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BLOW-UP SOLUTIONS FOR A NONLINEAR WAVE EQUATION WITH POROUS ACOUSTIC BOUNDARY CONDITIONS

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ABSTRACT. We study a nonlinear wave equation with porous acoustic boundary conditions in a bounded domain. We prove a finite time blow-up for certain solutions with positive initial energy.

1. INTRODUCTION

We consider the following system of nonlinear wave equations with porous acoustic boundary conditions:

$$u_{tt} - \Delta u + \alpha(x)u + \phi(u_t) = j_1(u) \quad \text{in } \Omega \times [0, T), \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \Gamma_0 \times (0, T), \tag{1.2}$$

$$u_t(x,t) + f(x)z_t + g(x)z = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{1.3}$$

$$\frac{\partial u}{\partial \nu} - h(x)z_t + \rho(u_t) = j_2(u) \quad \text{on } \Gamma_1 \times (0, T), \tag{1.4}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.5}$$

$$z(x,0) = z_0(x), \quad x \in \Gamma_1, \tag{1.6}$$

where Ω is a bounded domain in R^n $(n \geq 1)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint. Let ν be the unit normal vector pointing to the exterior of Ω and $\alpha : \Omega \to \mathbb{R}$, $f, g, h : \overline{\Gamma_1} \to \mathbb{R}$ and $j_1, j_2 : \mathbb{R} \to \mathbb{R}$ are given functions.

The system (1.1)–(1.6) is a model of nonlinear wave equations with acoustic boundary conditions which are described by (1.3) and (1.4). These boundary conditions were introduced by Morse and Ingard [12] and developed by Beale and Rosencrans [2, 3, 4]. In recent years, questions related to wave equations with acoustic boundary conditions have been treated by many authors [5, 10, 11, 13, 14, 15, 16]. For example, Frota and Larkin [11] studied (1.1)–(1.6) with $\phi = \rho = j_1 = j_2 = 0$ and they established the exponential decay result for suitably defined solutions. Recently, as $j_1 = j_2 = 0$, Graber [6, 7] showed that the systems (1.1)-(1.6) generates a well-posed dynamical system by using semigroup theory. When one considers the presence of the double interaction between source and damping terms, both in the interior of Ω and on the boundary Γ_1, the analysis becomes more difficult. Very

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recently, Graber and Said-Houari [8] studied this challenging problem and obtained several results in local existence, global existence, the decay rate and blow-up results. Particularly, in the absence of boundary source, that is $j_2 = 0$, for certain initial data, the authors proved that the solution is unbounded and grows as an exponential function. However, the possibility of the solution that blows up in finite time is not addressed in that paper. Therefore, the intention of this paper is to investigate the blow-up phenomena of solutions for system (1.1)-(1.6) without imposing the boundary source. In this way, we can extend this unbounded result of [8] to a blow-up result with positive initial energy.

The content of this paper is organized as follows. In section 2, we state the local existence result and the energy identity which is crucial in establishing the blow-up result in finite time. In section 3, we study the blow-up problem for the initial energy being positive.

2. Preliminaries

In this section, we present some material which will be used throughout this work. First, we introduce the set

$$H^1_{\Gamma_0} = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\},$$

and endow $H^1_{\Gamma_0}$ with the Hilbert structure induced by $H^1(\Omega)$, we have that $H^1_{\Gamma_0}$ is a Hilbert space. For simplicity, we denote $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{p,\Gamma} = \|\cdot\|_{L^p(\Gamma)}$, $1 \le p \le \infty$, $\|u\|^2_\alpha = \|\nabla u\|^2_2 + \int_\Omega \alpha(x)u^2(x)dx$ and $\|u\|^2_{gh} = \int_{\Gamma_1} g(x)h(x)u^2(x)d\Gamma$. The following assumptions for problem (1.1)-(1.6) were used in [8].

- (A1) The functions $j_1(s) = |s|^{p-1}s$ and $j_2(s) = 0$, where $p \ge 1$ is such that $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{p+1}(\Omega)$.
- (A2) $\phi, \rho : \mathbb{R} \to \mathbb{R}$ are continuous and increasing functions with $\phi(0) = \rho(0) = 0$. In addition, there exist positive constants a_i and b_i $i = r, q$ such that

$$a_r|s|^{r+1} \le \phi(s)s \le b_r|s|^{r+1}, \ r \ge 1, \tag{2.1}$$

$$a_q|s|^{q+1} \le \rho(s)s \le b_q|s|^{q+1}, \ q \ge 1. \tag{2.2}$$

- (A3) The functions α, f, g, h are essentially bounded such that $f > 0$, $g > 0$, $h > 0$ and $\alpha \ge 0$. (If $\alpha = 0$, Γ_0 is assumed to have a non-empty interior such that the Poincarè inequality is applicable.)

Next, the energy function associated with problem (1.1)–(1.6), with $j_2 = 0$, is defined as

$$E(t) = \frac{1}{2}\|u_t\|^2_2 + \frac{1}{2}\Big(\|u\|^2_\alpha + \|z\|^2_{gh}\Big) - \frac{1}{p+1}\|u\|^{p+1}_{p+1}. \tag{2.3}$$

Then, we are ready to state the following local existence result and energy identity.

Lemma 2.1 ([8]). *Suppose that* (A1)–(A3) *hold, and that* $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$ *and* $z_0 \in L^2(\Gamma_1)$. *Then the system* (1.1)–(1.6) *with* $j_2 = 0$ *admits a unique solution* (u, z) *such that, for* $T > 0$,

$$u \in C([0,\ T); H^1_{\Gamma_0}(\Omega)) \cap C^1([0,\ T); L^2(\Omega)), \quad z \in C([0,T); L^2(\Gamma_1)).$$

Moreover, the energy satisfies

$$E(0) = E(T) + \int_0^T \int_\Omega \phi(u_t)u_t dxdt + \int_0^T \int_{\Gamma_1} (\rho(u_t)u_t + fhz_t^2)d\Gamma dt. \tag{2.4}$$

Note that (2.4) shows that the energy is a non-increasing function along trajectories.

3. Blow-up of Solutions

In this section, we state and prove our main result. First, we define a functional G which helps in establishing desired results. Let

$$G(x) = \frac{1}{2}x^2 - \frac{B_1^{p+1}}{p+1}x^{p+1}, \quad x > 0,$$

where $B_1^{-1} = \inf\{\|\nabla u\|_2 : u \in H^1_{\Gamma_0}(\Omega),\ \|u\|_{p+1} = 1\}$. Then, G has a maximum at $\lambda_1 = B_1^{-\frac{p+1}{p-1}}$ with the maximum value

$$E_1 \equiv G(\lambda_1) = (\frac{1}{2} - \frac{1}{p+1})\lambda_1^2.$$

The next Lemma will play an important role in proving our result.

Lemma 3.1 ([8]). *Suppose that* (A1)–(A3) *hold, and that* $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$ *and* $z_0 \in L^2(\Gamma_1)$. *Let* (u, z) *be a solution of* (1.1)–(1.6) *with* $j_2 = 0$. *Assume that* $E(0) < E_1$ *and* $\|\nabla u_0\|_2 > \lambda_1$. *Then there exists* $\lambda_2 > \lambda_1$ *such that, for all* $t \geq 0$,

$$(\|u\|_\alpha^2 + \|z\|_{gh}^2)^{1/2} \geq \lambda_2, \tag{3.1}$$

$$\|u\|_{p+1} \geq B_1\lambda_2. \tag{3.2}$$

Now, we are ready to state and prove our main result. Our proof technique follows the arguments of [8] and some estimates obtained in [9].

Theorem 3.2. *Suppose that* (A1)–(A3) *hold, and that* $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$ $z_0 \in L^2(\Gamma_1)$. *Assume further that* $p > \max(r, 2q - 1)$. *Then any solution of* (1.1)-(1.6) *with* $j_2 = 0$*and satisfying* $E(0) < E_1$ *and* $\|\nabla u_0\|_2 > \lambda_1$ *blows up at a finite time.*

Proof. We suppose that the solution exists for all time and we reach to a contradiction. To achieve this, we set

$$H(t) = E_1 - E(t), \quad t \geq 0. \tag{3.3}$$

Then, by (2.4), we see that $H'(t) \geq 0$. From (3.1), the definition of $E(t)$ and $E_1 = (\frac{1}{2} - \frac{1}{p+1})\lambda_1^2$, we deduce that, for all $t \geq 0$,

$$0 < H(0) \leq H(t) \leq E_1 - \frac{1}{2}\lambda_2^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \leq \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \tag{3.4}$$

Let

$$A(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega uu_t dx - \frac{\varepsilon}{2}\int_{\Gamma_1} fhz^2 d\Gamma - \varepsilon\int_{\Gamma_1} huz d\Gamma, \tag{3.5}$$

where ε is a positive constant to be specified later and

$$0 < \sigma < \min\{\frac{p-r}{r(p+1)}, \frac{p-1}{2(p+1)}\}. \tag{3.6}$$

Then

$$\begin{aligned} A'(t) &= (1-\sigma)H(t)^{-\sigma}H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|u\|_\alpha^2 + \varepsilon\|u\|_{p+1}^{p+1} \\ &\quad + \varepsilon\int_{\Gamma_1} hg|z(x,t)|^2 d\Gamma - \varepsilon\int_\Omega u\phi(u_t)dx \\ &\quad - \varepsilon\int_{\Gamma_1} u\rho(u_t)d\Gamma + 2\varepsilon H(t) - 2\varepsilon H(t) \\ &= (1-\sigma)H(t)^{-\sigma}H'(t) + 2\varepsilon\|u_t\|_2^2 + \frac{\varepsilon(p-1)}{p+1}\|u\|_{p+1}^{p+1} + 2\varepsilon\|z\|_{gh}^2 \\ &\quad - \varepsilon\int_\Omega u\phi(u_t)dx - \varepsilon\int_{\Gamma_1} u\rho(u_t)d\Gamma - 2\varepsilon E_1 + 2\varepsilon H(t). \end{aligned} \tag{3.7}$$

We observe from (3.2) that

$$E_1 = E_1(B_1^{p+1}\lambda_2^{p+1})(B_1^{p+1}\lambda_2^{p+1})^{-1} \le E_1\|u\|_{p+1}^{p+1}(B_1^{p+1}\lambda_2^{p+1})^{-1}. \tag{3.8}$$

Inserting (3.8) into (3.7), we have

$$\begin{aligned} A'(t) &\ge (1-\sigma)H(t)^{-\sigma}H'(t) + 2\varepsilon\|u_t\|_2^2 + \varepsilon c_1\|u\|_{p+1}^{p+1} + 2\varepsilon\|z\|_{gh}^2 \\ &\quad - \varepsilon\int_\Omega u\phi(u_t)dx - \varepsilon\int_{\Gamma_1} u\rho(u_t)d\Gamma + 2\varepsilon H(t), \end{aligned} \tag{3.9}$$

where

$$c_1 = \frac{p-1}{p+1} - 2E_1(B_1^{p+1}\lambda_2^{p+1})^{-1} > \frac{p-1}{p+1} - 2E_1(B_1^{p+1}\lambda_1^{p+1})^{-1} = 0.$$

By (2.1), Hölder inequality and Young's inequality, we see that, for $\delta_1 > 0$,

$$\Big|\int_\Omega u\phi(u_t)dx\Big| \le \frac{b_r\delta_1^{r+1}}{r+1}\|u\|_{r+1}^{r+1} + \frac{b_r r\delta_1^{-\frac{r+1}{r}}}{r+1}\|u_t\|_{r+1}^{r+1}, \tag{3.10}$$

A substitution of (3.10) into (3.9) leads to

$$\begin{aligned} A'(t) &\ge (1-\sigma)H(t)^{-\sigma}H'(t) + 2\varepsilon\|u_t\|_2^2 + \varepsilon c_1\|u\|_{p+1}^{p+1} \\ &\quad + 2\varepsilon\|z\|_{gh}^2 - \varepsilon\int_{\Gamma_1} u\rho(u_t)d\Gamma \\ &\quad - \varepsilon\Big(\frac{b_r\delta_1^{r+1}}{r+1}\|u\|_{r+1}^{r+1} + \frac{b_r r\delta_1^{-\frac{r+1}{r}}}{r+1}\|u_t\|_{r+1}^{r+1}\Big) + 2\varepsilon H(t), \end{aligned} \tag{3.11}$$

At this point, for a large positive constant M_1 to be chosen later, picking δ_1 such that $\delta_1^{-\frac{r+1}{r}} = M_1 H(t)^{-\sigma}$ and using the fact

$$H'(t) \ge a_r\|u_t\|_{r+1}^{r+1} + a_q\|u_t\|_{q+1,\Gamma}^{q+1} + \int_{\Gamma_1} fhz_t^2 d\Gamma \tag{3.12}$$

by (2.4) and (A1) we have

$$\begin{aligned} A'(t) &\ge \big(1-\sigma-\frac{\varepsilon r b_r M_1}{a_r(r+1)}\big)H(t)^{-\sigma}H'(t) + 2\varepsilon\|u_t\|_2^2 + \varepsilon c_1\|u\|_{p+1}^{p+1} \\ &\quad + 2\varepsilon\|z\|_{gh}^2 - \varepsilon\int_{\Gamma_1} u\rho(u_t)d\Gamma - \frac{\varepsilon b_r M_1^{-r}}{r+1}H(t)^{\sigma r}\|u\|_{r+1}^{r+1} + 2\varepsilon H(t). \end{aligned} \tag{3.13}$$

In addition, using (3.4) and the inequality

$$\chi^\gamma \le \chi + 1 \le \big(1+\frac{1}{\omega}\big)(\chi+\omega), \quad \forall \chi \ge 0,\ 0 < \gamma \le 1,\ \omega > 0,$$

with $\chi = \frac{1}{p+1}\|u\|_{p+1}^{p+1}$ and $\omega = H(0)$ and noting that $p > r$, and $0 < \sigma r + \frac{r+1}{p+1} \le 1$ by (3.6), we have

$$\begin{aligned} H(t)^{\sigma r}\|u\|_{r+1}^{r+1} &\le c_2 H(t)^{\sigma r}\Big(\|u\|_{p+1}^{p+1}\Big)^{\frac{r+1}{p+1}} \\ &\le c_3\Big(\frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big)^{\sigma r + \frac{r+1}{p+1}} \\ &\le c_3 d\Big(\frac{1}{p+1}\|u\|_{p+1}^{p+1} + H(t)\Big), \end{aligned} \tag{3.14}$$

where $c_2 = \mathrm{vol}(\Omega)^{\frac{p-r}{p+1}}$ and $c_3 = (p+1)^{\frac{r+1}{p+1}} \cdot c_2$ and $d = 1 + \frac{1}{H(0)}$. Combining (3.14) with (3.13), we obtain

$$\begin{aligned} A'(t) \ge (1 - \sigma - \frac{\varepsilon r b_r M_1}{a_r(r+1)})H(t)^{-\sigma}H'(t) + 2\varepsilon\|u_t\|_2^2 + \varepsilon(c_1 - c_4)\|u\|_{p+1}^{p+1} \\ + 2\varepsilon\|z\|_{gh}^2 + \varepsilon(2 - (p+1)c_4)H(t) - \varepsilon\int_{\Gamma_1} u\rho(u_t)d\Gamma. \end{aligned} \tag{3.15}$$

with $c_4 = \frac{b_r c_3 d M_1^{-r}}{(p+1)(r+1)}$. Next, we will follow the arguments as in [9] to estimate the last term on the right hand side of (3.15). For this purpose, let us recall the following trace and interpolation theorems [1, 17]

$$\|u\|_{q+1,\Gamma} \le C\|u\|_{W^{s,q+1}}, \tag{3.16}$$

which holds for some positive constant C, $q \ge 0$, $0 < s < 1$ and $s > \frac{1}{q+1}$.

$$W^{1-\theta,\tau}(\Omega) = [H^1(\Omega), L^{p+1}(\Omega)]_\theta, \tag{3.17}$$

where $\frac{1}{\tau} = \frac{1-\theta}{2} + \frac{\theta}{p+1}$, $\theta \in [0,1]$ and $[\cdot,\cdot]_\theta$ denotes the interpolation bracket. We note from $q \ge 1$ and $p > 2q-1$ that $\frac{1}{q+1} \le \frac{p-1}{2(p-q)} < 1$. Then, we choose β satisfying

$$\frac{p-1}{2(p-q)} \le \beta < 1 \tag{3.18}$$

and select θ such that

$$1 - \theta = \frac{1}{\beta(q+1)}, \quad \tau = \frac{2(p+1)}{(1-\theta)(p+1) + 2\theta},$$

which imply that $1 - \theta > \frac{1}{q+1}$ and $\tau \ge q+1$. From (3.16), (3.17) and Young's inequality, we have

$$\begin{aligned} \|u\|_{q+1,\Gamma} &\le C\|u\|_{W^{1-\theta,q+1}(\Omega)} \le C\|u\|_{W^{1-\theta,\tau}(\Omega)} \le C\|u\|_\alpha^{1-\theta}\|u\|_{p+1}^\theta \\ &= C\|u\|_\alpha^{\frac{1}{\beta(q+1)}}\|u\|_{p+1}^{1-\frac{1}{\beta(q+1)}} \\ &\le C\Big(\|u\|_\alpha^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{2\beta^2(q+1)-2\beta}{(2\beta^2-1)(q+1)}}\Big), \end{aligned}$$

where C is a generic positive constant. Further, as in [9], there exists β satisfying (3.18) such that $\frac{2\beta^2(q+1)-2\beta}{(2\beta^2-1)(q+1)} = \frac{(p+1)\beta}{q+1}$. Thus,

$$\|u\|_{q+1,\Gamma} \le C\Big(\|u\|_\alpha^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}}\Big). \tag{3.19}$$

Besides, we observe from the definition of $E(t)$ by (2.3) and $E(0) < E_1$ that $\frac{1}{2}\|u\|_\alpha^2 \le E_1 + \frac{1}{p+1}\|u\|_{p+1}^{p+1}$. Hence, by (2.2) and (3.19), we have

$$\begin{aligned}|\int_{\Gamma_1} u\rho(u_t)d\Gamma| &\le b_q\|u\|_{q+1,\Gamma}\|u_t\|_{q+1,\Gamma}^q \le c_5(\|u\|_\alpha^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}})\|u_t\|_{q+1,\Gamma}^q \\ &\le c_6(E_1 + \frac{1}{p+1}\|u\|_{p+1}^{p+1})^{\frac{\beta}{q+1}}\|u_t\|_{q+1,\Gamma}^q,\end{aligned}$$

with $c_5 = b_qC$ and $c_6 = c_5(2^{\frac{\beta}{q+1}} + (p+1)^{\frac{\beta}{q+1}})$. Thus, using Young's inequality, further requiring σ such that $0 < \sigma < \frac{1-\beta}{q+1}$, and exploiting (3.4) and (3.12), we obtain, for $\delta > 0$,

$$\begin{aligned}&|\int_{\Gamma_1} u\rho(u_t)d\Gamma| \\ &\le c_6\Big(E_1 + \frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big)^{\frac{\beta-1}{q+1}}\Big(E_1 + \frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big)^{\frac{1}{q+1}}\|u_t\|_{q+1,\Gamma}^q \\ &\le c_6\Big(E_1 + \frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big)^{\frac{\beta-1}{q+1}}\Big[\delta\Big(E_1 + \frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big) + c_\delta\|u_t\|_{q+1,\Gamma}^{q+1}\Big] \\ &\le c_6\delta H(0)^{\frac{\beta-1}{q+1}}\Big(E_1 + \frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big) + c_\delta c_6 a_q^{-1}H(0)^{\frac{\beta-1}{q+1}+\sigma}H(t)^{-\sigma}H'(t).\end{aligned}$$

Thus, (3.15) becomes

$$\begin{aligned}A'(t) &\ge (1-\sigma-\varepsilon c_7)H(t)^{-\sigma}H'(t) + 2\varepsilon\|u_t\|_2^2 \\ &\quad + \varepsilon\Big(c_1 - c_4 - \frac{c_6\delta H(0)^{\frac{\beta-1}{q+1}}}{p+1}\Big)\|u\|_{p+1}^{p+1} \\ &\quad + 2\varepsilon\|z\|_{gh}^2 + \varepsilon(2-(p+1)c_4)H(t) - c_6\delta\varepsilon H(0)^{\frac{\beta-1}{q+1}}E_1,\end{aligned}$$

where $c_7 = \frac{rb_rM_1}{a_r(r+1)} + c_\delta c_6 a_q^{-1}H(0)^{\frac{\beta-1}{q+1}+\sigma}$. Employing the estimate (3.8) again, we arrive at

$$\begin{aligned}A'(t) &\ge (1-\sigma-\varepsilon c_7)H(t)^{-\sigma}H'(t) + 2\varepsilon\|u_t\|_2^2 + \varepsilon(c_1 - c_4 - \delta c_8)\|u\|_{p+1}^{p+1} \\ &\quad + 2\varepsilon\|z\|_{gh}^2 + \varepsilon(2-(p+1)c_4)H(t),\end{aligned}$$

where

$$c_8 = c_6H(0)^{\frac{\beta-1}{q+1}}(\frac{1}{p+1} + E_1(B_1^{p+1}\lambda_2^{p+1})^{-1}).$$

Now, we choose M_1 large enough such that $2-(p+1)c_4 > 0$ and $c_1 - c_4 > \frac{c_1}{2}$. Once M_1 is fixed, we select δ small enough such that $\frac{c_1}{2} - \delta c_8 > 0$. Then, pick ε small enough such that $1-\sigma-\varepsilon c_7 \ge 0$ and $A(0) > 0$. Thus, there exists $K > 0$ such that

$$\begin{gathered}A'(t) \ge \varepsilon K(\|u_t\|_2^2 + \|u\|_{p+1}^{p+1} + H(t) + \|z\|_{gh}^2), \\ A(t) \ge A(0) > 0, \quad \text{for } t \ge 0.\end{gathered} \tag{3.20}$$

On the other hand, from the result of Graber et al [8, Lemma 6.5], we have

$$A(t)^{\frac{1}{1-\sigma}} \le c_9\Big(\|u_t\|_2^2 + \|u\|_{p+1}^{p+1} + H(t) + \|z\|_{gh}^2\Big), \quad t \ge 0. \tag{3.21}$$

Combining (3.21) with (3.20), we obtain

$$A'(t) \ge c_{10}A(t)^{\frac{1}{1-\sigma}}, \; t \ge 0, \tag{3.22}$$

where c_i, $i = 9, 10$, are positive constants. Thus, inequality (3.22) leads to a blow-up result in a finite time T with

$$0 < T \leq \frac{1-\sigma}{c_{10}\sigma A(0)^{\frac{\sigma}{1-\sigma}}}.$$

The proof is complete. □

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