Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 204, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

THE $(n-1)$-RADIAL SYMMETRIC POSITIVE CLASSICAL SOLUTION FOR ELLIPTIC EQUATIONS WITH GRADIENT

YONG ZHANG, QIANG XU, PEIHAO ZHAO

Abstract. In this article, we study the existence of the $(n-1)$-radial symmetric positive classical solution for elliptic equations with gradient. By some special techniques in two variables, we show a priori estimates, and then show the existence of a solution using a fixed point theorem.

1. Introduction

In this article, we consider the following boundary-value problem of a second-order elliptic equation,

$$\begin{aligned} -\Delta u &= f(x,u,\nabla u) \quad \text{in } \Omega, \\ u(x) &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 3$.

This type of equations have been studied by several authors. As the nonlinearity f depends on the gradient of the solution, solving (1.1) is not variational and the well developed critical point theory can not be applied directly. But if f has a special form, by changing variables, (1.1) can be transformed into a boundary-value problem which is independent of ∇u. For example, When $f(x,u,\nabla u) = g(u) + \lambda|\nabla u|^2 + \eta$, Ghergu and Rădulescu [8] used the above method to show the existence of positive classical solution under the assumption that g is decreasing and unbounded at the origin. A similar method appears in [1], where $f(x,u,\nabla u)$ has critical growth with respect to ∇u; see also [9, 20]. In addition, Chen and Yang [5] considered the existence of positive solutions for (1.1) on a smooth compact Riemannian manifold. As far as we know, the methods used to solve (1.1) are mainly sub and super-solution, fixed point theorems, Galerkin method, and topological degree, see, for instance, [2, 3, 7, 13, 17, 18, 19].

It is worth mentioning that de Figueiredo, Girardi and Matzeu [6] developed a quite different method of variational type. Firstly, for each $\omega \in H_0^1(\Omega)$, they considered the boundary problem

$$\begin{aligned} -\Delta u &= f(x,u,\nabla \omega) \quad \text{in } \Omega, \\ u(x) &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35J60, 35B09.
Key words and phrases. Elliptic equations; symmetric; positive solution; a priori estimates; fixed point theorem.

Submitted June 14, 2013. Published Spetember 16, 2013.

which is a variational problem. Under the assumptions that f has a superlinear subcritical growth at zero and at infinity with respect to the second variable, and f is locally Lipschitz continuous with the third variable, they proved that a weak solution u_ω of (1.2) exists by mountain-pass theorem. Then they have constructed a sequence $\{u_k\} \subset H_0^1(\Omega)$ as solutions of

$$\begin{gathered} -\Delta u_n = f(x, u_n, \nabla u_{n-1}) \quad \text{in } \Omega, \\ u_n(x) = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

and verified that $\{u_k\}$ converges to a solution of (1.1). However, this solution is just in $H_0^1(\Omega)$.

Additionally, the existence of classical solutions for (1.1) has been obtained by mountain-pass lemma and a suitable truncation method in [11], but the conditions imposed on f are very strong:

(1) f is locally Lipschitz continuous on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$,
(2) $\frac{f(x,t,\xi)}{t}$ converges to zero uniformly with respect to $x \in \Omega$, $\xi \in \mathbb{R}^n$ as t tends to zero,
(3) there exist $a_1 > 0$, $p \in (1, \frac{n+2}{n-2})$ and $r \in (0, 1)$ such that
$$|f(x, t, \xi)| \le a_1(1 + |t|^p)(1 + |\xi|^r), \quad \forall x \in \bar{\Omega},\ t \in \mathbb{R},\ \xi \in \mathbb{R}^n,$$
(4) there exist $\vartheta > 2$ and $a_2, a_3, t_0 > 0$ such that
$$0 < \vartheta F(x, t, \xi) \le t f(x, t, \xi), \quad \forall x \in \bar{\Omega},\ t \ge t_0,\ \xi \in \mathbb{R}^n,\ F(x, t, \xi) \ge a_2|t|^\vartheta - a_3;$$
$$F(x, t, \xi) \ge a_2|t|^\vartheta - a_3,$$
where $F(x, t, \xi) = \int_0^t f(x, s, \xi)ds$.

As far as we know, a few authors have paid attention to the radial solutions of (1.1); see for example [4, 7]. So we will limit us to the radially symmetric case and try to focus on some new methods to study (1.1). We consider the boundary-value problem (1.1) and assume the following:

(D1) Ω is a so-called $(n-1)$-symmetric domain in $\mathbb{R}^n (n \ge 3)$, that is, Ω is symmetric with respect to $x_1, x_2, \cdots, x_{n-1}$ and $0 \notin \overline{\Omega}$;
(F1) $f(x, u, \eta)$ is a nonnegative function satisfying $f(x, u, \eta) = f(r, x_n, u, |\eta|)$, where $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}$;
(F2) there exist $c_0 \ge 1$, $M > 0$, $p > 1$, $\tau \in (0, \frac{2p}{p+1})$ such that
$$u^p - M|\eta|^\tau \le f(x, u, \eta) \le c_0 u^p + M|\eta|^\tau, \quad \forall (x, u, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n;$$
(F3) $f(x, u, \eta) \in C^\beta(\Omega, \mathbb{R}, \mathbb{R}^n)$ for some $\beta \in (0, 1)$.

We remark that in [14], the constants p and τ belong to $(1, \frac{2(n-1)}{n-2})$ and $(1, \frac{2p}{p+1})$ respectively. Obviously, the conditions in (F2) are weaker than those in [14].

If the solution $u(x)$ is $(n-1)$-radial symmetric, that is $u(x) = u(r, x_n)$, then by (F1) Equation (1.1) can be transformed into the following elliptic equation in two variables:

$$\begin{gathered} -(u_{rr} + u_{x_n x_n}) = H(r, x_n, u, u_r, u_{x_n}), \quad \text{in } \Omega, \\ u(x) = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where $H(r, x_n, u, u_r, u_{x_n}) = f(r, x_n, u, |\nabla u|) + \frac{n-2}{r} u_r$. Motivated by the priori estimates mentioned in [14] and special technique for the equation in two variables developed in [10], we develop an approach which is distinct from the previous works, and shows the existence of the $(n-1)$-radial symmetric positive classical $C^{2,\beta}$-solutions of (1.1). Note that solution in [14] is just in $C^{1,\alpha}(\Omega)$.

The rest of this work is organized as follows. Motivated by [14] we give a priori estimates in section 2. In section 3 we show the existence of $(n-1)$-radial symmetric positive classical solutions with the help of [10].

2. A priori estimates

Compared with the reference [14], we should deal with the second term $\frac{n-2}{r}u_r$ of $H(r,x_n,u,u_r,u_{x_n})$ in (1.4) additionally, it is necessary to give a brief proof of the a priori estimates although the process is similar to that in [14].

Theorem 2.1. *Assume that* (D1) *and* (F2) *hold, and that* $\lambda<\lambda_0$ *for some* λ_0 *fixed. Then, for any* C^1*-solution* u *of the equation*

$$\begin{gathered} -(u_{rr}+u_{x_nx_n}) = H(r,x_n,u,u_r,u_{x_n})+\lambda, \quad \text{in } \Omega,\\ u(x)=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

there exists a positive constant C *such that* $\sup_\Omega u<C$.

To prove this theorem, we need the following lemmas.

Lemma 2.2. *Let* (D1) *hold and* $u(r,x_n)$ *be a positive weak* C^1*-solution of the inequality*

$$-(u_{rr}+u_{x_nx_n}) \ge u^p - M|\nabla u|^\tau + \frac{n-2}{r}u_r, \tag{2.2}$$

where $1<p$ *and* $0<\tau<2p/(p+1)$*. Take* $\gamma\in(0,p)$ *and* $\mu\in(0,\frac{2p}{p+1})$*. Denote by* B_{2R} *a ball of radius* $2R$ *contained in* Ω*, where* $R<R_0$ *and* R_0 *is a positive constant. Then there exists a positive constant* $C=C(p,\gamma,\mu,R_0)$ *such that*

$$\int_{B_R} u^\gamma \le CR^{2-2\gamma/(p-1)}, \tag{2.3}$$

$$\int_{B_R} |\nabla u|^\mu \le CR^{2-(p+1)\mu/(p-1)}. \tag{2.4}$$

Proof. We can assume that B_R is centered at $x_0\in\Omega$ and first focus on proving (2.3). Let ξ be a C^2-cut-off function on B_2 satisfying:

1. $\xi(x)=\xi(|x-x_0|)$, $0\le|x-x_0|\le 2$.
2. $\xi(x)=1$ for $|x-x_0|\le 1$.
3. ξ has compact support in B_2 and $0\le\xi\le 1$.
4. $|\nabla\xi|\le 2$.

Let $d=p-\gamma>0$ and $\phi=[\xi(\frac{x-x_0}{R})]^k u^{-d}$ as a test function for (2.2) (k to be fixed later). We obtain

$$-\int_\Omega (u_{rr}+u_{x_nx_n})\xi^k u^{-d} \ge \int_\Omega (u^p - M|\nabla u|^\tau + \frac{n-2}{r}u_r)\xi^k u^{-d}.$$

Integrating by parts and using that $|\nabla\xi^k| = k\xi^{k-1}|\nabla\xi| \le \xi^k\frac{2k}{R\xi}$, we obtain

$$\begin{aligned} &d\int_\Omega \xi^k u^{\gamma-p-1}|\nabla u|^2 + \int_\Omega \xi^k u^\gamma\\ &\le \int_\Omega u^{-d}|\nabla u||\nabla\xi^k| + M\int_\Omega |\nabla u|^\tau \xi^k u^{-d} - \int_\Omega \frac{n-2}{r}u_r\xi^k u^{-d} \end{aligned}$$

$$\le \int_\Omega u^{-d}|\nabla u|\xi^k \frac{2k}{R\xi} + M\int_\Omega |\nabla u|^\tau \xi^k u^{-d} + \frac{n-2}{\operatorname{dist}(0,\partial\Omega)}\int_\Omega |\nabla u|\xi^k u^{-d}.$$

Applying the Young inequality to the first right term, we have

$$\int_\Omega u^{-d}|\nabla u|\xi^k \frac{2k}{R\xi} \le \frac{d}{4}\int_\Omega \xi^k u^{\gamma-p-1}|\nabla u|^2 + CR^{-2}\int_\Omega \xi^{k-2}u^{\gamma-p+1},$$

so

$$\begin{aligned} &\frac{3}{4}d\int_\Omega \xi^k u^{\gamma-p-1}|\nabla u|^2 + \int_\Omega \xi^k u^\gamma \\ &\le CR^{-2}\int_\Omega \xi^{k-2}u^{\gamma-p+1} + M\int_\Omega |\nabla u|^\tau \xi^k u^{-d} + \frac{n-2}{\operatorname{dist}(0,\partial\Omega)}\int_\Omega |\nabla u|\xi^k u^{-d}. \end{aligned}$$

Next we focus on the case of $\gamma > p-1$. Take $k = \frac{2\gamma}{p-1}$. By using the Young inequality again, we have

$$CR^{-2}\int_\Omega \xi^{k-2}u^{\gamma-p+1} \le \frac{1}{4}\int_\Omega \xi^k u^\gamma + CR^{2-2\gamma/(p-1)}$$

and

$$\begin{aligned} M\int_\Omega |\nabla u|^\tau \xi^k u^{-d} &\le \frac{d}{4}\int_\Omega \xi^k u^{\gamma-p-1}|\nabla u|^2 + C\int_\Omega \xi^k u^t \\ &\le \frac{d}{4}\int_\Omega \xi^k u^{\gamma-p-1}|\nabla u|^2 + \frac{1}{4}\int_\Omega \xi^k u^\gamma + CR^{-2}, \end{aligned}$$

the second inequality holds becasue $t = (-d - \tau\frac{\gamma-p-1}{2})\frac{2}{2-\tau} < \gamma$, and

$$\begin{aligned} \frac{n-2}{\operatorname{dist}(0,\partial\Omega)}\int_\Omega |\nabla u|\xi^k u^{-d} &\le \frac{d}{4}\int_\Omega \xi^k u^{\gamma-p-1}|\nabla u|^2 + C\int_\Omega \xi^k u^{\gamma-p+1} \\ &\le \frac{d}{4}\int_\Omega \xi^k u^{\gamma-p-1}|\nabla u|^2 + \frac{1}{4}\int_\Omega \xi^k u^\gamma + CR^{-2}. \end{aligned}$$

So

$$\frac{d}{4}\int_\Omega \xi^k u^{\gamma-p-1}|\nabla u|^2 + \frac{1}{4}\int_\Omega \xi^k u^\gamma \le CR^{2-2\gamma/(p-1)}, \tag{2.5}$$

which gives (2.3).

If $\gamma = p-1$, (2.3) is obvious by the above arguments. For the case of $\gamma < p-1$, the following Höder inequality

$$\int_{B_R} u^\gamma \le CR^{2(1-\gamma)/(p-1)}\Big(\int_{B_R} u^{p-1}\Big)^{\gamma/(p-1)}$$

and the above argument yields to (2.3).

To prove (2.4), we use Höder inequality:

$$\int_{B_R} |\nabla u|^\mu \le \Big(\int_{B_R} u^{\gamma-p-1}|\nabla u|^2\Big)^{\mu/2}\Big(\int_{B_R} u^s\Big)^{1-\frac{\mu}{2}},$$

where $s = (p+1-\gamma)/(2-\mu)$. We can choose γ close enough to $p-1$ such that $s < p$, and then obtain (2.4) by combining (2.3) and (2.5). Thus we complete the proof. □

Lemma 2.3. *Let $u(r, x_n)$ be a nonnegative weak solution of the following inequality, in a domain Ω,*

$$|u_{rr} + u_{x_n x_n}| \le c(x)|\nabla u| + d(x)u + f(x),$$

where $c(x) \in L^{q'}(\Omega)$, $d, f \in L^q(\Omega)$, $q' > 2$ and $q \in (1,2)$. Then for every R such that $B_{2R} \subset \Omega$, there exists a constant $C = C(q, q', R^{1-\frac{2}{q'}}\|c\|_{L^{q'}}, R^{2-\frac{2}{q}}\|d\|_{L^q})$ such that

$$\sup_{B_R} u \le C(\inf_{B_R} u + R^{2-\frac{2}{q}}\|f\|_{L^q}).$$

Note that this lemma is of Harnack type; see [15] for more information on this type of inequalities. The next theorem is similar to [14, Theorem 2.3].

Theorem 2.4. *Let* (D) *hold and $R \le R_0$ such that $B_{2R} \subset \Omega$. Suppose $u(r, x_n)$ is a positive weak solution of the inequality*

$$u^p - M|\nabla u|^\tau + \frac{n-2}{r}u_r \le -(u_{rr} + u_{x_nx_n}) \le c_0u^p + M|\nabla u|^\tau + \frac{n-2}{r}u_r + \lambda,$$

where $p > 1$, $0 < \tau < \frac{2p}{p+1}$, $\lambda > 0$. Then there exists a constant $C = C(p, \tau, R_0, M)$ such that

$$\sup_{B_R} u \le C(\inf_{B_R} u + \lambda R^2).$$

Proof. From (2.4), we obtain

$$|u_{rr} + u_{x_nx_n}| \le c_0u^p + M|\nabla u|^\tau + \frac{n-2}{r}|\nabla u| + \lambda.$$

Take $f = \lambda$, $c = M|\nabla u|^{\tau-1} + \frac{n-2}{r}$ and $d = c_0u^{p-1}$. To prove this theorem, we only need to verify that

$$c(x) \in L^{q'}(B_{2R}), \quad d \in L^q(B_{2R}).$$

Note that $\frac{n-2}{r}$ obviously belongs to $L^{q'}(B_{2R})$, so we only need to prove $M|\nabla u|^{\tau-1} \in L^{q'}(B_{2R})$. By lemma 2.1, we have

$$\|M|\nabla u|^{\tau-1}\|_{L^{q'}} = M\Big(\int_{B_{(2R)}} |\nabla u|^\mu\Big)^{1/q'} \le CR^{\frac{2-(p+1)\mu/(p-1)}{q'}},$$

where $\mu = q'(\tau - 1)$ should satisfy $q'(\tau - 1) < \frac{2p}{p+1}$ for some $q' > 2$. Since $\tau < \frac{2p}{p+1}$ and $q' > 2$ can be close enough to 2, so we just need to verify

$$2\Big(\frac{2p}{p+1} - 1\Big) < \frac{2p}{p+1}.$$

The above inequality is obvious, that is to say, $c(x) \in L^{q'}(B_{2R})$.

For $d = c_0u^{p-1}$, by lemma 2.1 we have

$$\|d\|_{L^q(B_{2R})} = c_0\Big(\int_{B_{(2R)}} u^\gamma\Big)^{1/q} \le CR^{(2-2q)/q},$$

where $\gamma = (p-1)q$ should satisfy $(p-1)q < p$. By choosing $q > 1$ close enough to 1, we can get $(p-1)q < p$, that is, $d \in L^q(B_{2R})$. The proof is complete. □

For completeness, we sketch the proof of Theorem 2.1 which is similar as the proof of [14, Proposition 3.3].

Proof of Theorem 2.1. Suppose, by contradiction, that there exist $\lambda_n < \lambda_0$, $u_n > 0$ such that u_n is solution of (2.1) with λ substituted by λ_n and $\max_\Omega u_n \to \infty$. Let z_n be a point in Ω such that $u_n(z_n) = \max_\Omega u_n \triangleq S_n$. Denote $\delta_n = \operatorname{dist}(z_n, \partial\Omega)$. In order to prove there exists a $y_0 \in \Omega$ such that $u_n(y_0) \to \infty$, we proceed in three steps:

Step 1: There exists $c > 0$ such that $c < \delta_n S_n^{(p-1)/2}$. Define $w(x) = S_n^{-1} u_n(y)$, where $y = M_n x + z_n$, $M_n = S_n^{(1-p)/2}$. By easy computation and condition (F2), we obtain

$$\begin{aligned} -\Delta w_n(x) &= S_n^{-1} M_n^2 (H(M_n x + z_n, S_n w_n(x), S_n M_n^{-1} \nabla w_n(x)) + \lambda_n) \\ &\le c_0 w_n^p + M S_n^{-p} S_n^{\tau \frac{p+1}{2}} |\nabla w_n|^\tau + \frac{n-2}{\operatorname{dist}(0, \partial\Omega)} |\nabla w_n| + \lambda_n S_n^{-p}. \end{aligned}$$

Notice that $M S_n^{-p} S_n^{\tau \frac{p+1}{2}}$ and $\lambda_n S_n^{-p}$ tend to zero respectively as n tends to infinity, so

$$-\Delta w_n(x) \le c_0 w_n^p + |\nabla w_n|^\tau + \frac{n-2}{\operatorname{dist}(0, \partial\Omega)} |\nabla w_n| + 1.$$

By the regularity result in [12], there exists a constant C independent of n such that $\sup_\Omega w_n \le C$. Let $y_n \in \partial\Omega$ such that $d(z_n, y_n) = \delta_n$; then, by the mean value theorem, we have

$$1 = w_n(0) = w_n(0) - w_n(M_n^{-1}(y_n - z_n)) \le \sup_\Omega w_n M_n^{-1} \delta_n \le C M_n^{-1} \delta_n.$$

Thus, the first step is complete.

Step 2: There exists $\gamma > 0$ such that

$$\int_{B(z_n, \delta_n/2)} |u_n|^\gamma \to \infty.$$

By Theorem 2.4, we obtain

$$S_n = \max_{B(z_n, \delta_n/2)} u_n \le C\Big(\min_{B(z_n, \delta_n/2)} u_n + \lambda_n \frac{\delta_n^2}{4} \Big).$$

Since λ_n and δ_n are bounded, we obtain that $\min_{B(z_n, \delta_n/2)} u_n \ge c S_n$ for some $c > 0$. So

$$\int_{B(z_n, \delta_n/2)} |u_n|^\gamma \ge c S_n^\gamma \delta_n^2 \ge c S_n^\gamma S_n^{1-p}.$$

We can choose a $\gamma > p - 1$ such that $c S_n^\gamma S_n^{1-p} \to +\infty$. The proof of step 2 is complete.

Step 3: There exists a $y_0 \in \Omega$ such that $u_n(y_0) \to \infty$. Notice that $\partial\Omega$ is C^2 and compact boundary , so we can find $\varepsilon > 0$ independent of n and $y_n \in \Omega$ such that:

- $d(y_n, \partial\Omega) = 2\varepsilon$, for all $n \in \mathbb{N}$.
- $B(z_n, \frac{\delta_n}{2}) \subset B(y_n, 2\varepsilon)$, for all $n \in \mathbb{N}$.

By the weak Harnack inequality in [16] and step 2, we conclude that

$$\min_{B(y_n, \varepsilon)} u_n \ge c\Big(\int_{B(y_n, 2\varepsilon)} |u_n|^\gamma \Big)^{1/\gamma} \to +\infty.$$

Taking a subsequence $\{\tilde{y}_n\} \subset \{y_n\}$ such that $\tilde{y}_n \to y_0 \in \Omega$. For n large enough, we have $y_0 \in B(\tilde{y}_n, \varepsilon)$ and $u_n(y_0) \to \infty$, which contradicts with Theorem 2.4. Thus we obtain a priori estimate of solutions. □

3. Existence of positive classical $C^{2,\beta}$-solutions

Theorem 3.1. *Assume* (D1), (F1)–(F3) *hold. Then* (1.1) *admits an* $(n-1)$*-radial symmetric positive classical solution* $u(r,x_n) \in C^{2,\beta}(\Omega) \cap C^0(\overline{\Omega})$.

The following lemma mentioned in [10] will be used in our proof.

Lemma 3.2 ([10, Theorem 12.4])**.** *Let* u *be a bounded* $C^2(\Omega)$ *solution of*

$$Lu = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = f(x,y),$$

where L *is uniformly elliptic in a domain* $\Omega \subset \mathbb{R}^2$*, satisfying*

$$\lambda(\xi^2+\eta^2) \le a\xi^2 + 2b\xi\eta + c\eta^2 \le \Lambda(\xi^2+\eta^2), \quad \forall (\xi,\eta) \in \mathbb{R}^2,$$
$$\frac{\Lambda}{\lambda} \le \gamma$$

for some constant $\gamma \ge 1$*. Then for some* $\alpha = \alpha(\gamma) > 0$*, we have*

$$[u]^*_{1,\alpha} = \sup_{z_1,z_2\in\Omega} d_{1,2}^{1+\alpha} \frac{|Du(z_2) - Du(z_1)|}{|z_2 - z_1|^\alpha} \le C(|u|_0 + |\frac{f}{\lambda}|_0^{(2)}),$$

where $C = C(\gamma)$, $|\frac{f}{\lambda}|_0^{(2)} = \sup_{z\in\Omega} d_z^2 |\frac{f}{\lambda}|$, $d_z = \operatorname{dist}(z,\partial\Omega)$ *and* $d_{1,2} = min\{d_{z_1}, d_{z_2}\}$.

Since the conditions imposed on f in Theorem 3.1 are different from those in [10, Theorem 12.5], it is necessary to give the proof, although similar to that of [10, Theorem 12.5].

Proof of Theorem 3.1. We now proceed by truncation of H to reduce (1.4) to the case of bounded H. Namely, let ψ_N denote the function given by

$$\psi_N(t) = \begin{cases} t, & |t| \le N \\ N \operatorname{sign} t, & |t| > N, \end{cases}$$

and define the truncation of H by

$$H_N(r,x_n,u,u_r,u_{x_n}) = H(r,x_n,\psi_N(u),\psi_N(u_r),\psi_N(u_{x_n})).$$

From (F2), we have $|H_N| \le c_0N^p + MN^\tau + \frac{n-2}{\operatorname{dist}(0,\partial\Omega)}N = C_0$. Consider now the family of problems

$$\begin{gathered} -(u_{rr} + u_{x_nx_n}) = H_N(r,x_n,u,u_r,u_{x_n}) \quad \text{in } \Omega, \\ u(x) = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.1}$$

By Theorem 2.1, any solution u of (3.1) is subject to the bound $\tilde{M}$, independent of N,

$$\sup_\Omega |u| \le \tilde{M}. \tag{3.2}$$

Now we make the following observation. Let v be any bounded function with locally Hölder continuous first derivatives in Ω and $\tilde{H_N} = H_N(r,x_n,v,v_r,v_{x_n})$. Then the following linear problem

$$\begin{gathered} -(u_{rr} + u_{x_nx_n}) = \tilde{H_N} \quad \text{in } \Omega, \\ u(x) = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.3}$$

has a unique solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. We observe from classical priori estimates that

$$|u|_0 = \sup_\Omega |u| \le M_0.$$

Furthermore, if $\sup_\Omega |v| \leq M_0$, from lemma 3.1, we have

$$|u|^*_{1,\alpha} \leq C(|u|_0 + C_0(\mathrm{diam}(\Omega))^2) \leq C(M_0 + C_0(\mathrm{diam}(\Omega))^2) = K,$$

where C, α depend on M_0. So K depends on M_0, N and Ω.

Next, define the Banach space

$$C^{1,\alpha}_*(\Omega) = \{u \in C^{1,\alpha}(\Omega) | |u|^*_{1,\alpha;\Omega} < +\infty\}$$

and define a mapping T on the set

$$\mathbb{S} = \{v \in C^{1,\alpha}_* : |v|^*_{1,\alpha} \leq K, |v|_0 \leq M_0\}.$$

So $u = Tv$ is the unique solution of the linear Dirichlet problem (3.3). It is easy to show that $\mathbb{S}$ is convex and closed in the Banach space, and T is continuous in $C^1_* = \{u \in C^1(\Omega) | |u|^*_{1;\Omega} < +\infty$ and $T\mathbb{S}$ is precompact. So we may conclude from the Schauder fixed point theorem and Schauder estimates that T has a fixed point, $u_N = Tu_N$, $u_N \in C^{1,\alpha}_*(\Omega) \cap C^{2,\beta}(\Omega) \cap C^o(\bar{\Omega})$. This will provide a solution of the problem (3.1).

Furthermore, from lemma 3.1 we infer the estimate

$$[u_N]^*_{1,\alpha} \leq C(|u|_0 + |G_{HN}|^{(2)}_0).$$

By (F2) and (3.2), we obtain

$$[u_N]^*_{1,\alpha} \leq C(1 + [u_N]^*_1),$$

where $C = C(\tilde{M}, M, c_0, p, \tau, \mathrm{diam}(\Omega))$. Furthermore, the interpolation inequality yields the uniform bound which is independent of N,

$$[u_N]^*_{1,\alpha} \leq C = C(\tilde{M}, M, c_0, p, \tau, \mathrm{diam}(\Omega)).$$

By similar arguments as in the proof of [10, Theorem 12.5], it is easy to show there is a subsequence $\{u_n\}$ of $\{u_N\}$ which converges to a solution u of (1.4), and u also satisfies the boundary condition $u = 0$ on $\partial\Omega$. Since f is nonnegative, by comparison principles, u is positive. This completes the proof. □

Remark 3.3. If $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, Ω_1 and Ω_2 are symmetric and $0 \notin \overline{\Omega}$, $f(x, u, |\nabla u|) = f(r_1, r_2, u, |\nabla u|)$, where $r_1 = \sqrt{x_1^2 + x_2^2 + \cdots + x_k^2}$, $r_2 = \sqrt{x_{k+1}^2 + x_{k+2}^2 + \cdots + x_n^2}$. Under the conditions of (F2) and (F3), (1.1) admits an $(n-1)$-radial symmetric positive classical solution $u(r_1, r_2) \in C^{2,\beta}(\Omega) \cap C^0(\overline{\Omega})$. The proof is left to readers.

Acknowledgments. This work is partly supported by the National Natural Science Foundation of China (10971088) and Natural Science Foundation of Chizhou College (2013ZRZ002).

References

[1] B. Abdellaoui, A. Dall Aglio, I. Peral; *Some remarks on elliptic problems with critical growth in the gradient*, J. Differential Equations 222 (2006), 21–62.

[2] Claudianor O. Alves, Paulo C. Carriao, Luiz F. O. Faria; *Existence of solutions to singular elliptic equations with convection terms via the Galerkin method*, Electronic Journal of Differential Equations Vol. 2010 (2010), No. 12, 1–12.

[3] H. Amann, M. G. Crandall; *On some existence theorems for semilinear elliptic equations*, Indiana Univ. Math. J 27 (1978), 779–790.

[4] Giovanni Molica Bisci, Vicentiu Rădulescu; *Multiple symmetric solutions for a Neumann problem with lack of compactness*, C. R. Acad. Sci. Paris, Ser. I 351 (2013) 37–42.

[5] Wenjing Chen, Jianfu Yang; *Existence of positive solutions for quasilinear elliptic equation on Riemannian manifolds*, Differential Equations and Applications Vol 2 (2010), 569–574.
[6] D. G. de Figueiredo, M. Girardi, M. Matzeu; *Semilinear ellptic equations with dependence on the gradient via mountain-pass techniques*, Differential and Integral Equations 17 (2004), 119–126.
[7] D. G. de Figueiredo, J. Sánchez, P. Ubilla; *Quasilinear equations with dependence on the gradient*, Nonlinear Analysis 71 (2009), 4862–4868.
[8] M. Ghergu, V. Rădulescu; *Bifurcation for a class of singular elliptic problems with quadratic convection term*, C. R. Acad. Sci. Paris, Ser. I 338 (2004), 831–836.
[9] M. Ghergu, V.Rădulescu; *On a class of sublinear singular elliptic problems with convection term*, J. Math. Anal. Appl. 311 (2005) 635–646.
[10] D.Gilbarg, N. S. Trudinger; *Elliptic Partial Differential Equations of Second Oder*, second ed. Springer-Verlag, Berlin, 1983.
[11] M. Girardi, M. Matzeu; *Positive and negative solutions of a quasilinear elliptic equation by a Mountain Pass method and truncature techniques*, Nonlinear Analysis T.M.A. 59 (2004), 199–210.
[12] G. M. Lieberman; *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. 12 (1988) 1203–1219.
[13] Pohozaev S; *On equations of the type* $\Delta u = f(x, u, Du)$, Mat. Sb. 113 (1980), 324–338.
[14] D. Ruiz; *A priori estimates and existence of positive solutions for strongly nonlinear problems*, J. Differential Equations 199 (2004), 96–114.
[15] J. Serrin, H. Zou; *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. 189 (2002) 79–142.
[16] N. Trudinger; *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. 20 (1967) 721–747.
[17] X. Wang, Y. Deng; *Existence of multiple solutions to nonlinear elliptic equations in nondivergence form*, J. Math. Anal. and Appl. 189 (1995), 617–630.
[18] J. B. M. Xavier; *Some existence theorems for equations of the form* $-\Delta u = f(x, u, Du)$, Nonlinear Analysis T.M.A. 15 (1990), 59–67.
[19] Z. Yan; *A note on the solvability in* $W^{2,p}(\Omega)$ *for the equation* $-\Delta u = f(x, u, Du)$, Nonlinear Analysis T.M.A. 24 (1995), 1413–1416.
[20] Henghui Zou; *A priori estimates and existence for quasilinear elliptic equations* Calc. Var. Partial Differential Equations 33 (2008), no. 4, 417–437.

Yong Zhang
Department of Mathematics, Lanzhou University, Lanzhou, Gansu, 730000, China
Department of Mathematics, Chizhou College, Chizhou, Anhui, 247000, China
E-mail address: `zhangy12@lzu.edu.cn`

Qiang Xu
Department of Mathematics, Lanzhou University, Lanzhou, Gansu, 730000, China
E-mail address: `xuqiang09@lzu.edu.cn`

Peihao Zhao
Department of Mathematics, Lanzhou University, Lanzhou, Gansu, 730000, China
E-mail address: `zhaoph@lzu.edu.cn`