aligned} \tag{3.11}$$

By of $m_1 = r_3 + p - 1$ and the condition (H3), and combining (3.9)-(3.11), we have

$$\begin{aligned}
&\int_0^T |x''(t)|^p dt \\
&\le \overline{\alpha}r_2 \max_{t\in[0,T]} |x(t) - x(t - \tau_1(t, x(t)))||x|_\infty^{r_3}|x|_\infty^{p-1} + \overline{\beta}r_2^*|x|_\infty^{m_2+1} + \theta|x|_\infty + K \\
&\le \overline{\alpha}r_2\gamma\Big(\int_0^T |x'(s)|^p ds\Big)^{1/p}\Big[\frac{\int_0^T |x'(s)|ds}{2(1-A)} + \frac{B}{1-A}\Big]^{p-1} + \overline{\beta}r_2^*|x|_\infty^{m_2+1} + \theta|x|_\infty + K \\
&\le \overline{\alpha}r_2\gamma\Big(\int_0^T |x'(s)|^p ds\Big)^{1/p}\frac{D_{p-1}\Big(\int_0^T |x'(s)|ds\Big)^{p-1}}{2^{p-1}(1-A)^{p-1}} \\
&\quad + \overline{\alpha}r_2\gamma D_{p-1}\Big(\int_0^T |x'(s)|^p ds\Big)^{1/p}\frac{B^{p-1}}{(1-A)^{p-1}}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\theta T^{1/q}}{2(1-A)}\big(\frac{T}{\pi_p}\big)\Big(\int_0^T |x''(s)|^p ds\Big)^{1/p} \\
&+ \frac{r_2^*\overline{\beta} D_{m_2+1} T^{\frac{m_2+1}{q}}}{2^{m_2+1}(1-A)^{m_2+1}}\big(\frac{T}{\pi_p}\big)^{m_2+1}\big(\int_0^T |x''(s)|^p ds\big)^{\frac{m_2+1}{p}} \\
&+ \frac{r_2^*\overline{\beta} D_{m_2+1} B^{m_2+1}}{(1-A)^{m_2+1}} + \frac{B\theta}{1-A} + K \\
\le{}& \frac{D_{p-1}\overline{\alpha} r_2 T^{\frac{p-1}{q}}\gamma}{2^{p-1}(1-A)^{p-1}}\Big(\int_0^T |x'(s)|^p ds\Big) + \frac{D_{p-1}\overline{\alpha} r_2 B^{p-1}\gamma}{(1-A)^{p-1}}\Big(\int_0^T |x'(s)|^p ds\Big)^{1/p} \\
&+ \frac{r_2^*\overline{\beta} D_{m_2+1} T^{\frac{m_2+1}{q}}}{2^{m_2+1}(1-A)^{m_2+1}}\big(\frac{T}{\pi_p}\big)^{m_2+1}\big(\int_0^T |x''(s)|^p ds\big)^{\frac{m_2+1}{p}} \\
&+ \frac{\theta T^{1/q}}{2(1-A)}\big(\frac{T}{\pi_p}\big)\Big(\int_0^T |x''(s)|^p ds\Big)^{1/p} + C \\
\le{}& \frac{D_{p-1}\overline{\alpha} r_2 T^{\frac{p-1}{q}}\gamma}{2^{p-1}(1-A)^{p-1}}\big(\frac{T}{\pi_p}\big)^p\Big(\int_0^T |x''(s)|^p ds\Big) \\
&+ \frac{D_{p-1}\overline{\alpha} r_2 B^{p-1}\gamma}{(1-A)^{p-1}}\big(\frac{T}{\pi_p}\big)\Big(\int_0^T |x''(s)|^p ds\Big)^{1/p} \\
&+ \frac{r_2^*\overline{\beta} D_{m_2+1} T^{\frac{m_2+1}{q}}}{2^{m_2+1}(1-A)^{m_2+1}}\big(\frac{T}{\pi_p}\big)^{m_2+1}\Big(\int_0^T |x''(s)|^p ds\Big)^{\frac{m_2+1}{p}} \\
&+ \frac{\theta T^{1/q}}{2(1-A)}\big(\frac{T}{\pi_p}\big)\Big(\int_0^T |x''(s)|^p ds\Big)^{1/p} + C \\
={}& \Delta_1\Big(\int_0^T |x''(s)|^p ds\Big) + \Delta_2\Big(\int_0^T |x''(s)|^p ds\Big)^{\frac{m_2+1}{p}} \\
&+ \frac{D_{p-1}\overline{\alpha} r_2 B^{p-1}\gamma}{(1-A)^{p-1}}\big(\frac{T}{\pi_p}\big)\Big(\int_0^T |x''(s)|^p ds\Big)^{1/p} \\
&+ \frac{\theta T^{1/q}}{2(1-A)}\big(\frac{T}{\pi_p}\big)\Big(\int_0^T |x''(s)|^p ds\Big)^{1/p} + C,
\end{aligned}
\tag{3.12}
$$

where

$$C = \frac{r_2^*\overline{\beta} D_{m_2+1} B^{m_2+1}}{(1-A)^{m_2+1}} + \frac{B\theta}{1-A} + K.$$

If $m_2 = p-1$ and $\Delta_1 + \Delta_2 < 1$, then from (3.12) it follows that $\int_0^T |x''(t)|^p dt$ is bounded. If $m_2 < p-1$ and $\Delta_1 < 1$, then from $\frac{m_2+1}{p} < 1$ and (3.12) we see that $\int_0^T |x''(t)|^p dt$ is also bounded. Thus, there exists a constant $M > 0$ such that

$$\Big(\int_0^T |x''(t)|^p dt\Big)^{1/p} \le M,$$

which shows that there exist positive numbers M_0 and M_1 such that

$$|x|_\infty \le M_0, \quad |x'|_\infty \le M_1.$$

Let

$$\Omega = \{x(t) \in C_T^1 : ||x|| < \rho\},$$

where $\rho > \max\{1, M_0, M_1\}$. Then the homotopy equation (3.1) has no T-periodic solution on $\partial\Omega$. In addition,

$$\begin{aligned} F(\rho) &= -\frac{1}{T}[\int_0^T \alpha(t)g_1(\rho)dt + \int_0^T \beta(t)g_2(\rho)dt - \int_0^T p(t)dt] \\ &= -\frac{1}{T}g_1(\rho)\int_0^T \alpha(t)dt - \frac{1}{T}g_2(\rho)\int_0^T \beta(t)dt. \end{aligned}$$

It means that the second condition of Lemma 2.1 is satisfied, and $F(\rho)F(-\rho) < 0$ from (H1)(ii). Consequently, from Lemma 2.1 the equation (1.1) has at least one T-periodic solution in $\overline{\Omega}$. □

Remark 3.2. If we replace the conditions $\alpha(t) > 0$, $\int_0^T \beta(t)dt > 0$ with $\alpha(t) < 0$, $\int_0^T \beta(t)dt < 0$ or $\alpha(t) < 0$, $\int_0^T \beta(t)dt > 0$ or $\alpha(t) > 0$, $\int_0^T \beta(t)dt < 0$, we can obtain the same conclusion as Theorem 3.1.

Remark 3.3. Condition (H1) can be replaced by

(H1') For $i = 1, 2$, there are positive constants r_i, r_i^*, m_i, d with $m_2 \le p - 1$ and $m_1 > p - 1$ such that
- (i) $r_1|u|^{m_1} \le |g_1(u)| \le r_2|u|^{m_1}$ and $r_1^*|u|^{m_2} \le |g_2(u)| \le r_2^*|u|^{m_2}$ for all $|u| > d \ge 1$,
- (ii) $g_i(u)(\operatorname{sgn} u) > 0$ for all $|u| > d \ge 1$;

while the conclusion of Theorem 3.1 is still true.

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