Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 201, pp. 1–30.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

GLOBAL SOLUTIONS OF A MODEL OF PHASE TRANSITIONS FOR DISSIPATIVE THERMOVISCOELASTIC MATERIALS

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Abstract. We analyze a highly nonlinear system of partial differential equations that may be seen as a model for solidification or melting of certain viscoelastic materials subject to thermal effects; under the assumption that solid parts of the material may support damped vibrations. Phase change is controlled by a phase field equation with a potential including barriers at the pure solid and pure liquid states.

The present system is closely related to a model analyzed by Rocca and Rossi [23]. They proved the existence of local in time solutions (global in the one dimensional case) assuming values just in the mushy zone, and thus such local solutions do not allow regions of pure solid or pure liquid states, except in the special one-dimensional case where pure liquid state is also allowed.

By including a suitable dissipation in the previous model and assuming constant latent heat, in this work we are able to prove global in time existence even for solutions that may touch the potential barriers; that is, they allow regions with pure solid or pure liquid.

1. Introduction

In this article we consider a class of systems including as a particular case the following nonlinear system of partial differential equations:

$$\theta_t + l\chi_t - \Delta\theta = g \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$\chi_t - \Delta\chi + W'(\chi) \ni h(\theta - \theta_c) + \frac{|\eta(u)|^2}{2} \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$u_{tt} - \operatorname{div}\big((1-\chi)\eta(u) + \chi\eta(u_t)\big) + \nu(-\Delta)^2 u_t = f \quad \text{in } \Omega \times (0,T), \tag{1.3}$$

subjected to the boundary conditions

$$u = \Delta u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.4}$$

$$\partial_n \chi = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.5}$$

$$\partial_n \theta = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.6}$$

and initial conditions

$$\theta(0) = \theta_0 \quad \text{in } \Omega, \tag{1.7}$$

2000 *Mathematics Subject Classification.* 76A10, 35A01, 35B45, 35B50, 35M33, 80A22.
Key words and phrases. Nonlinear PDE system; degenerate PDE system; global solutions; uniqueness; phase transitions; thermoviscoelastic materials.

Submitted February 8, 2013. Published September 11, 2013.

$$\chi(0) = \chi_0 \quad \text{in } \Omega, \tag{1.8}$$
$$u(0) = u_0, \quad u_t(0) = v_0 \quad \text{in } \Omega, \tag{1.9}$$

which is a variant of the system treated in the work by Rocca and Rossi [21]; the differences are that in (1.1) we have the simpler term $l\chi_t$ instead of $\theta\chi_t$ as in Rocca and Rossi [21] and in (1.3) we have the extra term $\nu(-\Delta)^2 u_t$.

We remark that the previous system may be taught as a model for phase transition processes occurring in a viscoelastic material occupying a bounded domain $\Omega \subseteq \mathbb{R}^n$, $n = 1, 2, 3$, subject to thermal effects during a time interval $[0, T]$. In the last section, we will consider modeling aspects of the problem and, following the arguments in [21] and [13], show how these equations are be obtained.

The state variables are the absolute temperature θ, (θ_c being a given constant equilibrium temperature), an order parameter χ, which is the phase field that in the present model stands for the local proportion of the liquid phase in the material, and u, which is the vector of the small displacements.

In the previous system, equation (1.1) is the internal energy balance equation; g is a known heat source and $l > 0$ is the latent heat, which is assumed to be a given positive constant.

Equation (1.3), ruling the evolution of the displacement u, is the balance equation for macroscopic movements (also known as *stress-strain relation*). The expression $\eta(u)$ denotes the linearized symmetric strain tensor, which in the (spatially) three-dimensional case is given by $\eta_{ij}(u) := (u_{i,x_j} + u_{j,x_i})/2$, $i, j = 1, 2, 3$ (with the commas we denote space derivatives); the symbol div stands both for the scalar and for the vectorial divergence operator. Further, the term $(-\Delta)^2$ denotes the biharmonic operator, and f on the right-hand side may be interpreted as an exterior volume force applied to the body.

Observe that in the pure solid phase, corresponding to $\chi = 0$, equations (1.3) simplify to a system for elasticity with dissipation; in the pure liquid phase, corresponding to $\chi = 1$,equations (1.3) simplifies to a parabolic system with dissipation for the velocity u_t; in this last case, there is no incompressibility requirement and thus no pressure term. We remark that we are presently also analyzing models that require such incompressibility conditions.

Following Frémond's perspective, see [13], (1.1) and (1.3) are coupled with equation (1.2) for the microscopic movements for the phase variable χ. In (1.2), $|\eta(u)|^2$ is a short-hand for the colon product $\eta(u) : \eta(u)$; $h(\cdot)$ is a given suitable function, and we assume that the potential W is given by the sum of a smooth nonconvex function $\widehat{\gamma}$ and of a convex function $\widehat{\beta}$, with domain contained in $[0, 1]$ and differentiable in $(0, 1)$. Typical examples of functionals which we can include in our analysis are the logarithmic potential

$$W(r) := r\ln(r) + (1-r)\ln(1-r) - c_1 r^2 - c_2 r - c_3 \quad \forall r \in (0, 1), \tag{1.10}$$

where c_1 and c_2 are positive constants, as well as the double obstacle potential, given by the sum of the indicator function $I_{[0,1]}$ with a nonconvex $\widehat{\gamma}$. Note that in this way the values outside $[0, 1]$ (which indeed are not physically meaningful for the present order parameter χ, which is the liquid phase proportion) are excluded. The real valued function $h(\cdot)$ is a given; in several models $h(z) \equiv z$.

Before describing our results, let us briefly recall and comment some earlier works closely related to ours.

Material models taking into account microscopic movements as proposed by Frémond have been studied in several articles; for instance, for materials with viscoelastic properties, but not subject to phase change, we can mention the articles by Bonetti and Bonfanti [3, 4], which considered a linear viscoelasticity equation for the displacement u and a internal energy balance equation for the temperature θ. By using similar modeling ideas, the articles [5, 6, 7, 16] consider models for damaging phenomena by using a variable similar to as our χ and related to local proportion of damaged material; in Kuttler [16], a evolution model of quasiestatic reversible damage in visco-plastic materials is considered, while in Bonetti and Bonfanti [5] and Bonetti and Schimperna [6, 7] irreversible damage process were considered.

Models including phase change and also following Frémond point of view were analyzed in an article by Bonfanti el al [8] and in Stefanelli [25] (see also the references therein). We also mention the article by Rocca-Rossi [22], where they analyzed the one-dimensional case of a model including the full equation for the internal energy, that is, $\theta_t+\theta\chi_t-\Delta\theta = |\chi_t|^2+\chi|\eta(u_t)|^2+g$, and the other equations as in the present article, but with the parameter $\nu = 0$.

We stress that in the more nonlinear setting of [21], Rocca and Rossi were able to prove local in time existence of solutions (global in time for dimension one), but with restrictive conditions on the initial data for the phase parameter. In fact, the initial value χ_0 of the phase parameter is required to be separated from the potential barriers, i.e.,

$$0 < \min_{x\in\overline{\Omega}} \chi_0(x) \le \max_{x\in\overline{\Omega}} \chi_0(x) < 1,$$

and for the obtained local solutions the same property holds; thus, all the process occur in the mushy zone and strict phase transitions do not happen, which means that (1.2) hold as an equality. Global results were obtained just in the one dimensional case.

In this work, we are interested in proving the existence of global in time solutions for (1.1) - (1.6) with initial data χ_0 such that

$$0 \le \min_{x\in\overline{\Omega}} \chi_0(x) \le \max_{x\in\overline{\Omega}} \chi_0(x) \le 1,$$

and the same for the obtained solutions, allowing in this way the possibility of touching the potential barriers and thus pure solid and pure liquid regions.

To prove such result, we will introduce approximate problems corresponding to regularized versions of the original problem and depending on two strictly positive parameters; then we will prove the existence of solutions for such approximate problems by using Leray-Schauder fixed point theorem. After that, by deriving estimates that are uniform with respect of such parameters and taking the limits in a suitable order, we will obtain solutions for the original model as limits of the approximate solutions.

The main difficulty in applying Leray-Schauder fixed point theorem will be the handling of the term $|\eta(u)|^2/2$ in (1.2) and also the nonlinearities related to χ and u in (1.3). To overcome these difficulties, the approximate problems are constructed by using truncation operators

This work is organized as follow. Section 2 is dedicated to introduce some notation, to rewrite the initial boundary value problem related to equations (1.1)-(1.3) in a suitable formulation and to state the main result of this paper. In Section

3, we introduce a suitable approximate problem and in Section 4 and 5, we prove the existence of approximate solutions. In Section 6 we prove our main result. Finally, in Section 7, where we present some considerations on modeling aspects of the problem.

2. Preliminaries and statement of main results

In this section, we fix the notation, recall certain facts, and present a suitable operational formulation of Problem (1.1)-(1.9).

2.1. **Notation.** We suppose that $\Omega \subset \mathbb{R}^n$ is a bounded connected domain, with C^4-boundary $\partial\Omega$, and consider the following Sobolev spaces

$$H_0^1(\Omega) := \{v \in H^1(\Omega); v = 0 \text{ on } \partial\Omega\}, \quad H_0^2(\Omega) := \{v \in H^2(\Omega); v = 0 \text{ on } \partial\Omega\},$$
$$H_N^2(\Omega) := \{v \in H^2(\Omega); \partial_n v = 0 \text{ on } \partial\Omega\},$$

endowed with the norms of $H^1(\Omega)$ and $H^2(\Omega)$, respectively. Furthermore, we identify $L^2(\Omega)$ with its dual space $L^2(\Omega)'$, so that $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^1(\Omega)'$ with dense and continuous embeddings.

We will also use the following continuous Sobolev embeddings:

$$H^\alpha(\Omega) \subset W^{\beta,p}(\Omega) \quad \text{for } \alpha - \frac{n}{2} \geq \beta - \frac{n}{p}, \ \ \alpha, \beta \in \mathbb{R}, \ p \geq 1; \tag{2.1}$$

this inclusion is compact when the inequality is strict. In particular,

$$H^\alpha(\Omega) \subset H^{\alpha-\epsilon}(\Omega), \quad \text{compactly for } \epsilon > 0 \text{ and } \alpha \in \mathbb{R}. \tag{2.2}$$

The following interpolation result will be important for the derivation of certain estimates; it can be found for example in Brézis-Mironescu [10] in a more general formulation:

$$\|v\|_{H^2(\Omega)} \leq C\|v\|_{H^{2\alpha}(\Omega)}^{1/\alpha}\|v\|_{L^2(\Omega)}^{1-1/\alpha}, \quad \text{for all } \alpha > 1.$$

We denote by $A := -\Delta : D(A) = (H_0^1(\Omega) \cap H^2(\Omega))^n \subset (L^2(\Omega))^n \to (L^2(\Omega))^n$ the Laplacian operator, acting on each coordinate, with homogeneous boundary conditions.

For $\alpha \geq 0$, we consider the following Banach spaces given by the domain of the fractional powers of A: $D(A^{\alpha/2})$ endowed with norm

$$\|u\|_{H^\alpha(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \|A^{\alpha/2}u\|_{L^2(\mathbb{R}^n)}.$$

It is known that $D(A^{\alpha/2})$ is closed in $H^\alpha(\Omega)$ with the norm of $H^\alpha(\Omega)$, and that $D(A^{\alpha/2}) \subset H^\alpha(\Omega)$ with continuous injection.

Further, we introduce the operator $A_N : H^1(\Omega) \to H^1(\Omega)'$ realizing the Laplace operator $-\Delta$ with homogeneous Neumann boundary conditions, defined by

$$\langle A_N u, v\rangle := (\nabla u, \nabla v) \quad \forall u, v \in H^1(\Omega),$$

We denote by J the duality operator $A_N + I : H^1(\Omega) \to H^1(\Omega)'$ (I being the identity operator); in the sequel, we will make use of the relations

$$\langle Ju, u\rangle = \|u\|_{H^1(\Omega)}^2 \quad \forall u \in H^1(\Omega), \quad \langle J^{-1}v, v\rangle = \|v\|_{H^1(\Omega)'}^2 \quad \forall v \in H^1(\Omega)'.$$

We also require the operator $A^2 : H^2(\Omega) \to H^2(\Omega)'$ realizing the biharmonic operator $(-\Delta)^2$ with the Navier boundary conditions (i. e. $u = \Delta u = 0$), defined by

$$\langle A^2 u, v\rangle := (\Delta u, \Delta v) \quad \forall u, v \in H^2(\Omega).$$

2.2. **A family of viscoelastic problems, an operational formulation and an existence result.** Exactly as in Rocca and Rossi [21], we will state an operational formulation associated to a family of viscoelastic problems including as a particular case problem (1.1)-(1.6). For this, we need to introduce some notation and properties.

To generalize the elastic part of the problem, let $\phi : \Omega \to [0,1]$ be a bounded measurable function and let us consider the following continuous bilinear symmetric forms $a_\phi, b_\phi : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\begin{gathered} a_\phi(u,v) := \alpha_1 \int_\Omega \phi \operatorname{div}(u) \operatorname{div}(v) + 2\alpha_2 \sum_{i,j=1}^{3} \int_\Omega \phi \eta_{ij}(u)\eta_{ij}(v) \quad \forall u,v \in H_0^1(\Omega), \\ b_\phi(u,v) := \sum_{i,j=1}^{3} \int_\Omega \phi b_{ij}\eta_{ij}(u)\eta_{ij}(v) \quad \forall u,v \in H_0^1(\Omega). \end{gathered} \tag{2.3}$$

Here, the positive Lamé constants α_1, α_2 are related to the elastic properties of the material. Matrix (b_{ij}) is positive definite and called viscosity matrix; it is also related to the properties of the material being considered, cf. Rocca and Rossi [21]. We remark that for the problem stated in the Introduction, we have $\alpha_1 = 0$, $\alpha_2 = 1$, $b_{ii} = 1$ and $b_{ij} = 0$ for $i \neq j$ $i,j = 1,2,3$.

For a bounded ϕ, there exists some positive constant K_a such that

$$|a_\phi(u,v)| \le K_a \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall u,v \in H_0^1(\Omega). \tag{2.4}$$

Furthermore, by Korn's inequality (see e.g. Ciarlet [11, Theorem 6.3-3]), when $\inf_{x\in\Omega}(\phi(x)) > 0$ the forms $a_\phi(\cdot,\cdot)$ and $b_\phi(\cdot,\cdot)$ are $H_0^1(\Omega)$-elliptic; i.e., there exist $C_a, C_b > 0$ such that for all $u \in H_0^1(\Omega)$ there hold

$$a_\phi(u,u) \ge \inf_{x\in\Omega}(\phi(x)) C_a \|u\|^2_{H^1(\Omega)}, \tag{2.5}$$

$$b_\phi(u,u) \ge \inf_{x\in\Omega}(\phi(x)) C_b \|u\|^2_{H^1(\Omega)}. \tag{2.6}$$

We will also need the following elliptic regularity result (see e.g. Nečas [20] p. 260): there exist constants $C_\gamma, C_\delta > 0$ such that

$$C_\gamma \|v\|_{H^2(\Omega)} \le \|\operatorname{div}(\eta(v))\|_{L^2(\Omega)} \le C_\delta \|v\|_{H^2(\Omega)} \quad \forall v \in H_0^2(\Omega). \tag{2.7}$$

We denote by $\mathcal{H}(\eta\,\cdot) : H_0^1(\Omega) \to H^{-1}(\Omega)$ and $\mathcal{K}(\eta\,\cdot) : H_0^1(\Omega) \to H^{-1}(\Omega)$ the operators associated with a_η and b_η, respectively, namely

$$\langle \mathcal{H}(\eta v), w\rangle = a_\eta(v,w), \quad \langle \mathcal{K}(\eta v), w\rangle = b_\eta(v,w) \quad \forall v,w \in H_0^1(\Omega).$$

It can be checked via an approximation argument that the following regularity result holds:

$$\text{if } \eta \in H^2(\Omega) \text{ and } v \in H_0^2(\Omega), \text{ then } \mathcal{H}(\eta v), \mathcal{K}(\eta v) \in L^2(\Omega). \tag{2.8}$$

As for the potential W in (1.2), we assume that it is given by

$$W = \widehat{\beta} + \widehat{\gamma}, \tag{2.9}$$

where $\widehat{\gamma}$ is a regular function:

$$\widehat{\gamma} \in C^2([0,1]), \tag{2.10}$$

and $\widehat{\beta}$ satisfies

$$\widehat{\beta} : [0,1] \to [0,+\infty] \text{ is proper, l.s.c., convex,} \tag{2.11}$$

$$\widehat{\beta}|_{(0,1)} \in C^{1,1}_{\rm loc}(0,1). \tag{2.12}$$

Remark 2.1. As it is well known, condition (2.11) implies the existence of a positive constant $M \le +\infty$ such that $-M \le \widehat{\beta}(x)$ for any $x \in [0,1]$

We recall that both the logarithmic function $\widehat{\beta}(r) = r\ln(r) + (1-r)\ln(1-r)$, for $r \in (0,1)$ (cf. (1.10)), and the indicator function $\widehat{\beta} = I_{[0,1]}$ of the interval $[0,1]$ fulfil (2.11)-(2.12).

Hereafter, for the sake of simplicity of notation, we will denote the following subdifferentials as

$$\partial W = W', \quad \partial\widehat{\beta} = \beta, \quad \widehat{\gamma}' = \gamma,$$

so that (2.9) yields $W' = \beta + \gamma$.

By composition, the graph β induces a maximal monotone operator $\beta_{\rm ext} : {\rm dom}(\beta_{\rm ext}) \subset L^2(\Omega) \to L^2(\Omega)$, which is defined by the following: for each $g \in L^2(\Omega)$, $\beta_{\rm ext}(g) = \{z \in L^2(\Omega) : z(x) \in \beta(g(x))$ for *a.e.* $x \in \Omega\}$, with the domain ${\rm dom}(\beta_{\rm ext}) = \{g \in L^2(\Omega) : \beta_{\rm ext}(g) \ne \emptyset\}$. Analogously, again by composition, the graph β also induces a maximal monotone operator $\beta_{\rm ext1} : {\rm dom}(\beta_{\rm ext1}) \subset L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$, with similar definition and domain.

Consider for example the case of the logarithmic potential W described in (1.10); then $\widehat{\gamma} \equiv 0$ and $W = \widehat{\beta}$; then we have ${\rm dom}(\beta) = [0,1]$ and more explicitly:

$$\beta(r) = \partial\widehat{\beta}(r) = \begin{cases} \emptyset & \text{for } r < 0, \\ (-\infty, 0] & \text{for } r = 0, \\ \widehat{\beta}'(r) & \text{for } r \in (0,1), \\ [0, +\infty) & \text{for } r = 1, \\ \emptyset & \text{for } r > 1. \end{cases}$$

This means that the first requirement in order to a function $\chi_0 \in L^2(\Omega)$ belong to ${\rm dom}(\beta_{\rm ext})$ is that $0 \le \chi_0 \le 1$ *a.e.* in Ω. If this is the case, by considering the subsets of Ω defined by $\Omega_{[\chi_0=0]}$, $\Omega_{[0<\chi_0<1]}$ and $\Omega_{[\chi_0=1]}$, which are defined up to zero measure subsets, for $\chi_0 \in {\rm dom}(\beta_{\rm ext})$ we also must require $\beta_{\rm ext}(\chi_0) \ne \emptyset$. But with the previous notations, we have

$$\begin{aligned} \beta_{\rm ext}(\chi_0) = \big\{ & z \in L^2(\Omega) : z \le 0 \text{ in } \Omega_{[\chi_0=0]},\ z = \widehat{\beta}'(\chi_0) \text{ in } \Omega_{[0<\chi_0<1]}, \\ & z \ge 0 \text{ in } \Omega_{[\chi_0=1]} \big\}. \end{aligned}$$

Since $\Omega_{[\chi_0=0]}$ and $\Omega_{[\chi_0=1]}$ have finite measures, and thus we can take constant values with the proper sign for z in those subsets, the only requirement left for $\beta_{\rm ext}(g) \ne \emptyset$ is that $\int_{\Omega_{[0<\chi_)<1]}} |\widehat{\beta}'(\chi_0)|^2 < \infty$, which imposes growth conditions on χ_0 as we approach the boundary of $\Omega_{[0<\chi_0<1]}$. In other words, $\chi_0 \in {\rm dom}(\beta_{\rm ext})$ can assume values 0 and 1 in nontrivial regions, which correspond respectively to pure solid or pure liquid regions. Analougous considerations can be done for $\beta_{\rm ext1}$.

Important notation. For the rest of this article, as it is standard in the monotone operators theory, we will suppress the subscripts in the symbols of those induced operators and write simply β instead of $\beta_{\rm ext}$ or $\beta_{\rm ext1}$; the context will distinguish their usage.

In addition to the previous hypotheses, we assume that function h satisfies the following conditions

$$h \in C^1,\ h(0) = 0 \text{ and } h' \text{is bounded}. \tag{2.13}$$

When $h(\theta) = \theta$, equation (1.2) is exactly the same as the one considered by Rocca and Rossi [21].

By using the previous notation, system (1.1)-(1.3) can be written in abstract form as

$$\begin{gathered} \theta_t + l\chi_t + A_N\theta = g, \\ \chi_t + A_N\chi + \xi + \gamma(\chi) = h(\theta) + \frac{|\eta(u)|^2}{2}, \\ u_{tt} + \mathcal{H}((1-\chi)u) + \mathcal{K}(\chi u_t) + \nu A^2 u_t = f, \end{gathered}$$

for some $\xi \in \beta(\chi)$, and in the special case where the Lamé constants are $\alpha_1 = 0$, $\alpha_2 = 1$ and the elasticity matrix given by $b_{ii} = 1$ and $b_{ij} = 0$ for $i \neq j$ and $i, j = 1, 2, 3$.

This formulation tell us that system (1.1)-(1.3) can be considered as a special case of a even more general context. In fact, not only the elastic part can be generalized, by taking different elastic matrix and Lamé constants, but also the operators A and A_N could be any uniformly elliptic second order linear operators with sufficiently smooth coefficients independent of time; moreover, in the dissipation term in the third equation, one could consider other fractional powers of A instead of A^2.

However, some of these generalizations will not substantially change the mathematical arguments and, for simplicity of exposition, we will consider only the case where A has fractional powers in the abstract formulation corresponding to the problem described in the introduction.

In the sequel, we shall assume the following regularity assumptions on the problem data:

$$g \in H^1(0,T;L^2(\Omega)), \tag{2.14}$$

$$f \in L^2(0,T;L^2(\Omega)), \tag{2.15}$$

$$\theta_0 \in H^2_N(\Omega), \tag{2.16}$$

$$\chi_0 \in H^2_N(\Omega), \tag{2.17}$$

$$u_0 \in D(A^\alpha), \quad v_0 \in D(A^{\alpha/2}), \tag{2.18}$$

Differently from Rocca and Rossi [21], we assume that the initial datum χ_0 may touch the potential barriers; i.e.,

$$\begin{gathered} \chi_0 \in \operatorname{dom}(\beta), \\ 0 \leq \min_{x\in\overline{\Omega}} \chi_0(x) \leq \max_{x\in\overline{\Omega}} \chi_0(x) \leq 1. \end{gathered} \tag{2.19}$$

In this way, we are interested in solving the following problem:

Problem 2.2. Find functions $\theta, \chi, \xi : \Omega \times [0,T] \to \mathbb{R}$ and $u : \Omega \times [0,T] \to \mathbb{R}^3$ satisfying the initial conditions (1.7)-(1.9), $\chi \in \operatorname{dom}(\beta)$, $\xi \in \beta(\chi)$, and the equations

$$\theta_t + l\chi_t + A_N\theta = g \tag{2.20}$$

$$\chi_t + A_N\chi + \xi + \gamma(\chi) = h(\theta) + \frac{|\eta(u)|^2}{2} \tag{2.21}$$

$$u_{tt} + \mathcal{H}((1-\chi)u) + \mathcal{K}(\chi u_t) + \nu A^\alpha u_t = f. \tag{2.22}$$

For this problem, we can prove the following theorem, which is our main result.

Theorem 2.3. *Assume (2.14)-(2.19) hold. Then there exist a unique solution (θ, χ, ξ, u) for Problem 2.2 with the following regularity:*

$$\begin{gathered}\theta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\ \chi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\ \xi \in L^2(0,T;L^2(\Omega)), \quad \xi \in \beta(\chi),\\ u \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;D(A^{\alpha/2})).\end{gathered}$$

where $\alpha > 7/4$ if $n = 3$, $\alpha > 3/2$ if $n = 2$, and $\alpha \geq 1$ if $n = 1$.

3. Approximate problems and ideas for the proof the main theorem

In this section we will explain the two approximate problems that will have to be considered prior to the proof of our main theorem. A first approximate problem depend on two parameters, while the second one depends just one of these parameters. We will start by proving the existence of solution for the first problem, and then, by letting such parameters go to zero in a proper order, we will get the existence of subsequences of such solutions converging to the solution of the second and then to a solution of the original problem.

The first approximate problem is obtained as a regularized and truncated version of equations (2.20)-(2.22), depending on two parameters. For this, we first define two truncation operators. Given $\epsilon > 0$, we introduce $T_{1/\epsilon}$ defined as

$$T_{1/\epsilon}(s) = \begin{cases} s & \text{if } s \in [-\frac{1}{\epsilon}, \frac{1}{\epsilon}] \\ \frac{1}{\epsilon}\operatorname{sign}(s) & \text{otherwise.} \end{cases}$$

We will also need the truncation operator

$$\tau(s) = \begin{cases} 0 & \text{if } s < 0 \\ s & \text{if } s \in [0,1] \\ 1 & \text{if } s > 1. \end{cases}$$

Also, given $\mu > 0$, we consider the corresponding Yosida approximation of the maximal monotone operator β, which we denote β_μ.

Then, we consider the following regularized and truncated version of Problem 2.2:

Problem 3.1. Fix a small $\epsilon > 0$ and consider any $\mu > 0$. Find functions $\theta^\mu, \chi^\mu, \xi^\mu : \Omega \times [0,T] \to \mathbb{R}$ and $u^\mu : \Omega \times [0,T] \to \mathbb{R}^3$ satisfying

$$\begin{gathered}\theta^\mu_t + l\chi^\mu_t + A_N\theta^\mu = g\\ \chi^\mu_t + A_N\chi^\mu + \beta_\mu(\chi^\mu) + \gamma(\chi^\mu) = h(\theta^\mu) + T_{1/\epsilon}(\frac{|\eta(u^\mu)|^2}{2})\\ u^\mu_{tt} - \mathcal{H}(\tau(1-\chi^\mu)u^\mu) + \mathcal{K}(\tau(\chi^\mu)u^\mu_t) + \nu A^\alpha u^\mu_t = f\end{gathered} \tag{3.1}$$

subjected to the same boundary and initial conditions as in Problem 2.2.

For this problem, by using Leray-Schauder fixed point arguments, we will prove the following result.

Proposition 3.2. *Fix any small $\epsilon > 0$ and assume the conditions in Theorem 2.3. Then, for each $\mu > 0$, there exists $(\theta^\mu, \chi^\mu, u^\mu)$ solving Problem 2.2 and with the following regularity:*

$$\theta^\mu \in H^2(0,T;H^1(\Omega)') \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2_N(\Omega)),$$
$$\chi^\mu \in H^2(0,T;H^1(\Omega)') \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2_N(\Omega)),$$
$$u^\mu \in H^2(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;D(A^{\alpha/2})) \cap H^1(0,T;D(A^\alpha)).$$

Next, by using estimates for $(\theta^\mu, \chi^\mu, u^\mu)$ that are independent of μ, we will pass to the limit as $\mu \to 0+$, to find functions θ^ϵ, χ^ϵ, ξ^ϵ and u^ϵ, depending on ϵ, that satisfy $\chi^\epsilon \in dom(\beta)$ and $\xi^\epsilon \in \beta(\chi^\epsilon)$. This last fact implies in particular that $0 \le \chi^\epsilon(x,t) \le 1$ and thus $\tau(\chi^\epsilon) = \chi^\epsilon$ and $\tau(1-\chi^\epsilon) = 1-\chi^\epsilon$; that is, we can disregard the truncation operator τ when working with these limit functions.

By using these results, we will easily prove that θ^ϵ, χ^ϵ, ξ^ϵ and u^ϵ is in fact a solution of the following problem, which now has only one truncation operator, that is, $T_{1/\epsilon}$:

Problem 3.3. For any small $\epsilon > 0$, find functions $\theta^\epsilon, \chi^\epsilon, \xi^\epsilon : \Omega \times [0,T] \to \mathbb{R}$ and $u^\epsilon : \Omega \times [0,T] \to \mathbb{R}^3$ satisfying

$$\theta^\epsilon_t + l\chi^\epsilon_t + A_N\theta^\epsilon = g \tag{3.2}$$

$$\chi^\epsilon_t + A_N\chi^\epsilon + \xi^\epsilon + \gamma(\chi^\epsilon) = h(\theta^\epsilon) + T_{1/\epsilon}(\frac{|\eta(u\epsilon)|^2}{2}) \tag{3.3}$$
$$\xi^\epsilon \in \beta(\chi^\epsilon)$$
$$u^\epsilon_{tt} - \mathcal{H}((1-\chi^\epsilon)u^\epsilon) + \mathcal{K}(\chi^\epsilon u^\epsilon_t) + \nu A^\alpha u^\epsilon_t = f \tag{3.4}$$

subjected to the same boundary and initial conditions as in Problem 2.2.

As a next step, we will obtain suitable estimates independent of ϵ. With the help of such estimates, as $\epsilon \to 0+$ we will then extract a subsequence θ^ϵ, χ^ϵ and u^ϵ converging to a solution of Problem 2.2.

4. Existence of solutions of Problem 3.1

We will apply Leray-Schauder's fixed point theorem (see Ladyzhenskaya [20, p. 293]). For this, we construct an operator T_λ, $0 \le \lambda \le 1$, on the Banach space

$$B := H^1(0,T;L^2(\Omega)) \times H^1(0,T;W^{1,4}(\Omega)),$$

that will be a composition of two others operators, defined as follows.

Construction of the family of operators. Let $T^1_\lambda : B \to X$, $0 \le \lambda \le 1$, be the operator solution of the problem

$$\begin{gathered}\chi^\mu_t + A_N\chi^\mu + \beta_\mu(\chi^\mu) + \gamma(\chi^\mu) = \lambda(h(\overline{\theta}^\mu) + T_{1/\epsilon}(\frac{|\eta(\overline{u}^\mu)|^2}{2}))\\ \chi^\mu(0) = \chi_0\end{gathered} \tag{4.1}$$

where

$$X := W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2_N(\Omega)).$$

We will prove that this operator is well defined. To ease the notation, we define $\overline{\omega}^\mu := h(\overline{\theta}^\mu) + T_{1/\epsilon}(|\eta(\overline{u}^\mu)|^2/2)$. Note that for every $(\overline{\theta}^\mu, \overline{u}^\mu) \in B$, we have

$$\overline{\omega}^\mu \in H^1(0,T;L^2(\Omega)). \tag{4.2}$$

In particular, $\overline{\omega}^\mu \in L^2(0,T;L^2(\Omega))$, and therefore, thanks to Colli and Laurençot [12, Lemma 3.3], problem (4.1) has a unique solution

$$\chi^\mu \in H^1(0,T;L^2(\Omega)) \cap C^0([0,T];H^1(\Omega)) \cap L^2(0,T;H^2_N(\Omega)). \tag{4.3}$$

Further, in view of (2.10), (4.3) entails that

$$\gamma(\chi^\mu) \in H^1(0,T;L^2(\Omega)). \tag{4.4}$$

By proceeding as in the proof of Rocca and Rossi [21, Lemma 4.2], we test the equation in (4.1) by $(A_N\chi^\mu + \beta_\mu(\chi^\mu))_t$ and integrate in time. We obtain

$$\begin{aligned} &\int_0^t \|\nabla \chi^\mu_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|A_N\chi^\mu(t) + \beta_\mu(\chi^\mu(t))\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \beta'_\mu(\chi^\mu)|\chi^\mu_t|^2 \\ &\le \|\chi_0\|^2_{H^2(\Omega)} + \|\beta_\mu(\chi_0)\|^2_{L^2(\Omega)} + I_0, \end{aligned} \tag{4.5}$$

where we estimate I_0 as follows

$$\begin{aligned} I_0 &= \Big|\int_0^t\int_\Omega (\lambda\overline{\omega}^\mu - \gamma(\chi^\mu))(A_N\chi^\mu + \beta_\mu(\chi^\mu))_t\Big| \\ &\le \int_0^t\int_\Omega |(\lambda\overline{\omega}^\mu_t - \gamma'(\chi^\mu)\chi^\mu_t)(A_N\chi^\mu + \beta_\mu(\chi^\mu))| \\ &\quad + \int_\Omega |(\lambda\overline{\omega}^\mu(t) - \gamma(\chi^\mu(t)))(A_N\chi^\mu(t) + \beta_\mu(\chi^\mu(t)))| \\ &\quad + \int_\Omega |(\lambda\overline{\omega}^\mu(0) - \gamma(\chi_0))(A_N\chi_0 + \beta_\mu(\chi_0))| \\ &\le \frac{1}{4}(\|\chi_0\|^2_{H^2(\Omega)} + \|\beta_\mu(\chi_0)\|^2_{L^2(\Omega)} + \|A_N\chi^\mu(t) + \beta_\mu(\chi^\mu(t))\|^2_{L^2(\Omega)}) \\ &\quad + 2\|\overline{\omega}^\mu + \gamma(\chi^\mu)\|^2_{C^0(0,T;L^2(\Omega))} \\ &\quad + \frac{1}{2}\Big(\int_0^t \|A_N\chi^\mu + \beta_\mu(\chi^\mu)\|^2_{L^2(\Omega)} + \|\overline{\omega}^\mu + \gamma(\chi^\mu)\|^2_{H^1(0,T;L^2(\Omega))}\Big), \end{aligned} \tag{4.6}$$

where the last inequality follows from (4.2) and (4.4). By using the fact that $\beta_\mu(\chi_0) \in L^\infty(\Omega)$ and (4.5)-(4.6), we can apply Gronwall's lemma and easily deduce that

$$\|A_N\chi^\mu + \beta_\mu(\chi^\mu)\|_{L^\infty(0,T;L^2(\Omega))} + \|\chi^\mu_t\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{4.7}$$

Next, by using the monotonicity of β_μ we infer that

$$\|A_N\chi^\mu + \beta_\mu(\chi^\mu)\|^2_{L^\infty(0,T;L^2(\Omega))} \ge \|A_N\chi^\mu\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\beta_\mu(\chi^\mu)\|^2_{L^\infty(0,T;L^2(\Omega))}, \tag{4.8}$$

and therefore, from (4.7) and well-known elliptic regularity results, we have a estimate for χ^μ in $L^\infty(0,T;H^2_N(\Omega))$. Moreover, from (4.5), we have a bound for $\|\chi^\mu_t\|_{L^2(0,T;H^1(\Omega))}$.

By writing

$$\chi^\mu_t = -A_N\chi^\mu - \beta_\mu(\chi^\mu) - \gamma(\chi^\mu) + \lambda(h(\overline{\theta}^\mu) + T_{1/\epsilon}(\frac{|\eta(\overline{u}^\mu)|^2}{2})),$$

we can use the above estimates to also get the bound

$$\|\chi^\mu_t\|_{L^\infty(0,T;L^2(\Omega))} \le C.$$

Therefore, for any $\lambda \in [0,1]$, the operator T^1_λ is well defined.

Remark 4.1. When χ^μ is bounded in $H^1(0,T;L^2(\Omega))$ with respect to μ, we have the same for $\gamma(\chi^\mu)$. Moreover, when θ^μ is also bounded in $H^1(0,T;L^2(\Omega))$ with respect to μ, using that $T_{1/\epsilon}(|\eta(u^\mu)|^2/2)$ is bounded with respect to μ, we have the estimates (4.5)-(4.8) independent of μ. We shall see later that χ^μ and θ^μ satisfy these properties.

Next, let $T^2_\lambda : T^1_\lambda(B) \subseteq X \to B$ be the solution operator of the problem

$$\begin{gathered} \theta^\mu_t + l\chi^\mu_t + A_N\theta^\mu = \lambda g \\ u^\mu_{tt} + \mathcal{H}(\tau(1-\chi^\mu)u^\mu) + \mathcal{K}(\tau(\chi^\mu)u^\mu_t) + \nu A^\alpha u^\mu_t = \lambda f \\ \theta^\mu(0) = \theta_0 \\ u^\mu(0) = u_0 \\ u^\mu_t(0) = v_0 \end{gathered} \tag{4.9}$$

We also must prove that T^2_λ is well defined. For this, we start consider the first equation in (4.9) which does not depend of u^μ.

Note that we have $\lambda g - l\chi^\mu_t \in L^2(0,T;L^2(\Omega))$; thus, from the L^p-theory of parabolic equations (see Ladyzhenskaya [18, P. 180, Remark 6.3]), there exists a unique solution θ^μ of the first equation of (4.9) with the following regularity:

$$\theta^\mu \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)). \tag{4.10}$$

We now analyze the second equation. The existence and uniqueness for this equation follow from Galerkin method. In fact, let $\{w_i\}_{i\geq 1}$ be a "special" base for $(H^\alpha(\Omega))^n$; i.e., eigenfunctions associate to the problem

$$\begin{gathered} (A^\alpha w_i, v) = \lambda^\alpha_i (w_i, v), \quad \forall v, w_i \in (H^\alpha(\Omega))^n, \\ |w_i| = 1, \quad \lambda^\alpha_i \nearrow +\infty. \end{gathered}$$

Let V^m be the space spanned by $w_1, \dots, w_m$. For each $m \geq 1$, we are interested in seeking an approximate solution u^m of the second equation of (4.9) with your respective initial condition, in the following sense:

$$u^m(t) = \sum_{i=1}^m g_{i,m}(t) w_i$$

satisfies the following equations for all $v^m \in V^m$:

$$(u^m_{tt}, v^m) + a_{\tau(1-\chi)}(u^m, v^m) + b_{\tau(\chi)}(u^m_t, v^m) + (A^\alpha u^m_t, v^m) = (f, v^m), \tag{4.11}$$

$$u^m(0) = u_{0m}, \tag{4.12}$$

$$u^m_t(0) = v_{0m}, \tag{4.13}$$

where u_{0m} and v_{0m} are orthogonal projections in $(H^\alpha(\Omega))^n$ of u_0 and v_0 respectively, on the space V^m.

In this way, we obtain a system of linear ordinary differential equations, which has a unique solution for the well-known theory of ordinary differential equations.

We now must obtain *a priori* estimates for (4.11) independent of m. We take u^m_t as test function in (4.11) and integrate in time; after using (2.4)-(2.6) and the definition of τ, we obtain

$$\frac{1}{2}\|u^m_t(t)\|^2_{L^2(\Omega)} + C\int_0^t \|u^m_t\|^2_{H^\alpha(\Omega)}$$

$$\leq \frac{1}{2}\|v_{0m}\|^2_{L^2(\Omega)} + \epsilon\int_0^t \|u^m_t\|^2_{H^1(\Omega)} + C_\epsilon\int_0^t \|u^m\|^2_{H^1(\Omega)} + C_\epsilon\int_0^t \|f\|^2_{L^2(\Omega)}$$

Now, we note that, if H is any Hilbert space, by taken $C = 2T$ we have the inequality

$$\int_0^t \|z(s)\|^2_H ds \leq C\Big(\|z(0)\|^2_H + \int_0^t \Big(\int_0^s \|z_t(t_1)\|^2_H dt_1\Big) ds\Big), \tag{4.14}$$

Thus, by recalling that $\alpha \geq 1$ and taking ϵ sufficiently small, and then using (4.14) with $z = u^m$ and $H = H^1(\Omega)$ and Gronwall's lemma considering as its variable the expression $\int_0^s \|u^m_t(t_1)\|^2_{H^1(\Omega)} dt_1$, we obtain a bound for it. With this bound, we then conclude that

$$u^m_t \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;D(A^{\alpha/2}));$$

therefore, again by (4.14), we obtain

$$u^m \text{ is bounded in } W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;D(A^{\alpha/2})).$$

Now, we consider in (4.11) $A^\alpha u^m_t$ as test function and integrate in time to obtain

$$\int_0^t (u^m_{tt}, A^\alpha u^m_t) = \frac{1}{2}\|A^{\alpha/2}u^m_t(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|A^{\alpha/2}v_{0m}\|^2_{L^2(\Omega)}, \tag{4.15}$$

$$\int_0^t \int_\Omega \mathcal{H}(\tau(1-\chi)) \cdot A^\alpha u^m_t = I_1 + I_2, \tag{4.16}$$

where

$$\begin{aligned} |I_1| &= \Big|\int_0^t \int_\Omega \tau(1-\chi)\operatorname{div}(\eta(u^m)) \cdot A^\alpha u^m_t\Big| \\ &\leq C\int_0^t \|u^m_t\|_{H^{2\alpha}(\Omega)}\|u^m\|_{H^2(\Omega)} \\ &\leq \epsilon\int_0^t \|u^m_t\|^2_{H^{2\alpha}(\Omega)} + C_\epsilon\int_0^t \|u^m\|^2_{H^2(\Omega)} \end{aligned} \tag{4.17}$$

thanks to (2.7). Also,

$$\begin{aligned} |I_2| &= \Big|\int_0^t \int_\Omega \nabla\tau(1-\chi)\eta(u^m) \cdot A^\alpha u^m_t\Big| \\ &\leq C\int_0^t \|u^m_t\|_{H^{2\alpha}(\Omega)}\|\chi\|_{H^2(\Omega)}\|u^m\|_{H^2(\Omega)} \\ &\leq \epsilon\int_0^t \|u^m_t\|^2_{H^{2\alpha}(\Omega)} + C_\epsilon\int_0^t \|u^m\|^2_{H^2(\Omega)}, \end{aligned} \tag{4.18}$$

where we used that $\chi^\mu \in X$ and the continuous embedding (2.1).

Furthermore, by recalling the definition of operator $\mathcal{K}$, we have

$$\int_0^t \int_\Omega \mathcal{K}(\tau(\chi)u^m_t) \cdot A^\alpha u^m_t = I_3 + I_4, \tag{4.19}$$

where

$$
\begin{aligned}
|I_3| &= \Big|\int_0^t \int_\Omega \tau(\chi)\,\mathrm{div}(\eta(u^m))\cdot A^\alpha u_t^m\Big| \\
&\le C\int_0^t \|u_t^m\|_{H^2(\Omega)}\|A^\alpha u_t^m\|_{L^2(\Omega)} \\
&\le C\int_0^t \|A^\alpha u_t^m\|_{L^2(\Omega)}\|u_t^m\|_{H^{2\alpha}(\Omega)}^{1/\alpha}\|u_t^m\|_{L^2(\Omega)}^{(\alpha-1)/\alpha} \\
&\le \epsilon\int_0^t \|u_t^m\|_{H^{2\alpha}(\Omega)}^2 + C_\epsilon\int_0^t \|u_t^m\|_{L^2(\Omega)}^2,
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
|I_4| &= \Big|\int_0^t \int_\Omega A^\alpha u_t^m\cdot \eta(u_t^m)\nabla(\tau(\chi))\Big| \\
&\le C\int_0^t \|A^\alpha u_t^m\|_{L^2(\Omega)}\|u_t^m\|_{H^2(\Omega)}\|\chi\|_{H^2(\Omega)} \\
&\le C\int_0^t \|A^\alpha u_t^m\|_{L^2(\Omega)}\|u_t^m\|_{H^{2\alpha}(\Omega)}^{1/\alpha}\|u_t^m\|_{L^2(\Omega)}^{(\alpha-1)/\alpha} \\
&\le \epsilon\int_0^t \|u_t^m\|_{H^{2\alpha}(\Omega)}^2 + C_\epsilon\int_0^t \|u_t^m\|_{L^2(\Omega)}^2,
\end{aligned}
\tag{4.21}
$$

and

$$
\int_0^t\int_\Omega A^\alpha u_t^m\cdot A^\alpha u_t^m = \int_0^t \|A^\alpha u_t^m\|_{L^2(\Omega)}^2 \ge C\int_0^t \|u_t^m\|_{H^{2\alpha}(\Omega)}. \tag{4.22}
$$

Finally,

$$
\int_0^t\int_\Omega f\cdot A^\alpha u_t^m \le \epsilon\int_0^t \|A^\alpha u_t^m\|_{L^2(\Omega)}^2 + C_\epsilon\int_0^t \|f\|_{L^2(\Omega)}^2. \tag{4.23}
$$

Thus, we add (4.15)-(4.23) and apply the Gronwall' s lemma to conclude that

$$
u^m \text{is bounded in } W^{1,\infty}(0,T;D(A^{\alpha/2}))\cap H^1(0,T;D(A^\alpha)). \tag{4.24}
$$

Next, by taking $v^m = u_{tt}^m$ in (4.11), we easily obtain $\|u_{tt}^m\|_{L^2(\Omega)} \le \| -\mathcal{H}(\tau(1-\chi^m)u^m) - \mathcal{K}(\tau(\chi^m)u_t^m) - \nu A^\alpha u_t^m + \lambda f\|_{L^2(\Omega)}$. Thus, by using our previous estimates and (2.8), we obtain

$$
u_{tt}^m \text{ is bounded in } L^2(0,T;L^2(\Omega)). \tag{4.25}
$$

Therefore, by using Simon [24, Theorem 5, Corollary 4], we obtain

$$
u^m \rightharpoonup^* u^\mu \quad \text{in } H^2(0,T;L^2(\Omega))\cap W^{1,\infty}(0,T;D(A^{\alpha/2}))\cap H^1(0,T;D(A^\alpha)), \tag{4.26}
$$

$$
u^m \to u^\mu \quad \text{in } C^1(0,T;H^1(\Omega))\cap H^1(0,T;H^2(\Omega)). \tag{4.27}
$$

Thus, using (2.1) and (4.14), we conclude that $u^\mu \in H^1(0,T;W^{1,4}(\Omega))$.

To prove uniqueness, consider two solutions u_1^μ, u_2^μ of the second equation of (4.9) and define $u^\mu := u_1^\mu - u_2^\mu$; we have u^μ satisfying the equation

$$
u_{tt}^\mu + \mathcal{H}(\tau(1-\chi^\mu)u^\mu) + \mathcal{K}(\tau(\chi^\mu)u_t^\mu) + \nu A^2 u_t^\mu = 0. \tag{4.28}
$$

By testing this equation by u_t^μ, integrating and using (2.4) and (2.6), we obtain

$$
\frac{1}{2}\|u_t^\mu(t)\|_{L^2(\Omega)}^2 + C\int_0^t \|u_t^\mu\|_{H^2(\Omega)}^2 \le \epsilon\int_0^t \|u_t^\mu\|_{H^2(\Omega)}^2 + C_\epsilon\int_0^t \|u^\mu\|_{H^2(\Omega)}^2. \tag{4.29}
$$

Next, we choose $\epsilon > 0$ small enough and use (4.14) and Gronwall's lemma to get

$$\frac{1}{2}\|u_t^\mu(t)\|^2_{L^2(\Omega)} + C\int_0^t \|u_t^\mu\|^2_{H^2(\Omega)} \le 0;$$

i.e., $u_t^\mu = 0$ a. e. in $\Omega \times [0,T]$ and therefore, by (4.14), $u^\mu = 0$ which allows us to conclude that the second equation of (4.9) has unique solution.

We conclude that $(\theta^\mu, u^\mu) \in B$ and, therefore, that the operator T^2_λ is well defined for all $\lambda \in [0,1]$.

Thus, from our previous results, it is well defined the family of operators as $T_\lambda : B \to B$, $\lambda \in [0,1]$, as the composition

$$T_\lambda := T^2_\lambda \circ T^1_\lambda.$$

Continuity of the operator with respect to λ. In the following, we will prove that T_λ in λ is continuous with respect to λ, uniformly in bounded sets of B. To this end, consider $0 \le \lambda_1, \lambda_2 \le 1$, $\chi_i^\mu = T^1_{\lambda_i}(\overline{\theta}^\mu, \overline{u}^\mu)$, $(\theta_i^\mu, u_i^\mu) = T^2_{\lambda_i}(\chi_i^\mu)$, and define $(\theta^\mu, \chi^\mu, u^\mu) := (\theta_1^\mu - \theta_2^\mu, \chi_1^\mu - \chi_2^\mu, u_1^\mu - u_2^\mu)$. We have the triple $(\theta^\mu, \chi^\mu, u^\mu)$ fulfils a.e. in $\Omega \times (0,T)$

$$\theta_t^\mu + l\chi_t^\mu + A_N\theta^\mu = (\lambda_1 - \lambda_2)g \tag{4.30}$$

$$\chi_t^\mu + A_N\chi^\mu + \beta_\mu(\chi_1^\mu) - \beta_\mu(\chi_2^\mu) + \gamma(\chi_1^\mu) - \gamma(\chi_2^\mu) = (\lambda_1 - \lambda_2)\overline{\omega} \tag{4.31}$$

$$\begin{aligned} &u_{tt}^\mu + \mathcal{H}(\tau(1-\chi_1^\mu)u^\mu) + \mathcal{H}((\tau(1-\chi_1^\mu) - \tau(1-\chi_2^\mu))u_2^\mu) \\ &+ \mathcal{K}(\tau(\chi_1^\mu)u_t^\mu) + \mathcal{K}((\tau(\chi_1^\mu) - \tau(\chi_2^\mu))\partial_t u_2^\mu) + \nu A^\alpha u_t^\mu \\ &= (\lambda_1 - \lambda_2)f \end{aligned} \tag{4.32}$$

By multiplying (4.31) by χ_t^μ and integrating in time, it is not difficult to infer that

$$\int_0^t \|\chi_t^\mu\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\chi^\mu(t)\|^2_{L^2(\Omega)} \le C\int_0^t\int_\Omega |\chi_t^\mu||\chi^\mu| + (\lambda_1 - \lambda_2)\int_0^t\int_\Omega |\overline{\omega}||\chi_t^\mu|. \tag{4.33}$$

Now, we test (4.31) by χ^μ and integrate in time to obtain

$$\int_\Omega (\chi^\mu(t))^2 + \int_0^t\int_\Omega |\nabla\chi^\mu|^2 \le C\int_0^t\int_\Omega (\chi^\mu)^2 + (\lambda_1 - \lambda_2)\int_0^t\int_\Omega \overline{\omega}^2;$$

by using Gronwall's lemma, we then get

$$\int_\Omega (\chi^\mu(t))^2 + \int_0^t\int_\Omega |\nabla\chi^\mu|^2 \le C|\lambda_1 - \lambda_2|. \tag{4.34}$$

From (4.33) and (4.34), we conclude that

$$\|\chi^\mu\|_{L^2(0,T;H^1(\Omega))\cap H^1(0,T;L^2(\Omega))} \le C|\lambda_1 - \lambda_2|. \tag{4.35}$$

We test now (4.31) by $A_N\chi^\mu$ and integrate in time; after some computations we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla\chi^\mu(t)|^2 + \int_0^t\int_\Omega |A_N\chi^\mu|^2 \\ &\le C\int_0^t\int_\Omega |\chi^\mu||A_N\chi^\mu| + |\lambda_1 - \lambda_2|\int_0^t\int_\Omega |\overline{\omega}||A_N\chi^\mu| \\ &\le \epsilon\int_0^t\int_\Omega |A_N\chi^\mu|^2 + C_\epsilon\int_0^t\int_\Omega |\chi^\mu|^2 + C_\epsilon|\lambda_1 - \lambda_2|\int_0^t\int_\Omega |\overline{\omega}|^2, \end{aligned} \tag{4.36}$$

and by taking ϵ small enough, we obtain

$$\|\chi^\mu\|_{L^2(0,T;H^2(\Omega))} \le C|\lambda_1 - \lambda_2|. \tag{4.37}$$

Next, we test (4.31) by $(A_N\chi^\mu + \beta_\mu(\chi_1^\mu) - \beta(\chi_2^\mu))_t$ and integrate in time:

$$\begin{aligned}
&\int_0^t \|\nabla\chi_t^\mu\|_{L^2(\Omega)}^2 + \int_0^t\int_\Omega \chi_t^\mu(\beta_\mu'(\chi_1^\mu)\partial_t\chi_1^\mu - \beta_\mu'(\chi_2^\mu)\partial_t\chi_2^\mu) \\
&+ \frac{1}{2}\|A_N\chi^\mu(t) + \beta_\mu(\chi_1^\mu(t)) - \beta_\mu(\chi_2^\mu(t))\|_{L^2(\Omega)}^2 \\
&= \int_0^t\int_\Omega [(\lambda_1-\lambda_2)\overline{\omega} - (\gamma(\chi_1^\mu) - \gamma(\chi_2^\mu))](A_N\chi^\mu + \beta_\mu(\chi_1^\mu) - \beta_\mu(\chi_2^\mu))_t.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_0^t\int_\Omega [(\lambda_1-\lambda_2)\overline{\omega}_t - (\gamma'(\chi_1^\mu)\partial_t\chi_1^\mu - \gamma'(\chi_2^\mu)\partial_t\chi_2^\mu)](A_N\chi^\mu + \beta_\mu(\chi_1^\mu) - \beta_\mu(\chi_2^\mu)) \\
&+ \int_\Omega [(\lambda_1-\lambda_2)\overline{\omega}(t) - (\gamma(\chi_1^\mu(t)) - \gamma(\chi_2^\mu(t)))](A_N\chi^\mu(t) + \beta_\mu(\chi_1^\mu(t)) - \beta_\mu(\chi_2^\mu(t))) \\
&\le |\lambda_1-\lambda_2|\int_0^t \|\overline{\omega}_t\|_{L^2(\Omega)}\|A_N\chi^\mu + \beta_\mu(\chi_1^\mu) - \beta_\mu(\chi_2^\mu)\|_{L^2(\Omega)} \\
&\quad + C\int_0^t \|\partial_t\chi^\mu\|_{L^2(\Omega)}\|A_N\chi^\mu + \beta_\mu(\chi_1^\mu) - \beta_\mu(\chi_2^\mu)\|_{L^2(\Omega)} \\
&\quad + C\int_0^t \|\chi^\mu\|_{H^2(\Omega)}\|A_N\chi^\mu + \beta_\mu(\chi_1^\mu) - \beta_\mu(\chi_2^\mu)\|_{L^2(\Omega)} \\
&\quad + \|(\lambda_1-\lambda_2)\overline{\omega}(t) - (\gamma(\chi_1^\mu(t)) - \gamma(\chi_2^\mu(t)))\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{4}\|A_N\chi^\mu(t) + \beta_\mu(\chi_1^\mu(t)) - \beta_\mu(\chi_2^\mu(t))\|_{L^2(\Omega)}^2.
\end{aligned}$$

In view of (4.35) and (4.37), using that β_μ' is bounded and Gronwall's lemma, is not difficult to conclude that

$$\|\chi^\mu\|_{L^\infty(0,T;H^2(\Omega))\cap H^1(0,T;H^1(\Omega))} \le C|\lambda_1 - \lambda_2|. \tag{4.38}$$

Next, we test (4.30) by θ^μ and integrate in time:

$$\int_\Omega |\theta^\mu(t)|^2 + \int_0^t\int_\Omega |\nabla\theta^\mu|^2 \le |\lambda_1-\lambda_2|^2\int_0^t\int_\Omega g^2 + \int_0^t\int_\Omega (\chi_t^\mu)^2 + \int_0^t\int_\Omega (\theta^\mu)^2; \tag{4.39}$$

using Gronwall's lemma and (4.38), we then obtain

$$\|\theta^\mu\|_{L^2(0,T;H^1(\Omega))} \le C|\lambda_1 - \lambda_2|. \tag{4.40}$$

Next, we test (4.30) by θ_t^μ and integrate in time to get

$$\int_0^t\int_\Omega |\theta_t^\mu|^2 + \int_\Omega |\nabla\theta^\mu(t)|^2 \le C_\epsilon(\lambda_1-\lambda_2)^2\int_0^t\int_\Omega g^2 + C_\epsilon\int_0^t\int_\Omega (\chi_t^\mu)^2 + \epsilon\int_0^t\int_\Omega |\theta_t^\mu|^2. \tag{4.41}$$

By collecting (4.38)-(4.41), we conclude that

$$\|\theta^\mu\|_{L^\infty(0,T;H^1(\Omega))\cap H^1(0,T;L^2(\Omega))} \le C|\lambda_1 - \lambda_2|. \tag{4.42}$$

Finally, we test (4.32) by $A^2u_t^\mu$ and integrate in time. Each of the terms in the resulting identity can be estimated as follows.

$$\int_0^t\int_\Omega u_{tt}^\mu\cdot A^\alpha u_t^\mu = \frac{1}{2}\|A^{\alpha/2}u_t^\mu(t))\|_{L^2(\Omega)}^2 - \frac{1}{2}\|A^{\alpha/2}v_0\|_{L^2(\Omega)}^2, \tag{4.43}$$

$$\int_0^t\int_\Omega \mathcal{H}(\tau(1-\chi_1^\mu)u^\mu)\cdot A^\alpha u_t^\mu = G_1+G_2, \tag{4.44}$$

where

$$\begin{aligned}|G_1| &= \big|\int_0^t\int_\Omega \tau(1-\chi_1^\mu)\operatorname{div}(\eta(u^\mu))\cdot A^\alpha u_t^\mu\big|\\ &\le \epsilon\int_0^t\|A^\alpha u_t^\mu\|_{L^2(\Omega)}^2 + C_\epsilon\int_0^t\|1-\chi_1^\mu\|_{L^\infty(\Omega)}^2\|u^\mu\|_{H^2(\Omega)}^2\end{aligned} \tag{4.45}$$

and

$$\begin{aligned}|G_2| &= \big|\int_0^t\int_\Omega A^\alpha u_t^\mu\cdot\eta(u^\mu)\nabla(\tau(1-\chi_1^\mu))\big|\\ &\le \epsilon\int_0^t\|A^\alpha u_t^\mu\|_{L^2(\Omega)}^2 + C_\epsilon\|\chi_1^\mu\|_{L^\infty(0,T;H^2(\Omega))}^2\int_0^t\|u^\mu\|_{H^2(\Omega)}^2.\end{aligned} \tag{4.46}$$

We also have

$$\int_0^t\int_\Omega \mathcal{H}((\tau(1-\chi_1^\mu)-\tau(1-\chi_2^\mu))u_2^\mu)\cdot A^\alpha u_t^\mu = G_3+G_4, \tag{4.47}$$

where, similarly as previously done, we can estimate

$$\begin{aligned}|G_3| &\le C\int_0^t\|\chi^\mu\|_{L^\infty(\Omega)}\|A^\alpha u_t^\mu\|_{L^2(\Omega)}\|u_2^\mu\|_{H^2(\Omega)}\\ &\le \epsilon\int_0^t\|A^\alpha u_t^\mu\|_{L^2(\Omega)}^2 + C_\epsilon\|u_2^\mu\|_{L^\infty(0,T;H^2(\Omega))}\int_0^t\|\chi^\mu\|_{H^2(\Omega)}^2\end{aligned} \tag{4.48}$$

and

$$\begin{aligned}|G_4| &\le C\int_0^t\|u_2^\mu\|_{H^2(\Omega)}\|\chi^\mu\|_{H^2(\Omega)}\|A^\alpha u_t^\mu\|_{L^2(\Omega)}\\ &\le \epsilon\int_0^t\|A^\alpha u_t^\mu\|_{L^2(\Omega)}^2 + C_\epsilon\|u_2^\mu\|_{L^\infty(0,T;H^2(\Omega))}\int_0^t\|\chi^\mu\|_{H^2(\Omega)}^2.\end{aligned} \tag{4.49}$$

Furthermore, we have

$$\int_0^t\int_\Omega \mathcal{K}(\tau(\chi_1^\mu)u_t^\mu)\cdot A^\alpha u_t^\mu = G_5+G_6, \tag{4.50}$$

where

$$\begin{aligned}|G_5| &= \big|\int_0^t\int_\Omega \tau(\chi_1^\mu)\operatorname{div}(\eta(u_t^\mu))\cdot A^\alpha u_t^\mu\big|\\ &\le C\int_0^t\|u_t^\mu\|_{H^2(\Omega)}\|A^\alpha u_t^\mu\|_{L^2(\Omega)}\\ &\le \int_0^t\|A^\alpha u_t^\mu\|_{L^2(\Omega)}\|u_t^\mu\|_{H^{2\alpha}(\Omega)}^{1/\alpha}\|u_t^\mu\|_{L^2(\Omega)}^{(\alpha-1)/\alpha}\\ &\le \epsilon\int_0^t\|u_t^\mu\|_{H^{2\alpha}(\Omega)}^2 + C_\epsilon\int_0^t\|u_t^\mu\|_{L^2(\Omega)}^2\end{aligned} \tag{4.51}$$

and

$$\begin{aligned}|G_6| &= \Big|\int_0^t\int_\Omega A^\alpha u_t^\mu\cdot\eta(u_t^\mu)\nabla(\tau(\chi_1^\mu))\Big| \\ &\le C\int_0^t \|u_t^\mu\|_{H^2(\Omega)}\|A^\alpha u_t^\mu\|_{L^2(\Omega)} \\ &\le \int_0^t \|A^\alpha u_t^\mu\|_{L^2(\Omega)}\|u_t^\mu\|_{H^{2\alpha}(\Omega)}^{1/\alpha}\|u_t^\mu\|_{L^2(\Omega)}^{(\alpha-1)/\alpha} \\ &\le \epsilon\int_0^t \|u_t^\mu\|_{H^{2\alpha}(\Omega)}^2 + C_\epsilon\int_0^t \|u_t^\mu\|_{L^2(\Omega)}^2.\end{aligned} \tag{4.52}$$

Also we have

$$\int_0^t\int_\Omega \mathcal{K}((\tau(\chi_1^\mu)-\tau(\chi_2^\mu))\partial_t u_2^\mu)\cdot A^2 u_t^\mu = G_7 + G_8, \tag{4.53}$$

where

$$\begin{aligned}|G_7| &= \Big|\int_0^t\int_\Omega (\tau(\chi_1^\mu)-\tau(\chi_2^\mu))\operatorname{div}(\eta(\partial_t u_2^\mu))\cdot A^\alpha u_t^\mu\Big| \\ &\le C\int_0^t\int_\Omega |\chi^\mu\|\operatorname{div}(\eta(\partial_t u_2^\mu))\|A^\alpha u_t^\mu| \\ &\le C\int_0^t \|\chi^\mu\|_{H^2(\Omega)}\|\partial_t u_2^\mu\|_{H^2(\Omega)}\|A^\alpha u_t^\mu\|_{L^2(\Omega)} \\ &\le \epsilon\int_0^t \|A^\alpha u_t^\mu\|_{L^2(\Omega)}^2 + C_\epsilon\|\chi^\mu\|_{L^\infty(0,T;H^2(\Omega))}\int_0^t \|\partial_t u_2^\mu\|_{H^2(\Omega)}^2\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}|G_8| &= \Big|\int_0^t\int_\Omega A^\alpha u_t^\mu\cdot\eta(\partial_t u_2^\mu)\nabla(\tau(\chi_1^\mu)-\tau(\chi_2^\mu))\Big| \\ &\le C\int_0^t\int_\Omega |A^\alpha u_t^\mu||\eta(\partial_t u_2^\mu)||\nabla\chi^\mu| \\ &\le \epsilon\int_0^t \|A^\alpha u_t^\mu\|_{L^2(\Omega)}^2 + C_\epsilon\|\chi^\mu\|_{L^\infty(0,T;H^2(\Omega))}\int_0^t \|\partial_t u_2^\mu\|_{H^2(\Omega)}^2.\end{aligned} \tag{4.55}$$

Next,

$$\nu\int_0^t\int_\Omega A^\alpha u_t^\mu\cdot A^\alpha u_t^\mu = \nu\int_0^t \|A^\alpha u_t^\mu\|_{L^2(\Omega)}^2, \tag{4.56}$$

and finally,

$$(\lambda_1-\lambda_2)\int_0^t\int_\Omega f\cdot A^\alpha u_t^\mu \le C_\epsilon|\lambda_1-\lambda_2|\|f\|_{L^2(0,T;L^2(\Omega))}^2 + \epsilon\int_0^t \|A^\alpha u_t^\mu\|_{L^2(\Omega)}^2. \tag{4.57}$$

Thus, to obtain the estimates that come from testing (4.32) by $A^\alpha u_t^\mu$ and integrating in time, we add (4.43)-(4.57), then estimate the integral terms containing $\|u^\mu\|_{H^2(\Omega)}^2$ by using (4.14) and applying Gronwall's lemma to conclude that

$$\|u^\mu\|_{H^1(0,T;D(A^\alpha))} \le C|\lambda_1-\lambda_2|.$$

Therefore, from (2.1), we have

$$\|u^\mu\|_{H^1(0,T;W^{1,4}(\Omega))} \le C|\lambda_1-\lambda_2|. \tag{4.58}$$

From (4.42) and (4.58), we conclude that T_λ is continuous in λ, uniformly in bounded sets of B.

Estimates of all possible fixed points. We now estimate the set of all fixed points of T_λ in B. Each fixed point $(\theta^\mu, u^\mu) \in B$ satisfies the following problem:

$$\theta^\mu_t + l\chi^\mu_t + A_N\theta^\mu = \lambda g \quad \text{in } \Omega \times (0,T), \tag{4.59}$$

$$\chi^\mu_t + A_N\chi^\mu + \beta_\mu(\chi^\mu) + \gamma(\chi^\mu) = \lambda(h(\theta^\mu) + T_{1/\epsilon}(\frac{|\eta(u^\mu)|^2}{2})) \quad \text{in } \Omega \times (0,T), \tag{4.60}$$

$$u^\mu_{tt} - \mathcal{H}(\tau(1-\chi^\mu)u^\mu) + \mathcal{K}(\tau(\chi^\mu)u^\mu_t) + \nu A^\alpha u^\mu_t = \lambda f \quad \text{in } \Omega \times (0,T), \tag{4.61}$$

subjected to the conditions

$$\begin{gathered} \theta^\mu(0) = \theta_0 \quad \text{in } \Omega, \\ \chi^\mu(0) = \chi_0 \quad \text{in } \Omega, \\ u^\mu(0) = u_0, \quad u^\mu_t(0) = v_0 \quad \text{in } \Omega. \end{gathered}$$

We start by testing (4.60) by χ^μ and integrate in time to get

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\chi^\mu(t)|^2 + \int_0^t\int_\Omega |\nabla\chi^\mu|^2 \\ &\le C_1 + C\int_0^t\int_\Omega (\chi^\mu)^2 + \int_0^t\int_\Omega h(\theta^\mu)\chi^\mu + \frac{1}{2}\|\chi_0\|_{L^2(\Omega)}. \end{aligned} \tag{4.62}$$

Next, we test (4.59) by $\theta^\mu + l\chi^\mu$ and integrate in time to obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\theta^\mu(t) + l\chi^\mu(t)|^2 + \int_0^t\int_\Omega |\nabla\theta^\mu|^2 + l\int_0^t\int_\Omega \nabla\theta^\mu\nabla\chi^\mu \\ &= \frac{1}{2}\|\theta_0 + l\chi_0\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega g(\theta^\mu + l\chi^\mu). \end{aligned} \tag{4.63}$$

By multiplying (4.62) by $1 + \frac{1}{2}l^2$, adding it to (4.63) and using that $|h(\theta)| \le C|\theta|$ we reach

$$\begin{aligned} &\int_\Omega \Big(\frac{1}{6}(\theta^\mu(t))^2 + \frac{1}{2}(\chi^\mu(t))^2\Big) + \frac{1}{2}\int_0^t\int_\Omega (|\nabla\theta^\mu|^2 + |\nabla\chi^\mu|^2) \\ &\le C(\|\theta_0\|^2_{L^2(\Omega)} + \|\chi_0\|^2_{L^2(\Omega)}) + \int_\Omega \|g\|^2_{L^2(\Omega)} + C\int_0^t\int_\Omega (|\theta^\mu|^2 + |\chi^\mu|^2). \end{aligned} \tag{4.64}$$

By using Gronwall's lemma, we then conclude that

$$\chi^\mu, \theta^\mu \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{4.65}$$

with a bound in (4.65) that is independent of μ.

We now test (4.59) by χ^μ_t and integrate in time

$$\begin{aligned} &\int_0^t\int_\Omega |\chi^\mu_t|^2 + \frac{1}{2}\int_\Omega |\nabla\chi^\mu(t)|^2 + \int_\Omega \widehat{\beta}_\mu(\chi^\mu) \\ &\le C + \epsilon\int_0^t\int_\Omega |\chi^\mu_t|^2 + C_\epsilon\Big(\int_0^t\int_\Omega |\chi^\mu|^2 + \int_0^t\int_\Omega |\theta^\mu|^2\Big). \end{aligned} \tag{4.66}$$

By using (4.65), Remark 2.1, which implies that $-\int_\Omega \widehat{\beta}_\mu(\chi^\mu) \le M|\Omega| < +\infty$, and Gronwall's lemma again, we have

$$\chi^\mu_t \text{ is bounded in } L^2(0,T;L^2(\Omega)); \tag{4.67}$$

moreover, the bound (4.67) is independent of μ.

Next, we test (4.59) by θ_t^μ and integrate in time to get

$$\begin{aligned}&\int_0^t\int_\Omega|\theta_t^\mu|^2+\frac{1}{2}\int_\Omega|\nabla\theta^\mu(t)|^2\\&\le\|\nabla\theta_0\|_{L^2(\Omega)}^2+\frac{1}{2}\int_0^t\int_\Omega|\theta_t^\mu|^2+l\int_0^t\int_\Omega|\chi_t^\mu|^2+\int_0^t\int_\Omega g^2.\end{aligned}\tag{4.68}$$

From (4.67), we have $\theta_t^\mu\in L^2(0,T;L^2(\Omega))$, and therefore

$$\theta^\mu \text{ is bounded in } H^1(0,T;L^2(\Omega)),\tag{4.69}$$

with a bound (4.69) which is independent of μ.

We have one more estimate for χ^μ. From the fact that $\theta^\mu,\chi^\mu\in H^1(0,T;L^2(\Omega))$ and $T_{1/\epsilon}(|\eta(u^\mu)|/2)$ is bounded, similarly as was done to get (4.38), we obtain

$$\chi^\mu \text{ is bounded in } L^\infty(0,T;H_N^2(\Omega)).\tag{4.70}$$

Finally, by proceeding similarly as was done in (4.43)-(4.57) and using (4.70), we have $u^\mu\in H^1(0,T;D(A^\alpha))$, and

$$u^\mu \text{ is bounded in } H^1(0,T;W^{1,4}(\Omega)).\tag{4.71}$$

Therefore, we can conclude from (4.69)-(4.71) that there exist $M>0$ such that

$$\|(\theta^\mu,u^\mu)\|_B\le M\,.$$

Remark 4.2. As seen above in the estimate of all possible fixed points, the boundedness of (4.65), (4.67) and (4.69) are independent of μ. Follow from these fact that the boundedness of Remark 4.1 are satisfies.

Furthermore, if we have $T_{1/\epsilon}(|\eta(u^\epsilon)|^2/2)$ bounded in $L^2(0,T;L^2(\Omega))$ with respect to ϵ, then the boundedness (4.62)-(4.64), (4.66) and (4.68) are independent of ϵ, and therefore the boundedness (4.65), (4.67) and (4.69) are independent of ϵ.

Continuity and compactness. It remains to prove that for each $\lambda\in[0,1]$ T_λ is continuous and compact on B. Before proving these properties, we need some estimates. We start by differentiating (4.1) in time, testing it by $J^{-1}(\chi_{tt}^\mu)$, and integrating the result in time:

$$\int_0^t\|\chi_{tt}^\mu\|_{H^1(\Omega)'}^2\le G_9+G_{10}+G_{11}+G_{12},\tag{4.72}$$

where

$$G_9=\Big|\int_0^t\int_\Omega\nabla\chi_t^\mu\nabla J^{-1}(\chi_{tt}^\mu)\Big|\le\int_0^t\|\chi_t^\mu\|_{H^1(\Omega)}^2+\frac{1}{4}\int_0^t\|\chi_{tt}^\mu\|_{H^1(\Omega)'}^2,\tag{4.73}$$

$$\begin{aligned}G_{10}&=\Big|\int_0^t\int_\Omega(\beta_\mu'(\chi^\mu)+\gamma'(\chi^\mu))\chi_t^\mu J^{-1}(\chi_{tt}^\mu)\Big|\\&\le\|\beta_\mu'(\chi^\mu)+\gamma'(\chi^\mu)\|_{L^\infty(\Omega\times(0,T))}\int_0^t\|J^{-1}(\chi_{tt}^\mu)\|_{L^2(\Omega)}\|\chi_t^\mu\|_{L^2(\Omega)}\\&\le\frac{1}{4}\int_0^t\|\chi_{tt}^\mu\|_{H^1(\Omega)'}^2+C\int_0^t\|\chi_t^\mu\|_{L^2(\Omega)}^2,\end{aligned}\tag{4.74}$$

$$G_{11}=\Big|\int_0^t\int_\Omega h'(\theta^\mu)\theta_t^\mu J^{-1}(\chi_{tt}^\mu)\Big|\le\frac{1}{8}\int_0^t\|\chi_{tt}^\mu\|_{H^1(\Omega)'}^2+C\int_0^t\|\theta_t^\mu\|_{L^2(\Omega)}^2,\tag{4.75}$$

and

$$\begin{aligned} G_{12} &= \Big|\int_0^t\int_\Omega T'_{1/\epsilon}\big(\frac{|\eta(u^\mu)|^2}{2}\big)\eta(u_t^\mu)\mathcal{R}_e\eta(u^\mu)J^{-1}(\chi_{tt}^\mu)\Big| \\ &\le C\|\eta(u^\mu)\|_{L^\infty(0,T;L^4(\Omega))}\int_0^t\|\eta(u_t^\mu)\|_{L^4(\Omega)}\|\chi_{tt}^\mu\|_{H^1(\Omega)'} \\ &\le \frac{1}{8}\int_0^t\|\chi_{tt}^\mu\|^2_{H^1(\Omega)'} + C\|u^\mu\|^2_{H^1(0,T;H^2(\Omega))}\int_0^t\|u_t^\mu\|^2_{H^2(\Omega)}. \end{aligned} \tag{4.76}$$

We remark that to estimate G_{10} we used the fact that β'_μ and γ' are Lipschitz functions, and that $\chi^\mu \in L^\infty(0,T;H^2_N(\Omega))$; in G_{11} we used that h' is limited. Collecting (4.72)-(4.76), we conclude that

$$\chi^\mu \text{ is bounded in } H^2(0,T;H^1(\Omega)'). \tag{4.77}$$

We differentiate the first equation in (4.9), multiply by $J^{-1}(\theta_{tt}^\mu)$, integrate in time and using (4.77), similarly as we did in in the (4.72)-(4.76), we obtain that

$$\theta^\mu \text{ is bounded in } H^2(0,T;H^1(\Omega)'). \tag{4.78}$$

We test the first equation in (4.9) by θ_t^μ and integrate in time to obtain

$$\begin{aligned} \int_0^t\int_\Omega(\theta_t^\mu)^2 + \frac{1}{2}\int_\Omega|\nabla\theta^\mu(t)|^2 &\le \frac{1}{2}\|\nabla\theta^\mu(0)\|^2_{L^2(\Omega)} + C\int_0^t\int_\Omega g^2 + \frac{1}{4}\int_0^t\int_\Omega(\theta_t^\mu)^2 \\ &\quad + C\int_0^t\int_\Omega(\chi_t^\mu)^2 + \frac{1}{4}\int_0^t\int_\Omega(\theta_t^\mu)^2. \end{aligned} \tag{4.79}$$

By recalling (4.7) and applying Gronwall's lemma, we infer that

$$\theta^\mu \text{ is bounded in } L^\infty(0,T;H^1(\Omega)). \tag{4.80}$$

Next, we test the first equation in (4.9) by $A_N\theta_t^\mu$ and integrate by parts, to obtain

$$\begin{aligned} &\int_0^t\int_\Omega|\nabla\theta_t^\mu|^2 + \frac{1}{2}\int_\Omega|A_N\theta^\mu(t)|^2 \\ &\le \frac{1}{2}\|A_N\theta^\mu(0)\|^2_{L^2(\Omega)} + \Big|\int_0^t\int_\Omega gA_N\theta_t^\mu\Big| + \Big|l\int_0^t\int_\Omega\chi_t^\mu A_N\theta_t^\mu\Big| \\ &\le \|\theta_0\|^2_{H^2(\Omega)} + C\|g\|^2_{H^1(0,T;L^2(\Omega))} + \frac{1}{2}\int_0^t\int_\Omega|A_N\theta^\mu|^2 + \frac{1}{4}\int_\Omega|A_N\theta^\mu(t)|^2 \\ &\quad + C\int_0^t\int_\Omega|\nabla\chi_t^\mu|^2 + \frac{1}{2}\int_0^t\int_\Omega|\nabla\theta_t^\mu|^2. \end{aligned} \tag{4.81}$$

By recalling (4.7), using (4.10), (4.80) and using again Gronwall's lemma, we obtain

$$\theta^\mu \text{ is bounded in } H^1(0,T;H^1(\Omega))\cap L^\infty(0,T;H^2_N(\Omega)). \tag{4.82}$$

Now, by writing $\theta_t^\mu = -l\chi_t^\mu - A_N\theta^\mu + \lambda g$ and using the above estimates, we obtain

$$\theta_t^\mu \text{ is bounded in } L^\infty(0,T;L^2(\Omega)). \tag{4.83}$$

Finally, arguing in the same way as to obtain (4.26)-(4.27), we obtain

$$u^\mu \text{ is bounded in } H^2(0,T;L^2(\Omega))\cap W^{1,\infty}(0,T;D(A^{\alpha/2}))\cap H^1(0,T;D(A^\alpha)). \tag{4.84}$$

Now we can prove the compactness and continuity of the operator. The compactness of T_λ on B easily follow from (4.78)-(4.84) and Simon [24, Theorem 5, Corollary 4.].

To show that T_λ is continuous in B, we fix a sequence such that

$$\{(\overline{\theta}^\mu_n, \overline{u}^\mu_n)\} \to (\overline{\theta}, \overline{u}) \text{ strongly in } H^1(0,T;L^2(\Omega)) \times H^1(0,T;W^{1,4}(\Omega)), \tag{4.85}$$

and we let $\chi^\mu_n := T^1_\lambda(\overline{\theta}^\mu_n, \overline{u}^\mu_n)$ for all $n \in \mathbb{N}$.

From (4.3), (4.7)-(4.8) and (4.77), we have

$$\begin{aligned}\chi^\mu_n \in H^2(0,T;H^1(\Omega)') \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \\ \cap L^\infty(0,T;H^2_N(\Omega)).\end{aligned}$$

It follows from Simon [24, Theorem 5, Corollary 4], that there exist a subsequence, which for simplicity of notations we do not relabel, and a function χ^μ such that the following convergences hold for any $1 \le p < \infty$ and any $\rho > 0$:

$$\begin{gathered}\chi^\mu_n \to \overline{\chi}^\mu \quad \text{in } C^1(0,T;H^1(\Omega)') \cap C^0(0,T;H^{2-\rho}(\Omega)), \\ \chi^\mu_n \to \overline{\chi}^\mu \quad \text{in } W^{1,p}(0,T;L^2(\Omega)) \cap H^1(0,T;H^{1-\rho}(\Omega)) \cap L^p(0,T;H^2_N(\Omega)), \\ \chi^\mu_n \rightharpoonup^* \overline{\chi}^\mu \quad \text{in } H^2(0,T;H^1(\Omega)') \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \\ \cap L^\infty(0,T;H^2_N(\Omega)).\end{gathered} \tag{4.86}$$

Hence, χ^μ satisifes the initial condition (1.8).

By using (4.85)-(4.86), it is easy to pass to the limit in (4.1) and conclude that

$$\chi^\mu = T^1_\lambda(\overline{\theta}^\mu, \overline{u}^\mu).$$

Therefore, we infer that convergences in (4.86) hold along the whole sequence $\{\chi^\mu_n\}$.

We now consider the sequence $(\theta^\mu_n, u^\mu_n) := T^2_\lambda(\chi^\mu_n) = T_\lambda(\overline{\theta}^\mu_n, \overline{u}^\mu_n)$. Collecting (4.78)-(4.83), we obtain a uniform bound for

$$\begin{aligned}\theta^\mu_n \in H^2(0,T;H^1(\Omega)') \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \\ \cap L^\infty(0,T;H^2_N(\Omega)).\end{aligned}$$

In view of (4.84) and Simon [24, Theorem 5, Corollary 4] , we deduce that there exist suitable subsequences (which we do not relabel) of $\{\theta^\mu_n\}$ and $\{u^\mu_n\}$ and two limit functions θ^μ and u^μ such that for all $1 \le p < \infty$ and for all $\rho > 0$

$$\begin{gathered}u^\mu_n \to u^\mu \quad \text{in } H^1(0,T;H^{2-\rho}(\Omega)) \cap W^{1,p}(0,T;H^1(\Omega)) \cap C^1(0,T;H^{1-\rho}(\Omega)), \\ u^\mu_n \rightharpoonup^* u^\mu \quad \text{in } H^2(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;D(A^{\alpha/2})) \cap H^1(0,T;D(A^\alpha)),\end{gathered} \tag{4.87}$$

while for $\{\theta^\mu_n\}$ and θ^μ the same convergences as in (4.86) hold true. In particular, $u^\mu_n \to u^\mu$ in $H^1(0,T;W^{1,4}(\Omega))$, whence

$$(\theta^\mu_n, u^\mu_n) \to (\theta^\mu, u^\mu) \quad \text{in } H^1(0,T;L^2(\Omega)) \times H^1(0,T;W^{1,4}(\Omega)). \tag{4.88}$$

It follows from (4.87) that u^μ complies with the initial condition (1.9). By combining (4.86) and (4.87) and arguing in the same way as in the corresponding proof in Rocca and Rossi [21], we infer that the pair (u^μ, χ^μ) satisfies the second equation in (4.9) on $\Omega \times (0,T)$. In the same way, convergences (4.86) for $\{\chi^\mu_n\}$ and $\{\theta^\mu_n\}$ allow us to conclude that (θ^μ, χ^μ) fulfils the first equation in (4.9) on $\Omega \times (0,T)$ and that θ^μ complies with the initial condition (1.7). Finally, we deduce that

$$(\theta^\mu, u^\mu) = T^2_\lambda(\chi^\mu) = T_\lambda(\overline{\theta}^\mu, \overline{u}^\mu),$$

and that (4.88) holds along the whole sequence $\{(\theta_n^\mu, u_n^\mu)\}$. Hence, the operator T_λ is continuous and compact with respect to B.

Remark 4.3. At this point is important to observe that from Remarks 4.1 and 4.2, χ^μ is bounded in $H^1(0,T;H^1(\Omega))$ independently of μ. Thus, the estimates (4.79)-(4.81) are independent of μ, and therefore the bounds in (4.82) and (4.83) are also independent of μ.

Conclusion. Since when $\lambda = 0$, it is trivial that we have unique solution for each equation in (4.1) and (4.9), from the previous proved results all the required conditions to use Leray-Schauder fixed point theorem are met. Thus, operator T_λ has a fixed point (θ^μ, u^μ) in $\lambda = 1$, i. e., there exist a triple $(\theta^\mu, \chi^\mu, u^\mu)$ that satisfies (3.1) with the following regularity

$$\theta^\mu \in H^2(0,T;H^1(\Omega)') \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H_N^2(\Omega)),$$
$$\chi^\mu \in H^2(0,T;H^1(\Omega)') \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H_N^2(\Omega)),$$
$$u^\mu \in H^2(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;D(A^{\alpha/2})) \cap H^1(0,T;D(A^\alpha)),$$

and Propositon 3.2 is proved.

5. Existence of solutions of Problem 3.3

Now, it is necessary to obtain suitable estimates uniform in μ for the solutions of Problem 3.1. This will allow us to pass to the limit as $\mu \searrow 0$. From Remarks 4.1, 4.2 and 4.3, we have

$$\chi^\mu, \theta^\mu \text{ is bounded in } W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H_N^2(\Omega)),$$

and in particular

$$\beta_\mu(\chi^\mu) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)).$$

By proceeding similarly as in (4.15)-(4.25), we obtain

$$u^\mu \text{is bounded in } H^2(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;D(A^{\alpha/2})) \cap H^1(0,T;D(A^\alpha)).$$

Hence, in view of Simon [24, Theorem 5, Corollary 4], and maximal monotone operator properties, we have the following convergences (along subsequences)

$$\theta^\mu \rightharpoonup^* \theta^\epsilon \quad \text{in } W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H_N^2(\Omega)), \tag{5.1}$$
$$\theta^\mu \to \theta^\epsilon \quad \text{in } C(0,T;H^1(\Omega)), \tag{5.2}$$
$$\chi^\mu \rightharpoonup^* \chi^\epsilon \quad \text{in } W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H_N^2(\Omega)), \tag{5.3}$$
$$\chi^\mu \to \chi^\epsilon \quad \text{in } C(0,T;H^1(\Omega)), \tag{5.4}$$
$$\beta_\mu(\chi^\mu) \rightharpoonup \xi^\epsilon \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{5.5}$$
$$u^\mu \rightharpoonup^* u^\epsilon \quad \text{in } H^2(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;D(A^{\alpha/2})) \cap H^1(0,T;D(A^\alpha)), \tag{5.6}$$
$$u^\mu \to u^\epsilon \quad \text{in } C^1(0,T;H^1(\Omega)) \cap H^1(0,T;D(A^{\alpha/2})). \tag{5.7}$$

Next, we observe that due to the monotonicity of β_μ, we have

$$(\beta_\mu(\chi) - \beta_\mu(\chi^\mu)\,, \chi - \chi^\mu) \geq 0 \quad \forall \chi \in \mathrm{dom}(\beta).$$

By using (5.4) and (5.5) and taking $\mu \to 0+$ in this last inequality we obtain

$$(\beta_0(\chi) - \xi^\epsilon\,, \chi - \chi^\epsilon) \geq 0 \quad \forall \chi \in \mathrm{dom}(\beta),$$

where $\beta_0(\chi)$ denotes the element of $\beta(\chi)$ with minimal norm and we have used the fact that β_μ is the Yosida approximation of β and Brezis [9, Prop. 2.6 (iii), p. 28].

But β_0 is a principal section of β, according to Brezis [9, Prop. 2.7. p. 29, Definition 2.3]), and thus the bove inequality implies

$$\chi^\epsilon \in \operatorname{dom}(\beta) \quad \text{and} \quad \xi^\epsilon \in \beta(\chi^\epsilon). \tag{5.8}$$

By using arguments similar to the ones used to prove that the operator T_λ was continuous and compact, we can pass the limit as $\mu \to 0+$ in the equations of Problem 3.1 and get that $(\theta^\epsilon, \chi^\epsilon, u^\epsilon)$ satisfies

$$\theta^\epsilon \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2_N(\Omega)), \tag{5.9}$$

$$\chi^\epsilon \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2_N(\Omega)), \tag{5.10}$$

$$\xi^\epsilon \in L^2(0,T;L^2(\Omega)), \quad \xi^\epsilon \in \beta(\chi^\epsilon), \tag{5.11}$$

$$u^\epsilon \in H^2(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;D(A^{\alpha/2})) \cap H^1(0,T;D(A^\alpha)), \tag{5.12}$$

and moreover it solves Problem 3 (see (3.2)-(3.4)) since (5.11) implies that $\tau(\chi^\epsilon) = \chi^\epsilon$ and $\tau(1-\chi^\epsilon) = 1-\chi^\epsilon$, and with the obtained convergences it is easy to verify that the initial conditions are met.

6. Proof of Theorem 2.3

6.1. **Existence of solutions of Problem 2.2.** To prove Theorem 2.3, it will be necessary to obtain estimates independent of ϵ. For this, we proceeding as in (4.62)-(4.64), to obtain

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{6}(\theta^\epsilon(t))^2 + \frac{1}{2}(\chi^\epsilon(t))^2\Big) + \frac{1}{2}\int_0^t \int_\Omega (|\nabla\theta^\epsilon|^2 + |\nabla\chi^\epsilon|^2) \\
&\le \int_0^t \|g\|^2_{L^2(\Omega)} + C\int_0^t (\|\theta^\epsilon\|^2_{L^2(\Omega)} + \|\chi^\epsilon\|^2_{L^2(\Omega)} + \|u^\epsilon\|^2_{H^1(\Omega)}) \\
&\quad + C(\|\theta_0\|^2_{L^2(\Omega)} + \|\chi_0\|^2_{L^2(\Omega)}).
\end{aligned} \tag{6.1}$$

We test now (3.4) by u_t^ϵ and integrate in time to obtain, in view of (2.4), (2.6) and (2.2), and that $0 \le \chi^\epsilon \le 1$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u_t^\epsilon(t)|^2 + C\int_0^t \|u_t^\epsilon\|_{H^\alpha(\Omega)} \\
&\le \frac{1}{2}\|v_0\|_{L^2(\Omega)} + \varepsilon\int_0^t \|u_t^\epsilon\|^2_{H^1(\Omega)} + C_\varepsilon\Big(\int_0^t \|u^\epsilon\|_{H^1(\Omega)} + \int_0^t \|f\|^2_{L^2(\Omega)}\Big).
\end{aligned} \tag{6.2}$$

Adding (6.1) and (6.2), choosing $\varepsilon > 0$ small enough and using Gronwall's lemma, we obtain

$$u^\epsilon \text{ bounded in } W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;D(A^{\alpha/2})). \tag{6.3}$$

and

$$\theta^\epsilon, \chi^\epsilon \text{ bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

Next, from (4.66) and (6.3), we have

$$\chi_t^\epsilon \text{ is bounded in } L^2(0,T;L^2(\Omega)). \tag{6.4}$$

Estimates (4.68) and (6.4) thus gives that

$$\theta_t^\epsilon \text{ is bounded in } L^2(0,T;L^2(\Omega)).$$

Hence, we have

$$\theta^\epsilon, \chi^\epsilon \text{ is bounded in } H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

Finally, we test equation (3.3) of the Problem 3.3 by $A_N\chi^\epsilon + \xi^\epsilon$ and integrate in time to obtain

$$\begin{aligned} \int_0^t \|A_N\chi^\epsilon + \xi^\epsilon\|^2_{L^2(\Omega)} &\le \epsilon \int_0^t \|A_N\chi^\epsilon + \xi^\epsilon\|^2_{L^2(\Omega)} + C_\epsilon\Big(\int_0^t \|\chi^\epsilon_t\|^2_{L^2(\Omega)} \\ &\quad + \int_0^t \|\chi^\epsilon\|^2_{L^2(\Omega)} + \int_0^t \|\theta^\epsilon\|^2_{L^2(\Omega)} + \int_0^t \|u^\epsilon\|^4_{W^{1,4}(\Omega)}\Big). \end{aligned}$$

In view of estimates getting above and monotony properties of the operator β, we obtain

$$\xi^\epsilon \text{ is bounded in } L^2(0,T;L^2(\Omega)).$$

Therefore, by Simon [24, Theorem 5, Corollary 4], we obtain the following convergences:

$$\theta^\epsilon \rightharpoonup \theta \quad \text{in } H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{6.5}$$

$$\theta^\epsilon \to \theta \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{6.6}$$

$$\chi^\epsilon \rightharpoonup \chi \quad \text{in } H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{6.7}$$

$$\chi^\epsilon \to \chi \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{6.8}$$

$$\xi^\epsilon \rightharpoonup \xi \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{6.9}$$

$$u^\epsilon \rightharpoonup^* u \quad \text{in } W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;D(A^{\alpha/2})), \tag{6.10}$$

$$u^\epsilon \to u \quad \text{in } C(0,T;H^1(\Omega)). \tag{6.11}$$

Next, for $g \in L^1(Q)$, by using the dominated convergence theorem, we easily get that $T_{1/\epsilon}g \to g$ in $L^1(Q)$. Moreover, when $g^\epsilon \to g$, by using the facts that $|g(x,t) - T_{1/\epsilon}(u^\epsilon)(x,t)| \le |g(x,t) - T_{1/\epsilon}(g)(x,t)| + |T_{1/\epsilon}(g)(x,t) - T_{1/\epsilon}(g^\epsilon)(x,t)|$ and that $|T_{1/\epsilon}(g)(x,t) - T_{1/\epsilon}(g^\epsilon)(x,t)| \le |g(x,t) - g^\epsilon(x,t)|$, we obtain that $T_{1/\epsilon}(g^\epsilon) \to g$ in $L^1(Q)$.

Since convergence (6.11) implies that $|\eta(u^\epsilon)|^2/2 \to |\eta(u)|^2/2$ in $C(0,T;L^1(\Omega))$, we can use the previous arguments to get that

$$T_{1/\epsilon}(|\eta(u^\epsilon)|^2/2) \to |\eta(u)|^2/2 \quad \text{in } L^1(Q).$$

The just obtained convergences are enough to pass to the limit as $\epsilon \to 0+$ in all the terms of the equations (3.2)-(3.4). Moreover, using (6.8) and (6.9), and proceeding exactly as we did to obtain (5.8), we obtain $\xi \in \beta(\chi)$. Also, we the obtained convergences it is standard to check that the initial conditions are met. Thus, we have all the required conditions for a solution and the existence result of our main theorem is proved.

6.2. **Uniqueness of solution of Problem 2.2.** We prove the uniqueness of solution for the Problem 2.2. Let us consider two solutions $(\theta_i, \chi_i, \xi_i, u_i)$, $i = 1,2$, of the Problem 2.2. Define $\theta := \theta_1 - \theta_2$, $\chi := \chi_1 - \chi_2$, $\xi := \xi_1 - \xi_2$ and $u := u_1 - u_2$; these functions satisfies the equations

$$\theta_t + l\chi_t + A_N\theta = 0, \tag{6.12}$$

$$\chi_t + A_N\chi + \xi + (\gamma(\chi_1) - \gamma(\chi_2)) = h(\theta_1) - h(\theta_2) + \frac{\eta(u_1):\eta(u)}{2} + \frac{\eta(u):\eta(u_2)}{2}, \tag{6.13}$$

$$u_{tt}+\mathcal{H}((1-\chi_1)u)-\mathcal{H}(\chi u_2)+\mathcal{K}(\chi_1 u_t)+\mathcal{K}(\chi\partial_t u_2)+\nu A^\alpha u_t=0. \tag{6.14}$$

We test equation (1.2) by χ and integrate in time to get

$$\begin{aligned}&\int_\Omega|\chi(t)|^2+\int_0^t\int_\Omega|\nabla\chi|^2+\int_0^t\int_\Omega\xi\chi+\int_0^t\int_\Omega(\gamma(\chi_1)-\gamma(\chi_2))\chi\\&=\int_0^t\int_\Omega(h(\theta_1)-h(\theta_2))\chi+\frac12\int_0^t\int_\Omega[\eta(u_1):\eta(u)+\eta(u):\eta(u_2)]\chi.\end{aligned}$$

By using the monotonicity of β and Lipschitz condition on γ and h, we obtain

$$\begin{aligned}&\int_\Omega|\chi(t)|^2+\int_0^t\int_\Omega|\nabla\chi|^2\\&\le C\int_0^t\int_\Omega|\chi|^2+C\int_0^t\int_\Omega|\theta||\chi|\\&\quad+C'\int_0^t(\|u_2\|_{H^\alpha(\Omega)}+\|\partial_t u_2\|_{H^\alpha(\Omega)})\|\chi\|_{L^2(\Omega)}\|u\|_{H^\alpha(\Omega)}.\end{aligned}\tag{6.15}$$

Next, we test equation (6.12) by $\theta+l\chi$ and integrated in time to conclude that

$$\frac12\int_0^t|\theta(t)+l\chi(t)|^2+\int_0^t\int_\Omega|\nabla\theta|^2+l\int_0^t\int_\Omega\nabla\theta\nabla\chi=0. \tag{6.16}$$

By multiplying (6.15) by $1+\frac12 l^2$ and adding with (6.16), we easily obtain

$$\begin{aligned}&\int_\Omega\big(\frac16(\theta(t))^2+\frac12(\chi(t))^2\big)+\frac12\int_0^t\int_\Omega(|\nabla\theta|^2+|\nabla\chi|^2)\\&\le C\int_0^t(\|\theta\|^2_{L^2(\Omega)}+\|\chi\|^2_{L^2(\Omega)}+\|u\|^2_{H^\alpha(\Omega)}).\end{aligned}\tag{6.17}$$

Finally, we test equation (6.14) by u_t and integrate in time to obtain

$$\frac12\int_\Omega|u_t(t)|^2+\nu\int_0^t\|u_t\|^2_{H^\alpha(\Omega)}\le I_5+I_6, \tag{6.18}$$

where

$$I_5=-\int_0^t\langle\mathcal{H}((1-\chi_1)u),u_t\rangle\le C\int_0^t\|1-\chi_1\|_{L^\infty(\Omega)}\|u\|_{H^1(\Omega)}\|u_t\|_{H^1(\Omega)} \tag{6.19}$$

and

$$\begin{aligned}I_6&=\int_0^t\langle\mathcal{H}(\chi u_2)-\mathcal{K}(\chi\partial_t u_2),u_t\rangle\\&\le C\int_0^t(\|u_2\|_{H^\alpha(\Omega)}+\|\partial_t u_2\|_{H^\alpha(\Omega)})\|\chi\|_{L^2(\Omega)}\|u_t\|_{H^\alpha(\Omega)}.\end{aligned}\tag{6.20}$$

From (6.18)-(6.20), we easily obtain

$$\begin{aligned}&\frac12\int_\Omega|u_t(t)|^2+C\int_0^t\|u_t\|^2_{H^\alpha(\Omega)}\\&\le\epsilon\int_0^t\|u_t\|^2_{H^\alpha(\Omega)}+C_\epsilon\int_0^t(\|u_2\|^2_{H^\alpha(\Omega)}+\|\partial_t u_2\|^2_{H^\alpha(\Omega)})\|\chi\|^2_{L^2(\Omega)}\\&\quad+C_\epsilon\int_0^t\|u\|^2_{H^\alpha(\Omega)}.\end{aligned}\tag{6.21}$$

Now, by adding (6.17) and (6.21) and choosing $\epsilon > 0$ small enough and also using Gronwall's lemma and inequality (4.14), we obtain

$$\begin{aligned}&\|\theta\|_{L^\infty(0,T;L^2(\Omega)\cap L^2(0,T;H^1(\Omega)} + \|\chi\|_{L^\infty(0,T;L^2(\Omega)\cap L^2(0,T;H^1(\Omega)}\\&+ \|u\|_{W^{1,\infty}(0,T;L^2(\Omega)\cap H^1(0,T;H^\alpha(\Omega)} \le 0.\end{aligned}$$

and therefore $\theta = \chi = u = 0$. To proof that $\xi = 0$, we just note that

$$\xi = h(\theta_1) - h(\theta_2) + \frac{\eta(u_1) : \eta(u)}{2} + \frac{\eta(u) : \eta(u_2)}{2} - \chi_t - A_N\chi - (\gamma(\chi_1) - \gamma(\chi_2)).$$

We conclude with these results that the Problem 2.2 has a unique solution, and therefore Theorem 2.3 is proved.

Remark 6.1. When the solidification process occur in a constant gravitational field $\vec{a}$, one may include the action of the buoyancy forces resulting from the small variations in the density due to differences in temperature. This is usually done by adding a further force in the momentum equation using the Boussinesq approximation $\theta\chi\vec{a}$ (with the reference temperature taken as zero). In this case, the equations for the problem are substituted by the following equations:

$$\begin{gathered}\theta_t + l\chi_t - \Delta\theta = g \quad \text{in } \Omega \times (0,T),\\ \chi_t - \Delta\chi + W'(\chi) \ni h(\theta - \theta_c) + \frac{|\eta(u)|^2}{2} \quad \text{in } \Omega \times (0,T),\\ u_{tt} - \operatorname{div}((1-\chi)\eta(u) + \chi\eta(u_t)) + \nu(-\Delta)^2 u_t + \theta\chi\vec{a} = f \quad \text{in } \Omega \times (0,T),\end{gathered}$$

subjected to the same boundary and an initial conditions as before.

The same sort of generalizations could be considered in this case, and exactly the same results as before hold true. In fact, the Boussinesq term $\theta\chi\vec{a}$ brings no further difficulties in the derivation of the estimates.

7. Considerations concerning modeling

In this section we comment on modeling aspects by following arguments that are very similar to the ones presented in Rocca and Rossi [21, pp. 3332-335]; specifically, by using the generalized Principle of Virtual Power introduced by Frémond (cf. [13]), we derive a model closely related to the one considered in this article . For simplicity of exposition and to ease the comparison with the model in [21], in this section we take $h(\theta) = \theta$ in equation (1.2).

We start by taking a free energy functional of form

$$\psi(u(\cdot), \chi(\cdot), \theta(\cdot)) = \int_\Omega \Psi(\eta(u), \chi, \nabla\chi, \theta)\, dx,$$

where the volumetric free energy density Ψ is exactly as in Rocca and Rossi [21, p. 3332]:

$$\begin{aligned}\Psi(\eta(u), \chi, \nabla\chi, \theta) = c_V\theta(1 - \log\theta) - \frac{\lambda}{\theta_C}(\theta - \theta_C)\chi + \frac{(1-\chi)\eta(u)\mathcal{R}_e\eta(u)}{2}\\ + W(\chi) + \frac{\nu}{2}|\nabla\chi|^2,\end{aligned}$$

where for simplicity of exposition we assume that $\partial W()$ is univalent.

Next, we take a pseudo-potential of dissipation as

$$\phi(u(\cdot), \chi(\cdot), \theta(\cdot)) = \int_\Omega \Phi(\eta(u_t), \chi_t, \nabla\theta)\, dx,$$

where the density of the pseudo-potential of dissipation Φ is of the form

$$\Phi(\eta(u_t), \chi_t, \nabla\theta) = \Phi_{RR}(\eta(u_t), \chi_t, \nabla\theta) + \frac{1}{2}|\mathcal{L}(\eta(u_t))|^2,$$

where $\mathcal{L} : D(\mathcal{L}) \subset (L^2(\Omega))^n \to (L^2(\Omega))^n$ is a linear operator, and Φ_{RR} is exactly the the density of the pseudo-potential of dissipation used by Rocca and Rossi in [21, p. 3332]:

$$\Phi_{RR}(\eta(u_t), \chi_t, \nabla\theta) = \frac{1}{2}|\chi_t|^2 + \frac{\chi}{2}\eta(u_t)\mathcal{R}_v\eta(u_t) + \frac{|\nabla\theta|^2}{2\theta}$$

Also as in [21], hereafter, for simplicity in notations, we set $c_V = \nu = \lambda/\theta_C = 1$ and incorporate the term $\chi\theta_C$ in $W(\chi)$.

To derive the equations for the model, we recall that the equation for the macroscopic motion (momentum equation) can be obtained by the principle of virtual power, which gives

$$u_{tt} - div\ \sigma = f \quad \text{in } \Omega \times (0, T), \tag{7.1}$$

where f is the exterior volume force and σ is the stress tensor given by the sum of the non dissipative stress given by the constitutive law $\sigma^{nd} = D_{\eta(u)}\Psi$ and the dissipative stress given by the constitutive law $\sigma^d = D_{\eta(u)}\Phi = D_{\eta(u)}\Phi_{RR} + \mathcal{L}^*\mathcal{L}(\eta(u_t))$, where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. Thus, we obtain the stress tensor as

$$\begin{aligned}\sigma &= \sigma^{nd} + \sigma^d = D_{\eta(u)}\Psi + D_{\eta(u)}\Phi \\ &= (1-\chi)\mathcal{R}_e\eta(u) + \big(\chi\mathcal{R}_v\eta(u_t) + \mathcal{L}^*\mathcal{L}(\eta(u_t))\big).\end{aligned}$$

Thus, the momentum equation (7.1) becomes

$$u_{tt} - \operatorname{div}((1-\chi)\mathcal{R}_e\eta(u) + \chi\mathcal{R}_v\eta(u_t)) - \operatorname{div}(\mathcal{L}^*\mathcal{L}(\eta(u_t))) = f$$

Next, according to the generalized principle of virtual power [13], the equation for the microscopic motions is given by

$$B - \operatorname{div}(\mathbf{H}) = 0, \tag{7.2}$$

where B is the sum of the a non dissipative part, $B^{nd} = D_\chi\Psi$ and a dissipative part, $B^d = D_{\chi_t}\Phi = D_{\chi_t}\Phi_{RR}$. Then, B is given by

$$B = B^{nd} + B^d = D_\chi\Psi + D_{\chi_t}\Phi = \Big(-\theta - \frac{\eta(u)\mathcal{R}_e\eta(u)}{2} + W'(\chi)\Big) + \chi_t,$$

and $\mathbf{H}$ is the sum of the a non dissipative part, $\mathbf{H}^{nd} = D_{\nabla\chi}\Psi$, and a dissipative part, $\mathbf{H}d = D_{\nabla\chi}\Phi = D_{\nabla\chi}\Phi_{RR}$. Thus, $\mathbf{H}$ is given by

$$\mathbf{H} = \mathbf{H}^{nd} + \mathbf{H}^d = D_{\nabla\chi}\Psi + D_{\nabla\chi}\Phi = \nabla\chi + 0.$$

Therefore, the equation for the microscopic motions (7.2) is reduced to

$$\chi_t - \Delta\chi + W'(\chi) = \theta + \frac{\eta(u)\mathcal{R}_e\eta(u)}{2}$$

Next, the internal energy equation is

$$e_t + \operatorname{div}\mathbf{q} = g + \sigma : \eta(u_t) + B\chi_t + \mathbf{H}\cdot\nabla\chi_t, \tag{7.3}$$

where the internal energy density is

$$e = \Psi + \theta s,$$

where $s = -D_\theta \Psi = -\log\theta - \chi$ is the entropy density of the system, and the heat flux $\mathbf{q}$ is given by the constitutive relation

$$\mathbf{q} = -\theta D_{\nabla\theta}\Phi = -\theta D_{\nabla\theta}\Phi_{RR} = -\nabla\theta.$$

Thus, the internal energy balance equation (7.3) becomes

$$\theta_t + \theta\chi_t - \Delta\theta = g + \chi\eta(u_t)\mathcal{R}_v\eta(u_t) + \mathcal{L}^*\mathcal{L}(\eta(u_t)) : \eta(u_t) + |\chi_t|^2$$

Then, the model derived from the previous potentials is associated with the system of equations

$$\begin{gathered}
\theta_t + \theta\chi_t - \Delta\theta = g + \chi\eta(u_t)\mathcal{R}_v\eta(u_t) + \mathcal{L}^*\mathcal{L}(\eta(u_t)) : \eta(u_t) + |\chi_t|^2 \quad \text{in } \Omega\times(0,T), \\
\chi_t - \Delta\chi + W'(\chi) = \theta + \frac{\eta(u)\mathcal{R}_e\eta(u)}{2} \quad \text{in } \Omega\times(0,T), \\
u_{tt} - \operatorname{div}((1-\chi)\eta(u) + \chi\eta(u_t)) - \operatorname{div}(\mathcal{L}^*\mathcal{L}(\eta(u_t))) = f \quad \text{in } \Omega\times(0,T),
\end{gathered} \tag{7.4}$$

We are not able to prove that this model (7.4) is locally thermodynamically consistent in the sense that the entropy production is nonnegative at any point; however, its total entropy production, weighted by the absolute temperature, is in fact nonnegative; that is, the model satisfies the following global temperature-weighted Clausius-Duhem inequality:

$$\int_\Omega \theta\ \Pi\ dx \geq 0,$$

where θ is the absolute (positive) temperature and Π denotes the density of the entropy production given by

$$\Pi = s_t + \operatorname{div}(\frac{\mathbf{q}}{\theta}) - \frac{g}{\theta}.$$

In fact, we observe that the internal energy equation (7.3) can be rewritten in terms of the entropy s as

$$\theta\Big(s_t + \operatorname{div}(\frac{\mathbf{q}}{\theta}) - \frac{g}{\theta}\Big) = \sigma^d : \eta(u_t) + B^d\chi_t - \frac{\mathbf{q}}{\theta}\cdot\nabla\theta.$$

Thus, by using this last identity and the previous definitions, we can write

$$\begin{aligned}
\int_\Omega \theta\Pi\,dx &= \int_\Omega \theta\Big(s_t + \operatorname{div}(\frac{\mathbf{q}}{\theta}) - \frac{g}{\theta}\Big)\,dx \\
&= \int_\Omega \big(D_{\eta(u)}\Phi_{RR} : \eta(u_t) + D_{\chi_t}\Phi_{RR}\chi_t + D_{\nabla\theta}\Phi_{RR}\cdot\nabla\theta\big)\,dx \\
&\quad + (\mathcal{L}^*\mathcal{L}(\eta(u_t))), \eta(u_t))_{L^2(\Omega)} \\
&= \int_\Omega \big(D_{\eta(u)}\Phi_{RR} : \eta(u_t) + D_{\chi_t}\Phi_{RR}\chi_t + D_{\nabla\theta}\Phi_{RR}\cdot\nabla\theta\big)\,dx \\
&\quad + |\mathcal{L}(\eta(u_t))|^2_{L^2(\Omega)} \geq 0
\end{aligned}$$

because by construction

$$(D_{\chi_t}\Phi_{RR}, D_{\nabla\theta}\Phi_{RR}, D_{\nabla\theta}\Phi_{RR}) \in \partial\Phi_{RR}(\eta(u_t), \chi_t, \nabla\theta),$$

$(0,0,0) \in \partial\Phi_{RR}(0,0,0)$ and Φ_{RR} is convex in all its variables; that is, the expression in the integral at the right-hand side of the last inequality is nonnegative.

As in Rocca and Rossi [21], by using the small perturbation assumption, see Germani [14], that the dissipative heat sources in the energy balance are small with respect to the external heating g; the higher order dissipative terms on the right-hand side of the first equation in (7.4) can be neglected. Assuming also that the latent heat is approximately constant l (take for instance $l = \theta_M$, where θ_M is the melting temperature), from (7.4) we derive the following perturbed model:

$$\begin{gathered}\theta_t + l\chi_t - \Delta\theta = g \quad \text{in } \Omega \times (0,T),\\ \chi_t - \Delta\chi + W'(\chi) = \theta + \frac{\eta(u)\mathcal{R}_e\eta(u)}{2} \quad \text{in } \Omega \times (0,T),\\ u_{tt} - \operatorname{div}((1-\chi)\eta(u) + \chi\eta(u_t)) - \operatorname{div}(\mathcal{L}^*\mathcal{L}(\eta(u_t))) = f \quad \text{in } \Omega \times (0,T),\end{gathered} \tag{7.5}$$

The total entropy production, weighted by the absolute temperature is also non-negative for this approximate model for slow solutions in the sense that χ_t is small enough such that

$$\int_\Omega \theta\Pi\,dx = \int_\Omega \theta\big(s_t + \operatorname{div}(\frac{\mathbf{q}}{\theta}) - \frac{g}{\theta}\big) = \int_\Omega \Big(\frac{|\nabla\theta|^2}{\theta^2} + (\theta_M - \theta)\chi_t\,dx\Big) \geq 0.$$

Next, we compare model (7.5) with the one analyzed in this paper. First, take $\mathcal{R}_e$ and $\mathcal{R}_v$ as identities and $\mathcal{L}(\cdot) = \nu^{1/2}(-\Delta)^{1/2}(\cdot)$, then the last term in the right-hand side of the third equation in (7.4) becomes $\operatorname{div}(\mathcal{L}^*\mathcal{L}(\eta(u_t))) = -\nu\operatorname{div}(\Delta\eta(u_t)) = -\nu\Delta\operatorname{div}(\eta(u_t))$. In this case, (7.5) becomes

$$\begin{gathered}\theta_t + l\chi_t - \Delta\theta = g \quad \text{in } \Omega \times (0,T),\\ \chi_t - \Delta\chi + W'(\chi) = \theta + \frac{|\eta(u)|^2}{2} \quad \text{in } \Omega \times (0,T),\\ u_{tt} - \operatorname{div}((1-\chi)\eta(u) + \chi\eta(u_t)) + \nu\Delta\operatorname{div}(\eta(u_t)) = f \quad \text{in } \Omega \times (0,T),\end{gathered} \tag{7.6}$$

which is expected to be mathematically related to the system presented in the Introduction. In fact, the term $h\theta_c$ can be incorporated in $W(\chi)$ by adding $h\theta_c\chi$ to it, and, since $\eta(u_t)$ is the symmetric part of ∇u_t, the term $\nu\Delta\operatorname{div}(\eta(u_t))$ in the third equation of (7.6) is mathematically related to the term $\nu(-\Delta)^2u_t$ in equation (1.3). Currently we are analyzing system (7.6); we remark, however, that the term $\nu\Delta\operatorname{div}(\eta(u_t))$ is harder to handle than the one present in (1.3).

Acknowledgments. W. V. Assunção was supported by grant 3608-04/2011 from CAPES/CEX/PNPD (Brazil). J. L. Boldrini was supported by grant 305467/2011-5 from CNPq (Brazil), and grant 2009/15098-0 from FAPESP (Brazil).

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