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STOKES PROBLEM WITH SEVERAL TYPES OF BOUNDARY CONDITIONS IN AN EXTERIOR DOMAIN

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Abstract. In this article, we solve the Stokes problem in an exterior domain of $\mathbb{R}^3$, with non-standard boundary conditions. Our approach uses weighted Sobolev spaces to prove the existence, uniqueness of weak and strong solutions. This work is based on the vector potentials studied in [7] for exterior domains, and in [1] for bounded domains. This problem is well known in the classical Sobolev spaces $W^{m,2}(\Omega)$ when Ω is bounded; see [3, 4].

1. Introduction and functional setting

Let Ω' denotes a bounded open in $\mathbb{R}^3$ of class $C^{1,1}$, simply-connected and with a connected boundary $\partial\Omega' = \Gamma$, representing an obstacle and Ω is its complement; i.e. $\Omega = \mathbb{R}^3 \setminus \overline{\Omega'}$. Then a unit exterior normal vector to the boundary can be defined almost everywhere on Γ; it is denoted by $\mathbf{n}$. The purpose of this paper is to solve the Stokes equation in Ω, with two types of non standard boundary conditions on Γ:

$$\begin{gathered} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{u} = \chi \quad \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} = g \quad \text{and} \quad \mathbf{curl}\,\mathbf{u}\times\mathbf{n} = \mathbf{h}\times\mathbf{n} \quad \text{on } \Gamma, \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{u} = \chi \quad \text{in } \Omega, \\ \pi = \pi_0, \quad \mathbf{u}\times\mathbf{n} = \mathbf{g}\times\mathbf{n} \quad \text{on} \quad \Gamma \quad \text{and} \quad \int_\Gamma \mathbf{u}\cdot\mathbf{n}\,d\sigma = 0. \end{gathered} \tag{1.2}$$

Since this problem is posed in an exterior domain, our approach is to use weighted Sobolev spaces. Let us begin by introducing these spaces. A point in Ω will be denoted by $\mathbf{x} = (x_1, x_2, x_3)$ and its distance to the origin by $r = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. We will use the weights

$$\rho = \rho(r) = (1 + r^2)^{1/2}.$$

For all m in $\mathbb{N}$ and all k in $\mathbb{Z}$, we define the weighted space

$$W_k^{m,2}(\Omega) = \{u \in \mathcal{D}'(\Omega) : \forall\lambda \in \mathbb{N}^3 : 0 \le |\lambda| \le m,\ \rho(r)^{k-m+|\lambda|} D^\lambda u \in L^2(\Omega)\},$$

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which is a Hilbert space with the norm

$$\|u\|_{W^{m,2}_k(\Omega)} = \Big(\sum_{|\lambda|=0}^{m} \|\rho^{k-m+|\lambda|} D^\lambda u\|^2_{L^2(\Omega)}\Big)^{1/2},$$

where $\|\cdot\|_{L^2(\Omega)}$ denotes the standard norm of $L^2(\Omega)$. We shall sometimes use the seminorm

$$|u|_{W^{m,2}_k(\Omega)} = \Big(\sum_{|\lambda|=m} \|\rho^{k} D^\lambda u\|^2_{L^2(\Omega)}\Big)^{1/2}.$$

In addition, it is established by Hanouzet in [11], for domains with a Lipschitz-continuous boundary, that $\mathcal{D}(\overline{\Omega})$ is dense in $W^{m,2}_k(\Omega)$. We set $\mathring{W}^{m,2}_k(\Omega)$ as the adherence of $\mathcal{D}(\Omega)$ for the norm $\|\cdot\|_{W^{m,2}_k(\Omega)}$. Then, the dual space of $\mathring{W}^{m,2}_k(\Omega)$, denoting by $W^{-m,2}_{-k}(\Omega)$, is a space of distributions. Furthermore, as in bounded domain, we have for $m=1$ or $m=2$,

$$\mathring{W}^{1,2}_k(\Omega) = \{v \in W^{1,2}_k(\Omega),\ \ v = 0 \quad \text{on } \Gamma\},$$
$$\mathring{W}^{2,2}_0(\Omega) = \{v \in W^{2,2}_0(\Omega),\ v = \frac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma\},$$

where $\frac{\partial v}{\partial \mathbf{n}}$ is the normal derivative of v. As a consequence of Hardy's inequality, the following Poincaré inequality holds: for $m=0$ or $m=1$ and for all k in $\mathbb{Z}$ there exists a constant C such that

$$\forall v \in \mathring{W}^{m,2}_k(\Omega), \quad \|v\|_{W^{m,2}_k(\Omega)} \le C|v|_{W^{m,2}_k(\Omega)}; \tag{1.3}$$

i.e., the seminorm $|\cdot|_{W^{m,2}_k(\Omega)}$ is a norm on $\mathring{W}^{m,2}_k(\Omega)$ equivalent to the norm $\|\cdot\|_{W^{m,2}_k(\Omega)}$.

In the sequel, we shall use the following properties. For all integers m and k in $\mathbb{Z}$, we have

$$\forall n \in \mathbb{Z} \quad \text{with} \quad n \le m-k-2, \quad \mathcal{P}_n \subset W^{m,2}_k(\Omega), \tag{1.4}$$

where $\mathcal{P}_n$ denotes the space of all polynomials (of three variables) of degree at most n, with the convention that the space is reduced to zero when n is negative. Thus the difference $m-k$ is an important parameter of the space $W^{m,2}_k(\Omega)$. We denote by $\mathcal{P}^\Delta_n$ the subspace of all harmonic polynomials of $\mathcal{P}_n$.

Using the derivation in the distribution sense, we can define the operators **curl** and div on $\mathbf{L}^2(\Omega)$. Indeed, let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $\mathcal{D}(\Omega)$ and its dual space $\mathcal{D}'(\Omega)$. For any function $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{L}^2(\Omega)$, we have for any $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \in \boldsymbol{\mathcal{D}}(\Omega)$,

$$\begin{aligned}\langle \mathbf{curl}\,\mathbf{v}, \boldsymbol{\varphi}\rangle &= \int_\Omega \mathbf{v}\cdot \mathbf{curl}\,\boldsymbol{\varphi}\, d\mathbf{x}\\ &= \int_\Omega \Big(v_1(\frac{\partial\varphi_3}{\partial x_2} - \frac{\partial\varphi_2}{\partial x_3}) + v_2(\frac{\partial\varphi_1}{\partial x_3} - \frac{\partial\varphi_3}{\partial x_1}) + v_3(\frac{\partial\varphi_2}{\partial x_1} - \frac{\partial\varphi_1}{\partial x_2})\Big)\, d\mathbf{x},\end{aligned}$$

and for any $\varphi \in \mathcal{D}(\Omega)$,

$$\langle \operatorname{div}\mathbf{v}, \varphi\rangle = -\int_\Omega \mathbf{v}\cdot\mathbf{grad}\,\varphi\, d\mathbf{x} = -\int_\Omega \Big(v_1\frac{\partial\varphi}{\partial x_1} + v_2\frac{\partial\varphi}{\partial x_2} + v_3\frac{\partial\varphi}{\partial x_3}\Big)\, d\mathbf{x}.$$

We note that the vector-valued Laplace operator of a vector field $\mathbf{v} = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\mathbf{v} = \mathbf{grad}(\operatorname{div}\mathbf{v}) - \mathbf{curl}\,\mathbf{curl}\,\mathbf{v}. \tag{1.5}$$

This leads to the following definitions:

Definition 1.1. For all integers $k \in \mathbb{Z}$, we define the space

$$\mathbf{H}^2_k(\mathbf{curl}, \Omega) = \{\mathbf{v} \in \mathbf{W}^{0,2}_k(\Omega); \mathbf{curl}\, \mathbf{v} \in \mathbf{W}^{0,2}_{k+1}(\Omega)\}\,,$$

with the norm

$$\|\mathbf{v}\|_{\mathbf{H}^2_k(\mathbf{curl},\Omega)} = \Big(\|\mathbf{v}\|^2{}_{\mathbf{W}^{0,2}_k(\Omega)} + \|\,\mathbf{curl}\ \mathbf{v}\|^2{}_{\mathbf{W}^{0,2}_{k+1}(\Omega)}\Big)^{1/2}.$$

Also we define the space

$$\mathbf{H}^2_k(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{W}^{0,2}_k(\Omega); \mathrm{div}\, \mathbf{v} \in W^{0,2}_{k+1}(\Omega)\}\,,$$

with the norm

$$\|\mathbf{v}\|_{\mathbf{H}^2_k(\mathrm{div},\Omega)} = \Big(\|\mathbf{v}\|^2_{\mathbf{W}^{0,2}_k(\Omega)} + \|\,\mathrm{div}\ \ \mathbf{v}\|^2_{W^{0,2}_{k+1}(\Omega)}\Big)^{1/2}.$$

Finally, we set

$$\mathbf{X}^2_k(\Omega) = \mathbf{H}^2_k(\mathbf{curl}, \Omega) \cap \mathbf{H}^2_k(\mathrm{div}, \Omega).$$

with the norm

$$\mathbf{X}^2_k(\Omega) = \Big(\|\mathbf{v}\|^2_{\mathbf{W}^{0,2}_k(\Omega)} + \|\,\mathrm{div}\, \mathbf{v}\|^2_{W^{0,2}_{k+1}(\Omega)} + \|\,\mathbf{curl}\, \mathbf{v}\|^2_{\mathbf{W}^{0,2}_{k+1}(\Omega)}\Big)^{1/2}.$$

These definitions will also be used with Ω replaced by $\mathbb{R}^3$.

The argument used by Hanouzet [11] to prove the denseness of $\mathcal{D}(\overline{\Omega})$ in $W^{m,2}_k(\Omega)$ can be easily adapted to establish that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in the space $\mathbf{H}^2_k(\mathrm{div}, \Omega)$ and in the space $\mathbf{H}^2_k(\mathbf{curl}, \Omega)$ and so in $\mathbf{X}^2_k(\Omega)$. Therefore, denoting by $\mathbf{n}$ the exterior unit normal to the boundary Γ, the normal trace $\mathbf{v} \cdot \mathbf{n}$ and the tangential trace $\mathbf{v} \times \mathbf{n}$ can be defined respectively in $H^{-1/2}(\Gamma)$ for the functions of $\mathbf{H}^2_k(\mathrm{div}, \Omega)$ and in $\mathbf{H}^{-1/2}(\Gamma)$ for functions of $\mathbf{H}^2_k(\mathbf{curl}, \Omega)$, where $H^{-1/2}(\Gamma)$ denotes the dual space of $H^{1/2}(\Gamma)$. They satisfy the trace theorems; i.e, there exists a constant C such that

$$\forall \mathbf{v} \in \mathbf{H}^2_k(\mathrm{div}, \Omega), \quad \|\mathbf{v} \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} \le C\|\mathbf{v}\|_{\mathbf{H}^2_k(\mathrm{div},\Omega)}, \tag{1.6}$$

$$\forall \mathbf{v} \in \mathbf{H}^2_k(\mathbf{curl}, \Omega), \quad \|\mathbf{v} \times \mathbf{n}\|_{\mathbf{H}^{-1/2}(\Gamma)} \le C\|\mathbf{v}\|_{\mathbf{H}^2_k(\mathbf{curl},\Omega)} \tag{1.7}$$

and the following Green's formulas holds: For any $\mathbf{v} \in \mathbf{H}^2_k(\mathrm{div}, \Omega)$ and $\varphi \in W^{1,2}_{-k}(\Omega)$

$$\langle \mathbf{v} \cdot \mathbf{n}, \varphi\rangle_\Gamma = \int_\Omega \mathbf{v} \cdot \nabla\varphi \, dx + \int_\Omega \varphi \,\mathrm{div}\, \mathbf{v} \, dx, \tag{1.8}$$

where $\langle , \rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. For any $\mathbf{v} \in \mathbf{H}^2_k(\mathbf{curl}, \Omega)$ and $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{-k}(\Omega)$

$$\langle \mathbf{v} \times \mathbf{n}, \boldsymbol{\varphi}\rangle_\Gamma = \int_\Omega \mathbf{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \, dx - \int_\Omega \mathbf{curl}\, \mathbf{v} \cdot \boldsymbol{\varphi} dx, \tag{1.9}$$

where $\langle , \rangle_\Gamma$ denotes the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$.

Remark 1.2. If $\mathbf{v}$ belongs to $\mathbf{H}^2_k(\mathrm{div}, \Omega)$ for some integer $k \ge 1$, then $\mathrm{div}\, \mathbf{v}$ is in $L^1(\Omega)$ and Green's formula (1.8) yields

$$\langle \mathbf{v} \cdot \mathbf{n}, 1\rangle_\Gamma = \int_\Omega \mathrm{div}\, \mathbf{v} \, dx \tag{1.10}$$

But when $k \le 0$, then $\mathrm{div}\, \mathbf{v}$ is not necessarily in $L^1(\Omega)$ and (1.10) is generally not valid. Note also that when $k \le 0$, $W^{0,2}_{-k-1}(\Omega)$ does not contain the constants.

The closures of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\mathbf{H}^2_k(\mathrm{div},\Omega)$ and in $\mathbf{H}^2_k(\mathbf{curl},\Omega)$ are denoted respectively by $\mathring{\mathbf{H}}^2_k(\mathbf{curl},\Omega)$ and $\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)$ and can be characterized respectively by

$$\mathring{\mathbf{H}}^2_k(\mathbf{curl},\Omega) = \{\mathbf{v}\in \mathbf{H}^2_k(\mathbf{curl},\Omega) : \mathbf{v}\times\mathbf{n} = \mathbf{0} \text{ on } \Gamma\},$$
$$\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega) = \{\mathbf{v}\in \mathbf{H}^2_k(\mathrm{div},\Omega) : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}.$$

Their dual spaces are characterized by the following propositions:

Proposition 1.3. *A distribution* $\mathbf{f}$ *belongs to* $[\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)]'$ *if and only if there exist* $\boldsymbol{\psi}\in\mathbf{W}^{0,2}_{-k}(\Omega)$ *and* $\chi\in W^{0,2}_{-k-1}(\Omega)$, *such that* $\mathbf{f} = \boldsymbol{\psi} + \mathbf{grad}\,\chi$. *Moreover*

$$\|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)]'} = \max\{\|\boldsymbol{\psi}\|_{\mathbf{W}^{0,2}_{-k}(\Omega)},\ \|\chi\|_{W^{0,2}_{-k-1}(\Omega)}\}. \tag{1.11}$$

Proof. Let $\boldsymbol{\psi}\in\mathbf{W}^{0,2}_{-k}(\Omega)$ and $\chi\in W^{0,2}_{-k-1}(\Omega)$, we have

$$\forall\mathbf{v}\in\boldsymbol{\mathcal{D}}(\Omega),\quad \langle\boldsymbol{\psi}+\mathbf{grad}\,\chi,\mathbf{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = \int_\Omega(\boldsymbol{\psi}\cdot\mathbf{v} - \chi\ \mathrm{div}\,\mathbf{v})\,\mathrm{d}\mathbf{x}.$$

Therefore, the linear mapping $\boldsymbol{\ell} : \mathbf{v} \longmapsto \int_\Omega(\boldsymbol{\psi}\cdot\mathbf{v} - \chi\ \mathrm{div}\,\mathbf{v})\mathrm{d}\mathbf{x}$ defined on $\boldsymbol{\mathcal{D}}(\Omega)$ is continuous for the norm of $\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)$. Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\mathbf{H}}^2_k(\mathrm{div},\,\Omega)$, $\boldsymbol{\ell}$ can be extended by continuity to a mapping still called $\boldsymbol{\ell}\in[\mathring{\mathbf{H}}^2_k(\mathrm{div},\,\Omega)]'$. Thus $\boldsymbol{\psi}+\mathbf{grad}\,\chi$ is an element of $[\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)]'$.

Conversely, Let $E = \mathbf{W}^{0,2}_k(\Omega)\times W^{0,2}_{k+1}(\Omega)$ equipped by the following norm

$$\|\mathbf{v}\|_E = (\|\mathbf{v}\|^2_{\mathbf{W}^{0,2}_k(\Omega)} + \|\,\mathrm{div}\,\mathbf{v}\|^2_{W^{0,2}_{k+1}(\Omega)})^{1/2}.$$

The mapping $T : \mathbf{v}\in\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)\to(\mathbf{v},\mathrm{div}\,\mathbf{v})\in E$ is an isometry from $\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)$ in E. Suppose $G = T(\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega))$ with the E-topology. Let $S = T^{-1} : G \to \mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)$. Thus, we can define the following mapping:

$$\mathbf{v}\in G\mapsto\langle\mathbf{f},S\mathbf{v}\rangle_{[\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)]'\times\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)}\quad \text{for } \mathbf{f}\in[\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)]'$$

which is a linear continuous form on G. Thanks to Hahn-Banach's Theorem, such form can be extended to a linear continuous form on E, denoted by Υ such that

$$\|\Upsilon\|_{E'} = \|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_k(\mathrm{div},\Omega)]'}. \tag{1.12}$$

From the Riesz's Representation Lemma, there exist functions $\boldsymbol{\psi}\in\mathbf{W}^{0,2}_{-k}(\Omega)$ and $\chi\in W^{0,2}_{-k-1}(\Omega)$, such that for any $\mathbf{v} = (\mathbf{v}_1, v_2)\in E$,

$$\langle\Upsilon,\mathbf{v}\rangle_{E'\times E} = \int_\Omega\mathbf{v}_1\cdot\boldsymbol{\psi}\,dx + \int_\Omega v_2\chi\,dx,$$

with $\|\Upsilon\|_{E'} = \max\{\|\boldsymbol{\psi}\|_{\mathbf{W}^{0,2}_{-k}(\Omega)}, \|\chi\|_{W^{0,2}_{-k-1}(\Omega)}\}$. In particular, if $\mathbf{v} = T\boldsymbol{\varphi}\in G$, where $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\Omega)$, we have

$$\langle\mathbf{f},\boldsymbol{\varphi}\rangle_{[\mathring{\mathbf{H}}^2_k(\mathrm{div},\,\Omega)]'\times\mathring{\mathbf{H}}^2_k(\mathrm{div},\,\Omega)} = \langle\boldsymbol{\psi}-\nabla\chi,\boldsymbol{\varphi}\rangle_{[\mathring{\mathbf{H}}^2_k(\mathrm{div},\,\Omega)]'\times\mathring{\mathbf{H}}^2_k(\mathrm{div},\,\Omega)},$$

and (1.11) follows imeddiatly from (1.12). □

We skip the proof of the following result as it is similar to that of Proposition 1.3.

Proposition 1.4. *A distribution* $\mathbf{f}$ *belongs to* $[\mathring{\mathbf{H}}^2_k(\mathbf{curl},\Omega)]'$ *if and only if there exist functions* $\boldsymbol{\psi} \in \mathbf{W}^{0,2}_{-k}(\Omega)$ *and* $\boldsymbol{\xi} \in \mathbf{W}^{0,2}_{-k-1}(\Omega)$*, such that* $\mathbf{f} = \boldsymbol{\psi} + \mathbf{curl}\,\boldsymbol{\xi}$*. Moreover*

$$\|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_k(\mathbf{curl},\Omega)]'} = \max\{\|\boldsymbol{\psi}\|_{\mathbf{W}^{0,2}_{-k}(\Omega)}, \|\boldsymbol{\xi}\|_{\mathbf{W}^{0,2}_{-k-1}(\Omega)}\}.$$

Definition 1.5. Let $\mathbf{X}^2_{k,N}(\Omega)$, $\mathbf{X}^2_{k,T}(\Omega)$ and $\mathring{\mathbf{X}}^2_k(\Omega)$ be the following subspaces of $\mathbf{X}^2_k(\Omega)$:

$$\mathbf{X}^2_{k,N}(\Omega) = \{\mathbf{v} \in \mathbf{X}^2_k(\Omega);\, \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\},$$
$$\mathbf{X}^2_{k,T}(\Omega) = \{\mathbf{v} \in \mathbf{X}^2_k(\Omega);\, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$
$$\mathring{\mathbf{X}}^2_k(\Omega) = \mathbf{X}^2_{k,N}(\Omega) \cap \mathbf{X}^2_{k,T}(\Omega).$$

2. Preliminary results

Now, we give some results related to the Dirichlet problem and Neumann problem which are essential to ensure the existence and the uniqueness of some vectors potentials and one usually forces either the normal component to vanish or the tangential components to vanish. We start by giving the definition of the kernel of the Laplace operator for any integer $k \in \mathbb{Z}$:

$$\mathcal{A}^{\Delta}_{k-1} = \{\chi \in W^{1,2}_{-k}(\Omega) : \Delta\chi = 0 \text{ in } \Omega \text{ and } \chi = 0 \text{ on } \Gamma\}.$$

In contrast to a bounded domain, the Dirichlet problem for the Laplace operator with zero data can have nontrivial solutions in an exterior domain; it depends upon the exponent of the weight. The result that we state below is established by Giroire in [10].

Proposition 2.1. *For any integer* $k \geq 1$*, the space* $\mathcal{A}^{\Delta}_{k-1}$ *is a subspace of all functions in* $W^{1,2}_{-k}(\Omega)$ *of the form* $v(p) - p$*, where* p *runs over all polynomials of* $\mathcal{P}^{\Delta}_{k-1}$ *and* $v(p)$ *is the unique solution in* $W^{1,2}_0(\Omega)$ *of the Dirichlet problem*

$$\Delta v(p) = 0 \quad in \quad \Omega \quad and \quad v(p) = p \quad on\ \Gamma. \tag{2.1}$$

The space $\mathcal{A}^{\Delta}_{k-1}$ *is a finite-dimentional space of the same dimension as* $\mathcal{P}^{\Delta}_{k-1}$ *and* $\mathcal{A}^{\Delta}_{k-1} = \{0\}$ *when* $k \leq 0$*.*

Our second proposition is established also by Giroire in [10], it characterizes the kernel of the Laplace operator with Neumann boundary condition. For any integer $k \in \mathbb{Z}$,

$$\mathcal{N}^{\Delta}_{k-1} = \{\chi \in W^{1,2}_{-k}(\Omega) : \Delta\chi = 0 \text{ in } \Omega \text{ and } \frac{\partial \chi}{\partial \mathbf{n}} = 0 \text{ on } \Gamma\}.$$

Proposition 2.2. *For any integer* $k \geq 1$*,* $\mathcal{N}^{\Delta}_{k-1}$ *the subspace of all functions in* $W^{1,2}_{-k}(\Omega)$ *of the form* $w(p) - p$*, where* p *runs over all polynomials of* $\mathcal{P}^{\Delta}_{k-1}$ *and* $w(p)$ *is the unique solution in* $W^{1,2}_0(\Omega)$ *of the Neumann problem*

$$\Delta w(p) = 0 \quad in\ \Omega \quad and \quad \frac{\partial w(p)}{\partial \mathbf{n}} = \frac{\partial p}{\partial \mathbf{n}} \quad on\ \Gamma. \tag{2.2}$$

Here also, we set $\mathcal{N}^{\Delta}_{k-1} = \{0\}$ *when* $k \leq 0$*;* $\mathcal{N}^{\Delta}_{k-1}$ *is a finite-dimentional space of the same dimension as* $\mathcal{P}^{\Delta}_{k-1}$ *and in particular,* $\mathcal{N}^{\Delta}_0 = \mathbb{R}$*.*

Next, the uniqueness of the solutions of Problem (1.1) and Problem (1.2) will follow from the characterization of the kernel. For all integers k in $\mathbb{Z}$, we define

$$\mathbf{Y}^2_{k,N}(\Omega) = \{\mathbf{w} \in \mathbf{X}^2_{-k,N}(\Omega) : \operatorname{div}\mathbf{w} = 0 \text{ and } \mathbf{curl}\,\mathbf{w} = \mathbf{0} \text{ in } \Omega\}$$
$$\mathbf{Y}^2_{k,T}(\Omega) = \{\mathbf{w} \in \mathbf{X}^2_{-k,T}(\Omega) : \operatorname{div}\mathbf{w} = 0 \text{ and } \mathbf{curl}\,\mathbf{w} = \mathbf{0} \text{ in } \Omega\}.$$

The proof of the following propositions can be easily deduced from [7].

Proposition 2.3. *Let $k \in \mathbb{Z}$ and suppose that Ω' is of class $C^{1,1}$, simply-connected and with a Lipshitz-continuous and connected boundary Γ.*

- *If $k < 1$, then $\mathbf{Y}^2_{k,N}(\Omega) = \{\mathbf{0}\}$.*
- *If $k \geq 1$, then $\mathbf{Y}^2_{k,N}(\Omega) = \{\nabla(v(p)-p),\, p \in \mathcal{P}^{\Delta}_{k-1}\}$, where $v(p)$ is the unique solution in $W^{1,2}_0(\Omega)$ of the Dirichlet problem (2.1).*

Proof. Let $k \in \mathbb{Z}$ and let $\mathbf{w} \in \mathbf{X}^2_{-k,N}(\Omega)$ such that $\operatorname{div}\mathbf{w} = 0$ and $\mathbf{curl}\,\mathbf{w} = \mathbf{0}$ in Ω. Then since Ω' is simply-connected, there exists $\chi \in W^{1,2}_{-k}(\Omega)$, unique up to an additive constant, such that $\mathbf{w} = \nabla\chi$. But $\mathbf{w} \times \mathbf{n} = \mathbf{0}$, hence, χ is constant on Γ (Γ is a connected boundary) and we choose the additive constant in χ so that $\chi = 0$ on Γ. Thus χ belongs to $\mathcal{A}^{\Delta}_{k-1}(\Omega)$. Due to Proposition 2.1, we deduce that if $k < 1$, χ is equal to zero and if $k \geq 1$, $\chi = v(p) - p$, where p runs over all polynomials of $\mathcal{P}^{\Delta}_{k-1}$ and $v(p)$ is the unique solution in $W^{1,2}_0(\Omega)$ of problem (2.1) and thus $\mathbf{w} = \nabla(v(p)-p)$. Now, to finish the proof we shall prove that $\nabla(v(p)-p)$ belongs to $\mathbf{Y}^2_{k,N}(\Omega)$ and this is a simple consequence of the definition of p and $v(p)$. □

We skip the proof of the following result as it is entirely similar to that of Proposition 2.3.

Proposition 2.4. *Let the assumptions of Proposition 2.4 hold.*

- *If $k < 1$, then $\mathbf{Y}^2_{k,T}(\Omega) = \{\mathbf{0}\}$.*
- *If $k \geq 1$, then $\mathbf{Y}^2_{k,T}(\Omega) = \{\nabla(w(p) - p),\, p \in \mathcal{P}^{\Delta}_{k-1}\}$, where $w(p)$ is the unique solution in $W^{1,2}_0(\Omega)$ of the Neumann problem (2.2)*

The imbedding results that we state below are established by Girault in [7]. The first imbedding result is given by the following theorem.

Theorem 2.5. *Let $k \leqslant 2$ and assume that Ω' is of class $C^{1,1}$. Then the space $\mathbf{X}^2_{k-1,T}(\Omega)$ is continuously imbedded in $\mathbf{W}^{1,2}_k(\Omega)$. In addition there exists a constant C such that for any $\boldsymbol{\varphi} \in \mathbf{X}^2_{k-1,T}(\Omega)$,*

$$\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,2}_k(\Omega)} \leq C\Big(\|\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_{k-1}(\Omega)} + \|\operatorname{div}\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_k(\Omega)}\Big). \tag{2.3}$$

If in addition, Ω' is simply-connected, there exists a constant C such that for all $\boldsymbol{\varphi} \in \mathbf{X}^2_{k-1,T}(\Omega)$ we have

$$\begin{aligned}\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,2}_k(\Omega)} \leq C(&\|\operatorname{div}\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_k(\Omega)}\\ &+ \sum_{j=2}^{N(-k)} |\int_\Gamma \boldsymbol{\varphi}\cdot\nabla w(q_j)\,d\sigma|),\end{aligned} \tag{2.4}$$

where $\{q_j\}_{j=2}^{N(-k)}$ denotes a basis of $\{q \in \mathcal{P}^{\Delta}_{-k} : q(\mathbf{0}) = 0\}$, $N(-k)$ denotes the dimension of $\mathcal{P}^{\Delta}_{-k}$ and $w(q_j)$ is the corresponding function of $\mathcal{N}^{\Delta}_{-k}$. Thus, the *seminorm in the right-hand side of* (2.4) *is a norm on* $\mathbf{X}^2_{k-1,T}(\Omega)$ *equivalent to the norm* $\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,2}_k(\Omega)}$.

The second imbedding result is given by the following theorem.

Theorem 2.6. *Let* $k \leqslant 2$ *and assume that* Ω' *is of class* $C^{1,1}$. *Then the space* $\mathbf{X}^2_{k-1,N}(\Omega)$ *is continuously imbedded in* $\mathbf{W}^{1,2}_k(\Omega)$. *In addition there exists a constant* C *such that for any* $\boldsymbol{\varphi} \in \mathbf{X}^2_{k-1,N}(\Omega)$,

$$\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,2}_k(\Omega)} \leq C\Big(\|\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_{k-1}(\Omega)} + \|\operatorname{div}\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_k(\Omega)}\Big). \tag{2.5}$$

If in addition, Ω' *is simply-connected and its boundary* Γ *is connected, there exists a constant* C *such that for all* $\boldsymbol{\varphi} \in \mathbf{X}^2_{k-1,N}(\Omega)$ *we have*

$$\begin{aligned}\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,2}_k(\Omega)} \leq C(&\|\operatorname{div}\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_k(\Omega)}\\ &+ |\int_\Gamma(\boldsymbol{\varphi}\cdot\mathbf{n})d\sigma| + \sum_{j=1}^{N(-k)} |\int_\Gamma(\boldsymbol{\varphi}\cdot\mathbf{n})q_j\,d\sigma|),\end{aligned} \tag{2.6}$$

where the term $|\int_\Gamma(\boldsymbol{\varphi}\cdot\mathbf{n})d\sigma|$ *can be dropped if* $k \neq 1$ *and where* $\{q_j\}_{j=1}^{N(-k)}$ *denotes a basis of* $\mathcal{P}^{\Delta}_{-k}$. *In other words, the seminorm in the right-hand side of* (2.6) *is a norm on* $\mathbf{X}^2_{k-1,N}(\Omega)$ *equivalent to the norm* $\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,2}_k(\Omega)}$.

Finally, let us recall the abstract setting of Babuška-Brezzi's Theorem (see Babuška [5], Brezzi [6] and Amrouche-Selloula [4]).

Theorem 2.7. *Let* X *and* M *be two reflexive Banach spaces and* X' *and* M' *their dual spaces. Let* a *be the continuous bilinear form defined on* $X \times M$, *let* $A \in \mathcal{L}(X;\ M')$ *and* $A' \in \mathcal{L}(M;\ X')$ *be the operators defined by*

$$\forall v \in X,\ \forall w \in M, \quad a(v,w) = \langle Av, w\rangle = \langle v, A'w\rangle$$

and $V = \ker A$. *The following statements are equivalent:*

(i) *There exist* $\beta > 0$ *such that*

$$\inf_{w\in M,\, w\neq 0}\ \sup_{v\in X,\, v\neq 0} \frac{a(v,w)}{\|v\|_X\|w\|_M} \geq \beta. \tag{2.7}$$

(ii) *The operator* $A : X/V \mapsto M'$ *is an isomophism and* $1/\beta$ *is the continuity constant of* A^{-1}.

(iii) *The operator* $A' : M \mapsto X'\perp V$ *is an isomophism and* $1/\beta$ *is the continuity constant of* $(A')^{-1}$.

Remark 2.8. As consequence, if the inf-sup condition (2.7) is satisfied, then we have the following properties:

(i) If $V = \{0\}$, then for any $f \in X'$, there exists a unique $w \in M$ such that

$$\forall v \in X,\ a(v,w) = \langle f, v\rangle \quad \text{and} \quad \|w\|_M \leq \frac{1}{\beta}\|f\|_{X'}. \tag{2.8}$$

(ii) If $V \neq \{0\}$, then for any $f \in X'$, satisfying the compatibility condition: $\forall v \in V,\ \langle f, v\rangle = 0$, there exists a unique $w \in M$ such that (2.8).

(iii) For any $g \in M'$, there exists $v \in X$, unique up an additive element of V, such that:

$$\forall w \in M,\ a(v,w) = \langle g, w\rangle \quad \text{and} \quad \|v\|_{X/V} \leq \frac{1}{\beta}\|g\|_{M'}.$$

3. Inequalities and inf-sup conditions

In this sequel, we prove some imbedding results. More precisely, we show that the results of Theorem 2.5 and the result of Theorem 2.6 can be extended to the case where the boundary conditions $\mathbf{v}\cdot\mathbf{n}=0$ or $\mathbf{v}\times\mathbf{n}=\mathbf{0}$ on Γ are replaced by inhomogeneous one. Next, we study some problems posed in an exterior domain which are essentials to prove the regularity of solutions for Problem (1.1) and Problem (1.2).

For any integer k in $\mathbb{Z}$, we introduce the following spaces:

$$\mathbf{Z}^2_{k,T}(\Omega)=\{\mathbf{v}\in\mathbf{X}^2_k(\Omega)\text{ and }\mathbf{v}\cdot\mathbf{n}\in H^{1/2}(\Gamma)\},$$
$$\mathbf{Z}^2_{k,N}(\Omega)=\{\mathbf{v}\in\mathbf{X}^2_k(\Omega)\text{ and }\mathbf{v}\times\mathbf{n}\in \mathbf{H}^{1/2}(\Gamma)\}$$

and

$$\mathbf{M}^2_{k,T}(\Omega)=\{\mathbf{v}\in\mathbf{W}^{1,2}_{k+1}(\Omega),\ \operatorname{div}\mathbf{v}\in W^{1,2}_{k+2}(\Omega),\ \mathbf{curl}\,\mathbf{v}\in\mathbf{W}^{1,2}_{k+2}(\Omega)\text{ and }\mathbf{v}\cdot\mathbf{n}\in H^{3/2}(\Gamma)\}.$$

Proposition 3.1. *Let $k=-1$ or $k=0$, then the space $\mathbf{Z}^2_{k,T}(\Omega)$ is continuously imbedded in $\mathbf{W}^{1,2}_{k+1}(\Omega)$ and we have the following estimate for any $\mathbf{v}$ in $\mathbf{Z}^2_{k,T}(\Omega)$:*

$$\|\mathbf{v}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)}\le C\big(\|\mathbf{v}\|_{\mathbf{W}^{0,2}_{k}(\Omega)}+\|\mathbf{curl}\,\mathbf{v}\|_{\mathbf{W}^{0,2}_{k+1}(\Omega)}+\|\operatorname{div}\mathbf{v}\|_{W^{0,2}_{k+1}(\Omega)}+\|\mathbf{v}\cdot\mathbf{n}\|_{H^{1/2}(\Gamma)}\big).\tag{3.1}$$

Proof. Let $k=-1$ or $k=0$ and let $\mathbf{v}$ any function of $\mathbf{Z}^2_{k,T}(\Omega)$. Let us study the Neumann problem

$$\Delta\chi=\operatorname{div}\mathbf{v}\text{ in }\Omega\quad\text{and}\quad\partial_n\chi=\mathbf{v}\cdot\mathbf{n}\text{ on }\Gamma.\tag{3.2}$$

It is shown in [7, Theorems 3.7 and 3.9], that Problem (3.2) has a unique solution χ in $W^{2,2}_{k+1}(\Omega)/\mathbb{R}$ if $k=-1$ and χ is unique in $W^{2,2}_{k+1}(\Omega)$ if $k=0$. With the estimate

$$\|\nabla\chi\|_{W^{1,2}_{k+1}(\Omega)}\le C\big(\|\operatorname{div}\mathbf{v}\|_{W^{0,2}_{k+1}(\Omega)}+\|\mathbf{v}\cdot\mathbf{n}\|_{H^{1/2}(\Gamma)}\big).\tag{3.3}$$

Let $\mathbf{w}=\mathbf{v}-\mathbf{grad}\,\chi$, then $\mathbf{w}$ is a divergence-free function. Since $\mathbf{W}^{1,2}_{k+1}(\Omega)\hookrightarrow\mathbf{W}^{0,2}_{k}(\Omega)$, then $\mathbf{w}\in\mathbf{X}^2_{k,T}(\Omega)$. Applying Theorem 2.5, we have $\mathbf{w}$ belongs to $\mathbf{W}^{1,2}_{k+1}(\Omega)$ and then $\mathbf{v}$ is in $\mathbf{W}^{1,2}_{k+1}(\Omega)$. According to Inequality (2.3), we obtain

$$\|\mathbf{w}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)}\le C\big(\|\mathbf{w}\|_{\mathbf{W}^{0,2}_{k}(\Omega)}+\|\mathbf{curl}\,\mathbf{w}\|_{\mathbf{W}^{0,2}_{k+1}(\Omega)}\big).$$

Then, inequality (3.1) follows directly from (3.3). □

Similarly, we can prove the following imbedding result.

Proposition 3.2. *Suppose that Ω' is of class $C^{2,1}$. Then the space $\mathbf{M}^2_{-1,T}(\Omega)$ is continuously imbedded in $\mathbf{W}^{2,2}_1(\Omega)$ and we have the following estimate for any $\mathbf{v}$ in $\mathbf{M}^2_{-1,T}(\Omega)$:*

$$\|\mathbf{v}\|_{\mathbf{W}^{2,2}_{1}(\Omega)}\le C\big(\|\mathbf{v}\|_{\mathbf{W}^{1,2}_{0}(\Omega)}+\|\mathbf{curl}\,\mathbf{v}\|_{\mathbf{W}^{1,2}_{1}(\Omega)}+\|\operatorname{div}\mathbf{v}\|_{W^{1,2}_{1}(\Omega)}+\|\mathbf{v}\cdot\mathbf{n}\|_{H^{3/2}(\Gamma)}\big).\tag{3.4}$$

Proof. Proceeding as in Proposition 3.1. Let $\mathbf{v}$ in $\mathbf{M}^2_{-1,T}(\Omega)$. Since Ω' is of class $C^{2,1}$, then according to [7, Theorem 3.9], there exists a unique solution χ

in $W_1^{3,2}(\Omega)/\mathbb{R}$ of Problem (3.2). Setting $\mathbf{w} = \mathbf{v} - \mathbf{grad}\,\chi$. Since $W_1^{2,2}(\Omega)$ is imbedded in $W_0^{1,2}(\Omega)$, it follows from [7, Corollary 3.16], that $\mathbf{w}$ belongs to $\mathbf{W}_1^{2,2}(\Omega)$ and moreover we have the estimate

$$\|\mathbf{w}\|_{\mathbf{W}_1^{2,2}(\Omega)} \le C\big(\|\mathbf{w}\|_{\mathbf{W}_0^{1,2}(\Omega)} + \|\,\mathbf{curl}\,\mathbf{w}\|_{\mathbf{W}_1^{1,2}(\Omega)}\big).$$

Then $\mathbf{v} = \mathbf{w} + \mathbf{grad}\,\chi$ belongs to $\mathbf{W}_1^{2,2}(\Omega)$ and we have the estimate (3.4). $\square$

Although we are under the Hilbertian case but the Lax-Milgram lemma is not always valid to ensure the existence of solutions. Thus, we shall establish two "inf-sup" conditions in order to apply Theorem 2.7. First recall the following spaces for all integers $k \in \mathbb{Z}$:

$$\mathbf{V}^2_{k,T}(\Omega) = \Big\{\mathbf{z} \in \mathbf{X}^2_{k,T}(\Omega) : \operatorname{div}\mathbf{z} = 0 \text{ in } \Omega \quad\text{and}\quad \int_\Gamma \mathbf{z}\cdot\nabla(w(q)-q)\,d\sigma = 0,\ \forall (w(q)-q) \in \mathcal{N}^{\Delta}_{-k-1}\Big\}$$

and

$$\mathbf{V}^2_{k,N}(\Omega) = \{\mathbf{z} \in \mathbf{X}^2_{k,N}(\Omega) : \operatorname{div}\mathbf{z} = 0 \text{ in } \Omega \text{ and } \int_\Gamma (\mathbf{z}\cdot\mathbf{n})q\,d\sigma = 0,\ \forall q \in \mathcal{P}^{\Delta}_{-k-1}\}.$$

The first "inf-sup" condition is given by the following lemma.

Lemma 3.3. *The following inf-sup Condition holds: there exists a constant $\beta > 0$, such that*

$$\inf_{\boldsymbol{\varphi}\in\mathbf{V}^2_{0,T}(\Omega),\,\boldsymbol{\varphi}\neq 0}\ \sup_{\boldsymbol{\psi}\in\mathbf{V}^2_{-2,T}(\Omega),\,\boldsymbol{\psi}\neq 0} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,d\mathbf{x}}{\|\boldsymbol{\psi}\|_{\mathbf{X}^2_{-2,T}(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{X}^2_{0,T}(\Omega)}} \ge \beta. \tag{3.5}$$

Proof. Let $\mathbf{g} \in \mathbf{W}^{0,2}_{-1}(\Omega)$ and let us introduce the Dirichlet problem

$$-\Delta\chi = \operatorname{div}\mathbf{g} \quad \text{in } \Omega, \quad \chi = 0 \quad \text{on } \Gamma.$$

It is shown in [7, Theorem 3.5], that this problem has a solution $\chi \in \mathring{W}^{1,2}_{-1}(\Omega)$ unique up to an element of $\mathcal{A}_0^{\Delta}$ and we can choose χ such that

$$\|\nabla\chi\|_{\mathbf{W}^{0,2}_{-1}(\Omega)} \le C\|\mathbf{g}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)}.$$

Set $\mathbf{z} = \mathbf{g} - \nabla\chi$. Then we have $\mathbf{z} \in \mathbf{W}^{0,2}_{-1}(\Omega)$, $\operatorname{div}\mathbf{z} = 0$ and we have

$$\|\mathbf{z}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)} \le C\|\mathbf{g}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)}. \tag{3.6}$$

Let $\boldsymbol{\varphi}$ any function of $\mathbf{V}^2_{0,T}(\Omega)$, by Theorem 2.5 we have $\boldsymbol{\varphi} \in \mathbf{X}^2_{0,T}(\Omega) \hookrightarrow \mathbf{W}^{1,2}_1(\Omega)$. Then due to (2.4) we can write

$$\|\boldsymbol{\varphi}\|_{\mathbf{X}^2_{0,T}(\Omega)} \le C\|\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{W}^{0,2}_1(\Omega)} = C \sup_{\mathbf{g}\in\mathbf{W}^{0,2}_{-1}(\Omega),\,\mathbf{g}\neq 0} \frac{\big|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\,\mathrm{d}\mathbf{x}\big|}{\|\mathbf{g}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)}}. \tag{3.7}$$

Using the fact that $\mathbf{curl}\,\boldsymbol{\varphi} \in \mathbf{H}^2_1(\operatorname{div},\Omega)$ and applying (1.8), we obtain

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\nabla\chi\,\mathrm{d}\mathbf{x} = 0. \tag{3.8}$$

Now, let $\lambda \in W_0^{1,2}(\Omega)$ the unique solution of the problem

$$\Delta\lambda = 0 \quad \text{in } \Omega \quad \text{and} \quad \lambda = 1 \quad \text{on } \Gamma.$$

It follows from [7, Lemma 3.11] that

$$\int_\Gamma \frac{\partial \lambda}{\partial \mathbf{n}}\, d\sigma = C_1 > 0.$$

Now, setting

$$\widetilde{\mathbf{z}} = \mathbf{z} - \frac{1}{C_1}\langle \mathbf{z}\cdot\mathbf{n}, 1\rangle_\Gamma \nabla\lambda.$$

It is clear that $\widetilde{\mathbf{z}} \in \mathbf{W}^{0,2}_{-1}(\Omega)$, $\operatorname{div}\widetilde{\mathbf{z}} = 0$ in Ω and that $\langle \widetilde{\mathbf{z}}\cdot\mathbf{n}, 1\rangle_\Gamma = 0$. Due to [7, Theorem 3.15], there exists a potential vector $\boldsymbol{\psi} \in \mathbf{W}^{1,2}_{-1}(\Omega)$ such that

$$\widetilde{\mathbf{z}} = \mathbf{curl}\,\boldsymbol{\psi}, \quad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\psi}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma. \tag{3.9}$$

and we have

$$\forall \mathbf{v}(q) \in \mathcal{N}_1^\Delta, \quad \int_\Gamma \boldsymbol{\psi}\cdot\nabla\mathbf{v}(q)\, d\sigma = 0. \tag{3.10}$$

In addition, we have the estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,2}_{-1}(\Omega)} \le C\|\widetilde{\mathbf{z}}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)} \le C\|\mathbf{z}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)}. \tag{3.11}$$

Using (3.10), we obtain that $\boldsymbol{\psi}$ belongs to $\mathbf{V}^2_{-2,T}(\Omega)$. Since $\boldsymbol{\varphi}$ is $\mathbf{H}^1$ in a neighborhood of Γ, then $\boldsymbol{\varphi}$ has an $\mathbf{H}^1$ extension in Ω' denoted by $\widetilde{\boldsymbol{\varphi}}$. Applying Green's formula in Ω', we obtain

$$0 = \int_{\Omega'} \operatorname{div}(\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}})\, \mathrm{d}\mathbf{x} = \langle \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\cdot\mathbf{n}, 1\rangle_\Gamma = \langle \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{n}, 1\rangle_\Gamma.$$

Using the fact that $\mathbf{curl}\,\boldsymbol{\varphi}$ in $\mathbf{H}^2_1(\operatorname{div},\Omega)$ and λ in $W^{1,2}_{-1}(\Omega)$ and applying (1.8), we obtain

$$0 = \langle \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{n}, 1\rangle_\Gamma = \langle \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{n}, \lambda\rangle_\Gamma = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\nabla\lambda\, \mathrm{d}\mathbf{x}. \tag{3.12}$$

Using (3.8) and (3.12), we deduce that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\, \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{z}\, \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\mathbf{z}}\, \mathrm{d}\mathbf{x}. \tag{3.13}$$

From (3.11), (3.6) and (3.13), we deduce that

$$\frac{\big|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\,\mathrm{d}\mathbf{x}\big|}{\|\mathbf{g}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)}} \le C\frac{\big|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\mathbf{z}}\,\mathrm{d}\mathbf{x}\big|}{\|\widetilde{\mathbf{z}}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)}} = C\frac{\big|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\big|}{\|\,\mathbf{curl}\,\boldsymbol{\psi}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)}}.$$

Applying again (2.4) and using (3.10), we obtain

$$\frac{\big|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\,\mathrm{d}\mathbf{x}\big|}{\|\mathbf{g}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)}} \le C\frac{\big|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\big|}{\|\boldsymbol{\psi}\|_{\mathbf{X}^2_{-2,T}(\Omega)}},$$

and the inf-sup Condition (3.5) follows immediately from (3.7). □

The second "inf sup" condition is given by the following lemma:

Lemma 3.4. *The following inf-sup Condition holds: there exists a constant $\beta > 0$, such that*

$$\inf_{\boldsymbol{\varphi}\in\mathbf{V}^2_{-2,N}(\Omega),\,\boldsymbol{\varphi}\neq 0}\ \sup_{\boldsymbol{\psi}\in\mathbf{V}^2_{0,N}(\Omega),\,\boldsymbol{\psi}\neq 0} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\, d\mathbf{x}}{\|\boldsymbol{\psi}\|_{\mathbf{X}^2_{0,N}(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{X}^2_{-2,N}(\Omega)}} \ge \beta. \tag{3.14}$$

Proof. The proof is similar to that of Lemma 3.3. Let $\mathbf{g} \in \mathbf{W}_1^{0,2}(\Omega)$ and let us introduce the generalized Neumann problem

$$\operatorname{div}(\nabla\chi - \mathbf{g}) = 0 \quad \text{in } \Omega \quad \text{and} \quad (\nabla\chi - \mathbf{g}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{3.15}$$

It follows from [10] that Problem (3.15) has a solution $\chi \in W_1^{1,2}(\Omega)$ and we have

$$\|\nabla\chi\|_{W_1^{0,2}(\Omega)} \leq C\|\mathbf{g}\|_{\mathbf{W}_1^{0,2}(\Omega)}.$$

Setting $\mathbf{z} = \mathbf{g} - \nabla\chi$, then we have $\mathbf{z} \in \mathring{\mathbf{H}}_1^2(\operatorname{div}, \Omega)$ and $\operatorname{div} \mathbf{z} = 0$ with the estimate

$$\|\mathbf{z}\|_{\mathbf{W}_1^{0,2}(\Omega)} \leqslant C\|\mathbf{g}\|_{\mathbf{W}_1^{0,2}(\Omega)}. \tag{3.16}$$

Let $\boldsymbol{\varphi}$ be any function of $\mathbf{V}_{-2,N}^2(\Omega)$. Due to Theorem 2.6, we have $\mathbf{X}_{-2,N}^2(\Omega) \hookrightarrow \mathbf{W}_{-1}^{1,2}(\Omega)$ and by (2.6) we can write

$$\|\boldsymbol{\varphi}\|_{\mathbf{X}_{-2,N}^2(\Omega)} \leq C\|\operatorname{\mathbf{curl}} \boldsymbol{\varphi}\|_{\mathbf{W}_{-1}^{0,2}(\Omega)} = C \sup_{\mathbf{g} \in \mathbf{W}_1^{0,2}(\Omega),\, \mathbf{g} \neq 0} \frac{\left|\int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{\varphi} \cdot \mathbf{g}\, \mathrm{d}\mathbf{x}\right|}{\|\mathbf{g}\|_{\mathbf{W}_1^{0,2}(\Omega)}}. \tag{3.17}$$

Observe that $\operatorname{\mathbf{curl}} \boldsymbol{\varphi}$ belongs to $\mathbf{H}_{-1}^2(\operatorname{div}, \Omega)$ with $\boldsymbol{\varphi} \times \mathbf{n} = \mathbf{0}$ on Γ and $\chi \in W_1^{1,2}(\Omega)$. Then using (1.8), we obtain

$$\int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{\varphi} \cdot \nabla\chi\, dx = \langle \operatorname{\mathbf{curl}} \boldsymbol{\varphi} \cdot \mathbf{n}, \chi \rangle_\Gamma = 0. \tag{3.18}$$

Due to [7, Proposition 3.12], there exists a potential vector $\boldsymbol{\psi} \in \mathbf{W}_1^{1,2}(\Omega)$ such that

$$\mathbf{z} = \operatorname{\mathbf{curl}} \boldsymbol{\psi}, \quad \operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\psi} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma, \tag{3.19}$$

$$\int_\Gamma \boldsymbol{\psi} \cdot \mathbf{n}\, d\sigma = 0. \tag{3.20}$$

In addition, we have

$$\|\boldsymbol{\psi}\|_{\mathbf{W}_1^{1,2}(\Omega)} \leq C\|\mathbf{z}\|_{\mathbf{W}_1^{0,2}(\Omega)}. \tag{3.21}$$

Then, we deduce that $\boldsymbol{\psi}$ belongs to $\mathbf{V}_{0,N}^2(\Omega)$. Using (3.16), (3.18) and (3.19), we deduce that

$$\frac{\left|\int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{\varphi} \cdot \mathbf{g}\, \mathrm{d}\mathbf{x}\right|}{\|\mathbf{g}\|_{\mathbf{W}_1^{0,2}(\Omega)}} \leq C \frac{\left|\int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{\varphi} \cdot \mathbf{z}\, \mathrm{d}\mathbf{x}\right|}{\|\mathbf{z}\|_{\mathbf{W}_1^{0,2}(\Omega)}} = C \frac{\left|\int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{\varphi} \cdot \operatorname{\mathbf{curl}} \boldsymbol{\psi}\, \mathrm{d}\mathbf{x}\right|}{\|\operatorname{\mathbf{curl}} \boldsymbol{\psi}\|_{\mathbf{W}_1^{0,2}(\Omega)}}.$$

Applying again (2.6) and using (3.20), we obtain

$$\frac{\left|\int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{\varphi} \cdot \mathbf{g}\, \mathrm{d}\mathbf{x}\right|}{\|\mathbf{g}\|_{\mathbf{W}_1^{0,2}(\Omega)}} \leq C \frac{\left|\int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{\varphi} \cdot \operatorname{\mathbf{curl}} \boldsymbol{\psi}\, \mathrm{d}\mathbf{x}\right|}{\|\boldsymbol{\psi}\|_{\mathbf{X}_{0,N}^2(\Omega)}},$$

and the inf-sup condition (3.14) follows immediately from (3.17). □

4. Elliptic problems with different boundary conditions

Next, we study the problem

$$\begin{gathered} -\Delta\boldsymbol{\xi} = \mathbf{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi} = 0 \quad \text{in } \Omega, \\ \boldsymbol{\xi} \times \mathbf{n} = \mathbf{g} \times \mathbf{n} \quad \text{on } \Gamma \quad \text{and} \quad \int_\Gamma (\boldsymbol{\xi} \cdot \mathbf{n}) q\, d\sigma = 0, \quad \forall q \in \mathcal{P}_k^\Delta. \end{gathered} \tag{4.1}$$

Proposition 4.1. *Let $k = -1$ or $k = 0$ and suppose that $\mathbf{g} \times \mathbf{n} = \mathbf{0}$ and let $\mathbf{f} \in [\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl}, \Omega)]'$ with $\operatorname{div} \mathbf{f} = 0$ in Ω and satisfying the compatibility condition*

$$\forall \mathbf{v} \in \mathbf{Y}^2_{1-k,N}(\Omega), \quad \langle \mathbf{f}, \mathbf{v} \rangle_{[\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl}\,\Omega)]' \times \mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl}\,\Omega)} = 0. \tag{4.2}$$

Then, Problem (4.1) has a unique solution in $\mathbf{W}^{1,2}_{-k}(\Omega)$ and we have

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_{-k}(\Omega)} \leqslant C \|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl},\Omega)]'}. \tag{4.3}$$

Moreover, if $\mathbf{f}$ in $\mathbf{W}^{0,2}_{-k+1}(\Omega)$ and Ω' is of class $C^{2,1}$, then the solution $\boldsymbol{\xi}$ is in $\mathbf{W}^{2,2}_{-k+1}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{2,2}_{-k+1}(\Omega)} \leqslant C \|\mathbf{f}\|_{\mathbf{W}^{0,2}_{-k+1}(\Omega)}. \tag{4.4}$$

Proof. (i) On the one hand, observe that Problem (4.1) is reduced to the variational problem: Find $\boldsymbol{\xi} \in \mathbf{V}^2_{-k-1,N}(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \mathbf{X}^2_{k-1,N}(\Omega), \quad \int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\, \mathrm{d}\mathbf{x} = \langle \mathbf{f},\, \boldsymbol{\varphi} \rangle_\Omega, \tag{4.5}$$

where the duality on Ω is

$$\langle \cdot,\, \cdot \rangle_\Omega = \langle \cdot,\, \cdot \rangle_{[\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl},\,\Omega)]' \times \mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl},\,\Omega)}.$$

On the other hand, (4.5) is equivalent to the problem: Find $\boldsymbol{\xi} \in \mathbf{V}^2_{-k-1,N}(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \mathbf{V}^2_{k-1,N}(\Omega), \quad \int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\, \mathrm{d}\mathbf{x} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_\Omega. \tag{4.6}$$

Indeed, every solution of (4.5) also solves (4.6). Conversely, assume that (4.6) holds, and let $\boldsymbol{\varphi} \in \mathbf{X}^2_{k-1,N}(\Omega)$. Let us solve the exterior Dirichlet problem

$$-\Delta \chi = \operatorname{div} \boldsymbol{\varphi} \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma. \tag{4.7}$$

It is shown in [7, Theorem 3.5] that problem (4.7) has a unique solution $\chi \in W^{2,2}_k(\Omega)/\mathcal{A}^\Delta_{-k}$.

First case. if $k = 0$, we set

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla \chi - \frac{1}{C_1} \langle \boldsymbol{\varphi} - \nabla \chi, 1 \rangle_\Gamma \nabla (v(1) - 1),$$

where $v(1)$ is the unique solution in $W^{1,2}_0(\Omega)$ of the Dirichlet problem (2.1) and

$$C_1 = \int_\Gamma \frac{\partial v(1)}{\partial \mathbf{n}}\, d\sigma.$$

It follows from [7, Lemma 3.11] that $C_1 > 0$ and since $\nabla(v(1) - 1)$ belongs to $\mathbf{Y}^2_{1,N}(\Omega)$, we deduce that $\widetilde{\boldsymbol{\varphi}}$ belongs to $\mathbf{V}^2_{-1,N}(\Omega)$.

Second case. if $k = -1$, for each polynomial p in $\mathcal{P}^\Delta_1$, we take $\widetilde{\boldsymbol{\varphi}}$ of the form

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla \chi - \nabla (v(p) - p)$$

where $v(p)$ is the unique solution in $W^{1,2}_0(\Omega)$ of the Dirichlet problem (2.1). The polynomial p is chosen to satisfy the condition

$$\int_\Gamma (\widetilde{\boldsymbol{\varphi}} \cdot \mathbf{n})\, q\, d\sigma = 0 \qquad \forall q \in \mathcal{P}^\Delta_1. \tag{4.8}$$

To show that this is possible, let T be a linear form defined by $T : \mathcal{P}_1^{\Delta} \to \mathbb{R}^4$,

$$
\begin{aligned}
T(p) = \Big(\int_\Gamma \frac{\partial(v(p)-p)}{\partial \mathbf{n}} d\sigma, \int_\Gamma \frac{\partial(v(p)-p)}{\partial \mathbf{n}}\, x_1 d\sigma,\\
\int_\Gamma \frac{\partial(v(p)-p)}{\partial \mathbf{n}}\, x_2 d\sigma, \int_\Gamma \frac{\partial(v(p)-p)}{\partial \mathbf{n}}\, x_3 d\sigma \Big),
\end{aligned}
$$

where $\{1, x_1, x_2, x_3\}$ denotes a basis of $\mathcal{P}_1^{\Delta}$. It is shown in the proof of [8, Theorem 7], that if

$$
\int_\Gamma \frac{\partial(v(p)-p)}{\partial \mathbf{n}}\, q\, d\sigma = 0 \quad \forall q \in \mathcal{P}_1^{\Delta},
$$

then $p = 0$. This implies that T is injective and so bijective. And so, there exists a unique p in $\mathcal{P}_1^{\Delta}$ so that condition (4.8) is satisfied and since $\nabla(v(p)-p)$ belongs to $\mathbf{Y}^2_{2,N}(\Omega)$, we prove that $\widetilde{\boldsymbol{\varphi}} \in \mathbf{V}^2_{-2,N}(\Omega)$.

Finally, using (4.2), we obtain for $k = 0$ and $k = -1$ that

$$
\langle \mathbf{f}, \nabla(v(p)-p)\rangle_\Omega = 0 \quad \text{and} \quad \langle \mathbf{f}, \nabla(v(1)-1)\rangle_\Omega = 0
$$

and as $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl}, \Omega)$, we obtain that

$$
\langle \mathbf{f}, \nabla \chi\rangle_\Omega = 0.
$$

Then we have

$$
\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{\varphi}}\, \mathrm{d}\mathbf{x} = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle_\Omega.
$$

Then Problem (4.5) and Problem (4.6) are equivalent. Now, to solve Problem (4.6), we use Lax-Milgram lemma for $k = 0$ and the inf-sup condition (3.14) for $k = -1$. Let us start by $k = 0$. We consider the bilinear form $\boldsymbol{a} : \mathbf{V}^2_{-1,N}(\Omega) \times \mathbf{V}^2_{-1,N}(\Omega) \to \mathbb{R}$ such that

$$
\boldsymbol{a}(\boldsymbol{\xi}, \boldsymbol{\varphi}) = \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\ \mathrm{d}\mathbf{x}.
$$

According to Theorem 2.6, $\boldsymbol{a}$ is continuous and coercive on $\mathbf{V}^2_{-1,N}(\Omega)$. Due to Lax-Milgram lemma, there exists a unique solution $\boldsymbol{\xi} \in \mathbf{V}^2_{-1,N}(\Omega)$ of Problem (4.6). Using again Theorem 2.6, we prove that this solution $\boldsymbol{\xi}$ belongs to $\mathbf{W}^{1,2}_0(\Omega)$ and the following estimate follows immediately

$$
\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_0(\Omega)} \leqslant C \|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)]'}. \tag{4.9}
$$

When $k = -1$, we have that Problem (4.6) satisfies the inf-sup condition (3.14). Let us consider the mapping $\boldsymbol{\ell} : \mathbf{V}^2_{-2,N}(\Omega) \to \mathbb{R}$ such that $\boldsymbol{\ell}(\boldsymbol{\varphi}) = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle_\Omega$. It is clear that $\boldsymbol{\ell}$ belongs to $(\mathbf{V}^2_{-2,N}(\Omega))'$ and according to Remark 2.8, there exists a unique solution $\boldsymbol{\xi} \in \mathbf{V}^2_{0,N}(\Omega)$ of Problem (4.6). Due to Theorem 2.6, we prove that this solution $\boldsymbol{\xi}$ belongs to $\mathbf{W}^{1,2}_1(\Omega)$. It follows from Remark 2.8 i) that

$$
\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_1(\Omega)} \leqslant C \|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{-2}(\mathbf{curl}, \Omega)]'}. \tag{4.10}
$$

(ii) We suppose in addition that $\mathbf{f}$ is in $\mathbf{W}^{0,2}_{-k+1}(\Omega)$ for $k = -1$ or $k = 0$ and Ω' is of class $C^{2,1}$ and we set $\mathbf{z} = \mathbf{curl}\, \boldsymbol{\xi}$, where $\boldsymbol{\xi} \in \mathbf{W}^{1,2}_{-k}(\Omega)$ is the unique solution of Problem (4.1). Then we have

$$
\mathbf{z} \in \mathbf{W}^{0,2}_{-k}(\Omega), \quad \mathbf{curl}\, \mathbf{z} = \mathbf{f} \in \mathbf{W}^{0,2}_{-k+1}(\Omega), \quad \operatorname{div} \mathbf{z} = 0 \quad \text{and} \quad \mathbf{z} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma
$$

and thus $\mathbf{z}$ belongs to $\mathbf{X}^2_{-k,T}(\Omega)$. By Theorem 2.5, we prove that $\mathbf{z}$ belongs to $\mathbf{W}^{1,2}_{-k+1}(\Omega)$ and using (2.4), we prove that $\mathbf{z}$ satisfies

$$\|\mathbf{z}\|_{\mathbf{W}^{1,2}_{-k+1}(\Omega)} \leqslant C\|\mathbf{f}\|_{\mathbf{W}^{0,2}_{-k+1}(\Omega)}. \tag{4.11}$$

As a consequence $\boldsymbol{\xi}$ satisfies

$$\boldsymbol{\xi} \in \mathbf{W}^{1,2}_{-k}(\Omega), \quad \mathbf{curl}\,\boldsymbol{\xi} \in \mathbf{W}^{1,2}_{-k+1}(\Omega), \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{and} \quad \boldsymbol{\xi}\times\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

Applying [7, Corollary 3.14], we deduce that $\boldsymbol{\xi}$ belongs to $\mathbf{W}^{2,2}_{-k+1}(\Omega)$ and using in addition the boundary condition of (4.1) we prove that

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{2,2}_{-k+1}(\Omega)} \leqslant C\|\,\mathbf{curl}\,\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_{-k+1}(\Omega)}. \tag{4.12}$$

Finally, estimate (4.4) follows from (4.11) and (4.12). □

Corollary 4.2. *Let $k=-1$ or $k=0$ and let $\mathbf{f} \in [\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl},\Omega)]'$ with $\operatorname{div}\mathbf{f} = 0$ in Ω and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ and satisfying the compatibility condition (4.2). Then, Problem (4.1) has a unique solution $\boldsymbol{\xi}$ in $\mathbf{W}^{1,2}_{-k}(\Omega)$ and we have:*

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_{-k}(\Omega)} \leqslant C\Big(\|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl},\Omega)]'} + \|\mathbf{g}\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big). \tag{4.13}$$

Moreover, if $\mathbf{f}$ in $\mathbf{W}^{0,2}_{-k+1}(\Omega)$, $\mathbf{g}$ in $\mathbf{H}^{3/2}(\Gamma)$ and Ω' is of class $C^{2,1}$, then the solution $\boldsymbol{\xi}$ is in $\mathbf{W}^{2,2}_{-k+1}(\Omega)$ and satisfies

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{2,2}_{-k+1}(\Omega)} \leqslant C\Big(\|\mathbf{f}\|_{\mathbf{W}^{0,2}_{-k+1}(\Omega)} + \|\mathbf{g}\times\mathbf{n}\|_{\mathbf{H}^{3/2}(\Gamma)}\Big). \tag{4.14}$$

Proof. Let $k=0$ or $k=-1$ and let $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$. We know that there exists $\boldsymbol{\xi}_0$ in $\mathbf{H}^1(\Omega)$ with compact support satisfying

$$\boldsymbol{\xi}_0 = \mathbf{g}_\tau \quad \text{on } \Gamma \quad \text{and} \quad \operatorname{div}\boldsymbol{\xi}_0 = 0 \quad \text{in } \Omega,$$

where $\mathbf{g}_\tau$ is the tangential component of $\mathbf{g}$ on Γ. Since support of $\boldsymbol{\xi}_0$ is compact, we deduce that $\boldsymbol{\xi}_0$ belongs to $\mathbf{W}^{1,2}_{-k}(\Omega)$ for $k=-1$ or $k=0$ and satisfies

$$\|\boldsymbol{\xi}_0\|_{\mathbf{W}^{1,2}_{-k}(\Omega)} \leq C\|\mathbf{g}_\tau\|_{\mathbf{H}^{1/2}(\Gamma)}. \tag{4.15}$$

Setting $\mathbf{z} = \boldsymbol{\xi} - \boldsymbol{\xi}_0$, then Problem (4.1) is equivalent to: find $\mathbf{z} \in \mathbf{W}^{1,2}_{-k}(\Omega)$ such that

$$\begin{gathered} -\Delta\mathbf{z} = \mathbf{f} + \Delta\boldsymbol{\xi}_0 \quad \text{and} \quad \operatorname{div}\mathbf{z} = 0 \quad \text{in } \Omega, \\ \mathbf{z}\times\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma \quad \text{and} \quad \int_\Gamma (\mathbf{z}\cdot\mathbf{n})\, q\, d\sigma = 0, \quad \forall q \in \mathcal{P}^\Delta_k. \end{gathered} \tag{4.16}$$

Observe that $\mathbf{F} = \mathbf{f} - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}_0$ belongs to $[\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl},\Omega)]'$. Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl},\,\Omega)$, we have for any $\mathbf{v} \in \mathbf{Y}^2_{1-k,N}(\Omega)$:

$$\langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}_0, \mathbf{v}\rangle_\Omega = \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}_0 \cdot \mathbf{curl}\,\mathbf{v}\, dx = 0.$$

Thus $\mathbf{F}$ satisfies the compatibility condition (4.2). Due to Proposition 4.1, there exists a unique $\mathbf{z} \in \mathbf{W}^{1,2}_{-k}(\Omega)$ solution of problem (4.16) such that

$$\|\mathbf{z}\|_{\mathbf{W}^{1,2}_{-k}(\Omega)} \leqslant C\|\mathbf{F}\|_{[\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl},\Omega)]'} \leq C\Big(\|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{k-1}(\mathbf{curl},\Omega)]'} + \|\,\mathbf{curl}\,\boldsymbol{\xi}_0\|_{\mathbf{W}^{0,2}_{-k}(\Omega)}\Big). \tag{4.17}$$

Then $\boldsymbol{\xi} = \mathbf{z} + \boldsymbol{\xi}_0$ belongs to $\mathbf{W}^{1,2}_{-k}(\Omega)$ is the unique solution of (5.8) and estimate (4.13) follows immediately from (4.15) and (4.17).

Regularity of the solution: Suppose in addition that Ω' is of class $C^{2,1}$, $\mathbf{f}$ in $\mathbf{W}^{0,2}_{-k+1}(\Omega)$ and $\mathbf{g}$ in $\mathbf{H}^{3/2}(\Gamma)$. Then the function $\boldsymbol{\xi}_0$ defined above belongs to $\mathbf{H}^2(\Omega)$ with compact support and thus $\boldsymbol{\xi}_0$ belongs to $\mathbf{W}^{2,2}_{-k+1}(\Omega)$ and we have

$$\|\boldsymbol{\xi}_0\|_{\mathbf{W}^{2,2}_{-k+1}(\Omega)} \le C\|\mathbf{g}_\tau\|_{\mathbf{H}^{3/2}(\Gamma)}. \tag{4.18}$$

Using again Proposition 4.1, we prove that $\mathbf{z}$ belongs to $\mathbf{W}^{2,2}_{-k+1}(\Omega)$ and satisfies

$$\|\mathbf{z}\|_{\mathbf{W}^{2,2}_{-k+1}(\Omega)} \leqslant C\|\mathbf{F}\|_{\mathbf{W}^{0,2}_{-k+1}(\Omega)}.$$

Then $\boldsymbol{\xi}$ is in $\mathbf{W}^{2,2}_{-k+1}(\Omega)$ and estimate (4.14) follows from (4.18). □

The next theorem solves an other type of exterior problem.

Theorem 4.3. *Let $k=-1$ or $k=0$ and let $\mathbf{v}$ belongs to $\mathbf{W}^{0,2}_{k}(\Omega)$. Then, the following problem*

$$\begin{gathered} -\Delta\boldsymbol{\xi} = \mathbf{curl}\,\mathbf{v} \quad \textit{and} \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \textit{in } \Omega, \\ \boldsymbol{\xi}\cdot\mathbf{n} = 0 \quad \textit{and} \quad (\mathbf{curl}\,\boldsymbol{\xi} - \mathbf{v})\times\mathbf{n} = \mathbf{0} \quad \textit{on } \Gamma, \\ \int_\Gamma \boldsymbol{\xi}\cdot\nabla(w(q)-q)\,d\sigma = 0, \quad \forall\,(w(q)-q)\in\mathcal{N}^{\Delta}_{-k} \end{gathered} \tag{4.19}$$

has a unique solution $\boldsymbol{\xi}$ in $\mathbf{W}^{1,2}_{k}(\Omega)$ and we have

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_{k}(\Omega)} \le C\|\mathbf{v}\|_{\mathbf{W}^{0,2}_{k}(\Omega)}. \tag{4.20}$$

Moreover, if $\mathbf{v}\in\mathbf{W}^{1,2}_{k+1}(\Omega)$ and Ω' is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{\xi}$ is in $\mathbf{W}^{2,2}_{k+1}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{2,2}_{k+1}(\Omega)} \le C\|\mathbf{v}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)}. \tag{4.21}$$

Proof. At first observe that if $\boldsymbol{\xi}\in\mathbf{W}^{1,2}_{k}(\Omega)$ is a solution of Problem (4.19) for $k=-1$ or $k=0$, then $\mathbf{curl}\,\boldsymbol{\xi}-\mathbf{v}$ belongs to $\mathbf{H}^2_k(\mathbf{curl},\Omega)$ and thus $(\mathbf{curl}\,\boldsymbol{\xi}-\mathbf{v})\times\mathbf{n}$ is well defined in Γ and belongs to $\mathbf{H}^{-1/2}(\Gamma)$.

On the other hand, note that (4.19) can be reduced to the variational problem: Find $\boldsymbol{\xi}\in\mathbf{V}^2_{k-1,T}(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\mathbf{X}^2_{-k-1,T}(\Omega) \quad \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}. \tag{4.22}$$

Indeed, every solution of (4.19) also solves (4.22). Conversely, let $\boldsymbol{\xi}\in\mathbf{V}^2_{k-1,T}(\Omega)$ a solution of the problem (4.22). Then,

$$\forall\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\Omega), \quad \langle\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}-\mathbf{curl}\,\mathbf{v},\,\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

Then

$$-\Delta\boldsymbol{\xi} = \mathbf{curl}\,\mathbf{v} \quad \text{in } \Omega. \tag{4.23}$$

Moreover, by the fact that $\boldsymbol{\xi}$ belongs to the space $\mathbf{V}^2_{k-1,T}(\Omega)$ we have $\operatorname{div}\boldsymbol{\xi}=0$ in Ω and $\boldsymbol{\xi}\cdot\mathbf{n}=0$ on Γ. Then, it remains to verify the boundary condition $(\mathbf{curl}\,\boldsymbol{\xi}-\mathbf{v})\times\mathbf{n}=\mathbf{0}$ on Γ. Now setting $\mathbf{z}=\mathbf{curl}\,\boldsymbol{\xi}-\mathbf{v}$, then $\mathbf{z}$ belongs to $\mathbf{H}^2_k(\mathbf{curl},\Omega)$. Therefore, (4.23) becomes

$$\mathbf{curl}\,\mathbf{z} = \mathbf{0} \quad \text{in } \Omega. \tag{4.24}$$

Let $\boldsymbol{\varphi} \in \mathbf{X}^2_{-k-1,T}(\Omega)$, by Theorem 2.5 we have $\mathbf{X}^2_{-k-1,T}(\Omega) \hookrightarrow \mathbf{W}^{1,2}_{-k}(\Omega)$. Thank's to (1.9) we obtain

$$\int_\Omega \mathbf{z} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x} = \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\mathbf{H}^{-1/2}(\Gamma)\times \mathbf{H}^{1/2}(\Gamma)} + \int_\Omega \mathbf{curl}\, \mathbf{z} \cdot \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}. \tag{4.25}$$

Compare (4.25) with (4.22) and using (4.24), we deduce that

$$\forall \boldsymbol{\varphi} \in \mathbf{X}^2_{-k-1,T}(\Omega), \quad \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_\Gamma = 0.$$

Let now $\boldsymbol{\mu}$ any element of the space $\mathbf{H}^{1/2}(\Gamma)$. As Ω' is bounded, we can fix once for all a ball B_R, centered at the origin and with radius R, such that $\overline{\Omega'} \subset B_R$. Setting $\Omega_R = \Omega \cap B_R$, then we have the existence of $\boldsymbol{\varphi}$ in $\mathbf{H}^1(\Omega_R)$ such that $\boldsymbol{\varphi} = \mathbf{0}$ on ∂B_R and $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$ on Γ, where $\boldsymbol{\mu}_t$ is the tangential component of $\boldsymbol{\mu}$ on Γ. The function $\boldsymbol{\varphi}$ can be extended by zero outside B_R and the extended function, still denoted by $\boldsymbol{\varphi}$, belongs to $\mathbf{W}^{1,p}_\alpha(\Omega)$, for any α since its support is bounded. Thus $\boldsymbol{\varphi}$, belongs to $\mathbf{W}^{1,2}_{-k}(\Omega)$. It is clear that $\boldsymbol{\varphi}$ belongs to $\mathbf{X}^2_{-k-1,T}(\Omega)$ and

$$\langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\mu} \rangle_\Gamma = \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\mu}_t \rangle_\Gamma = \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_\Gamma = 0. \tag{4.26}$$

This implies that $\mathbf{z} \times \mathbf{n} = \mathbf{0}$ on Γ which is the last boundary condition in (4.19).

On the other hand, let us introduce the problem: Find $\boldsymbol{\xi} \in \mathbf{V}^2_{k-1,T}(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \mathbf{V}^2_{-k-1,T}(\Omega) \quad \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}. \tag{4.27}$$

Problem (4.27) can be solved by Lax-Milgram lemma if $k = 0$ and by Lemma 3.3 if $k = -1$.

We start by the case $k = -1$. Observe that Problem (4.27) satisfies the inf-sup condition (3.5). Let consider the mapping $\boldsymbol{\ell} : \mathbf{V}^2_{0,T}(\Omega) \to \mathbb{R}$ such that $\boldsymbol{\ell}(\boldsymbol{\varphi}) = \int_\Omega \mathbf{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}$. It is clear that $\boldsymbol{\ell}$ belongs to $(\mathbf{V}^2_{0,T}(\Omega))'$ and according to Remark 2.8, there exists a unique solution $\boldsymbol{\xi} \in \mathbf{V}^2_{-2,T}(\Omega)$. Applying Theorem 2.5, we deduce that this solution $\boldsymbol{\xi}$ belongs to $\mathbf{W}^{1,2}_{-1}(\Omega)$. It follows from Remark 2.8 i) and Theorem 2.6 that

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_{-1}(\Omega)} \le C \|\boldsymbol{\ell}\|_{(\mathbf{V}^2_{0,T}(\Omega))'} \le C \|\mathbf{v}\|_{\mathbf{W}^{0,2}_{-1}(\Omega)}. \tag{4.28}$$

For $k = 0$, let us consider the bilinear form $\boldsymbol{b} : \mathbf{V}^2_{-1,T}(\Omega) \times \mathbf{V}^2_{-1,T}(\Omega) \to \mathbb{R}$ such that

$$\boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\varphi}) = \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}.$$

According to Theorem 2.5, $\boldsymbol{b}$ is continuous and coercive on $\mathbf{V}^2_{-1,T}(\Omega)$. Due to Lax-Milgram lemma, there exists a unique solution $\boldsymbol{\xi} \in \mathbf{V}^2_{-1,T}(\Omega)$ of Problem (4.27). Using again Theorem 2.5, we prove that this solution $\boldsymbol{\xi}$ belongs to $\mathbf{W}^{1,2}_0(\Omega)$ and estimate (4.20) follows immediately.

Next, we extend (4.27) to any test function in $\mathbf{X}^2_{-k-1,T}(\Omega)$. Let $\boldsymbol{\varphi} \in \mathbf{X}^2_{-k-1,T}(\Omega)$ and let us solve the exterior Neumann problem

$$\Delta \chi = \operatorname{div} \boldsymbol{\varphi} \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \chi}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma. \tag{4.29}$$

It is shown in [7, Lemma 3.7 and Theorem 3.9] that this problem has a unique solution χ in $W^{1,2}_{-k-1}(\Omega)$ if $k = -1$ and unique up to a constant if $k = 0$. Set

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla \chi. \tag{4.30}$$

It is clear that for $k=0$ and $k=-1$, $\int_\Gamma \widetilde{\boldsymbol{\varphi}}\cdot\nabla(w(q)-q)\,d\sigma = 0$ for any $(w(q)-q)\in \mathcal{N}_k^{\Delta}$. Then $\widetilde{\boldsymbol{\varphi}}$ belongs to $\mathbf{V}^2_{-k-1,T}(\Omega)$. Now, if (4.27) holds, we have

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}.$$

Hence, problem (4.22) and problem (4.27) are equivalent. This implies that problem (4.19) has a unique solution $\boldsymbol{\xi}$ in $\mathbf{W}^{1,2}_k(\Omega)$ for $k=0$ or $k=-1$.

Regularity. Now, we suppose that $\mathbf{v}\in\mathbf{W}^{1,2}_{k+1}(\Omega)\hookrightarrow\mathbf{W}^{0,2}_k(\Omega)$ and Ω' is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{\xi}\in\mathbf{W}^{1,2}_k(\Omega)$ the weak solution of (4.19) and we set $\mathbf{z}=\mathbf{curl}\,\boldsymbol{\xi}-\mathbf{v}$. It is clear that $\mathbf{z}$ belongs to $\mathbf{X}^2_{k,N}(\Omega)$. Applying Theorem 2.6, we obtain that $\mathbf{z}\in\mathbf{W}^{1,2}_{k+1}(\Omega)$ and using (2.5) and (4.20) we obtain that

$$\begin{aligned}\|\mathbf{z}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)} &\le C\Big(\|\mathbf{z}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\operatorname{div}\mathbf{z}\|_{W^{0,2}_{k+1}(\Omega)}\Big)\\ &\le C\Big(\|\mathbf{curl}\,\boldsymbol{\xi}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\mathbf{v}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\operatorname{div}\mathbf{v}\|_{W^{0,2}_{k+1}(\Omega)}\Big)\\ &\le C\|\mathbf{v}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)}.\end{aligned}\tag{4.31}$$

This implies that $\boldsymbol{\xi}$ satisfies

$$\boldsymbol{\xi}\in\mathbf{W}^{1,2}_k(\Omega),\quad \operatorname{div}\boldsymbol{\xi}=0\in\mathbf{W}^{1,2}_{k+1}(\Omega),\quad \mathbf{curl}\,\boldsymbol{\xi}\in\mathbf{W}^{1,2}_{k+1}(\Omega),\quad \boldsymbol{\xi}\cdot\mathbf{n}=0\quad\text{on }\Gamma.$$

Applying [7, Corollary 3.16], we deduce that $\boldsymbol{\xi}$ belongs to $\mathbf{W}^{2,2}_{k+1}(\Omega)$ and using (4.31), we obtain

$$\begin{aligned}\|\boldsymbol{\xi}\|_{\mathbf{W}^{2,2}_{k+1}(\Omega)} &\le C\Big(\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_k(\Omega)} + \|\mathbf{curl}\,\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)}\Big)\\ &\le C\Big(\|\mathbf{v}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\mathbf{z}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)} + \|\mathbf{v}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)}\Big)\\ &\le \|\mathbf{v}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)}.\end{aligned}$$

This completes the proof of the theorem. □

As consequence, we can prove other imbedding results. We start by the following theorem.

Theorem 4.4. *Let $k=-1$ or $k=0$. Then the space $\mathbf{Z}^2_{k,N}(\Omega)$ is continuously imbedded in $\mathbf{W}^{1,2}_{k+1}(\Omega)$ and we have the following estimate for any $\mathbf{v}$ in $\mathbf{Z}^2_{k,N}(\Omega)$:*

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)} \le C\big(&\|\mathbf{v}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\mathbf{curl}\,\mathbf{v}\|_{\mathbf{W}^{0,2}_{k+1}(\Omega)}\\ &+ \|\operatorname{div}\mathbf{v}\|_{W^{0,2}_{k+1}(\Omega)} + \|\mathbf{v}\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)}\big).\end{aligned}\tag{4.32}$$

Proof. Let $k=-1$ or $k=0$ and let $\mathbf{v}$ be any function of $\mathbf{Z}^2_{k,N}(\Omega)$. We set $\mathbf{z}=\mathbf{curl}\,\boldsymbol{\xi}-\mathbf{v}$ where $\boldsymbol{\xi}\in\mathbf{W}^{1,2}_k(\Omega)$ is the solution of the problem (4.19). Hence, $\mathbf{z}$ belongs to the space $\mathbf{X}^2_{k,N}(\Omega)$. By Theorem 2.6 and (2.5), $\mathbf{z}$ even belongs to $\mathbf{W}^{1,2}_{k+1}(\Omega)$ with the estimate

$$\|\mathbf{z}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)} \le C\big(\|\mathbf{z}\|_{\mathbf{W}^{0,2}_k(\Omega)} + \|\operatorname{div}\mathbf{z}\|_{W^{0,2}_{k+1}(\Omega)} + \|\mathbf{curl}\,\mathbf{z}\|_{\mathbf{W}^{0,2}_{k+1}(\Omega)}\big).\tag{4.33}$$

Then, it suffices to prove that $\mathbf{curl}\,\boldsymbol{\xi}\in\mathbf{W}^{1,2}_{k+1}(\Omega)$ to obtain $\mathbf{v}\in\mathbf{W}^{1,2}_{k+1}(\Omega)$. Setting $\boldsymbol{\omega}=\mathbf{curl}\,\boldsymbol{\xi}$. It is clear that

$$\int_\Gamma \boldsymbol{\omega}\cdot\mathbf{n}\,d\sigma = 0\tag{4.34}$$

and then $\boldsymbol{\omega}$ satisfies

$$\begin{gathered}-\Delta\boldsymbol{\omega} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} \quad \text{and} \quad \operatorname{div}\boldsymbol{\omega} = 0 \quad \text{in } \Omega \\ \boldsymbol{\omega}\times\mathbf{n} = \mathbf{v}\times\mathbf{n} \quad \text{on } \Gamma \quad \text{and} \quad \int_\Gamma (\boldsymbol{\omega}\cdot\mathbf{n})q\,d\sigma = 0,\,\forall\, q\in\mathcal{P}^{\Delta}_{-k-1}.\end{gathered} \tag{4.35}$$

Note that $\mathbf{curl}\,\mathbf{v} \in \mathbf{W}^{0,2}_{k+1}(\Omega)$ then $\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}$ is in $[\mathring{\mathbf{H}}^2_{-k-2}(\mathbf{curl},\Omega)]'$ and we have $\mathbf{v}\times\mathbf{n}\in\mathbf{H}^{1/2}(\Gamma)$. Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\mathbf{H}}^2_{-k-2}(\mathbf{curl},\Omega)$, we prove that

$$\forall \boldsymbol{\phi}\in\mathbf{Y}^2_{k+2,N}(\Omega), \quad \langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{v},\boldsymbol{\phi}\rangle_{[\mathring{\mathbf{H}}^2_{-k-2}(\mathbf{curl},\Omega)]'\times\mathring{\mathbf{H}}^2_{-k-2}(\mathbf{curl},\Omega)} = 0.$$

Due to Corollary 4.2, the function $\boldsymbol{\omega}$ belongs to $\mathbf{W}^{1,2}_{k+1}(\Omega)$ and satisfies the estimate

$$\begin{aligned}\|\boldsymbol{\omega}\|_{\mathbf{W}^{1,2}_{k+1}(\Omega)} &\le C\big(\|\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}\|_{[\mathring{\mathbf{H}}^2_{-k-2}(\mathbf{curl},\Omega)]'} + \|\mathbf{v}\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)}\big) \\ &\le C\big(\|\mathbf{curl}\,\mathbf{v}\|_{\mathbf{W}^{0,2}_{k+1}(\Omega)} + \|\mathbf{v}\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)}\big).\end{aligned} \tag{4.36}$$

Finally, estimate (4.32) can be deduced by using inequalities (4.33) and (4.36). □

Before giving the second imbedding result, we need to introduce the following space for any integer k in $\mathbb{Z}$,

$$\begin{aligned}\mathbf{M}^2_{k,N}(\Omega) = \Big\{&\mathbf{v}\in\mathbf{W}^{1,2}_{k+1}(\Omega),\ \operatorname{div}\mathbf{v}\in W^{1,2}_{k+2}(\Omega),\ \mathbf{curl}\,\mathbf{v}\in\mathbf{W}^{1,2}_{k+2}(\Omega), \\ &\mathbf{v}\times\mathbf{n}\in\mathbf{H}^{3/2}(\Gamma)\Big\}.\end{aligned}$$

Proposition 4.5. *Suppose that Ω' is of class $C^{2,1}$. Then the space $\mathbf{M}^2_{-1,N}(\Omega)$ is continuously imbedded in $\mathbf{W}^{2,2}_1(\Omega)$ and we have the following estimate for any $\mathbf{v}$ in $\mathbf{M}^2_{-1,N}(\Omega)$,*

$$\|\mathbf{v}\|_{\mathbf{W}^{2,2}_1(\Omega)} \le C\big(\|\mathbf{v}\|_{\mathbf{W}^{1,2}_0(\Omega)} + \|\mathbf{curl}\,\mathbf{v}\|_{\mathbf{W}^{1,2}_1(\Omega)} + \|\operatorname{div}\mathbf{v}\|_{W^{1,2}_1(\Omega)} + \|\mathbf{v}\times\mathbf{n}\|_{\mathbf{H}^{3/2}(\Gamma)}\big). \tag{4.37}$$

Proof. The proof is very similar to that of Theorem 4.4. Let $\mathbf{v}$ be any function of $\mathbf{M}^2_{-1,N}(\Omega)$ and set $\mathbf{z} = \mathbf{curl}\,\boldsymbol{\xi} - \mathbf{v}$ where $\boldsymbol{\xi}\in\mathbf{W}^{2,2}_0(\Omega)$ is the solution of the problem (4.19). According to [7, Corollary 3.14], we prove that $\mathbf{z}$ belongs to $\mathbf{W}^{2,2}_1(\Omega)$ with the estimate

$$\|\mathbf{z}\|_{\mathbf{W}^{2,2}_1(\Omega)} \le C\big(\|\mathbf{z}\|_{\mathbf{W}^{1,2}_0(\Omega)} + \|\operatorname{div}\mathbf{z}\|_{W^{1,2}_1(\Omega)} + \|\mathbf{curl}\,\mathbf{z}\|_{\mathbf{W}^{1,2}_1(\Omega)}\big). \tag{4.38}$$

Then, it suffices to prove that $\mathbf{curl}\,\boldsymbol{\xi}\in\mathbf{W}^{2,2}_1(\Omega)$ to obtain $\mathbf{v}\in\mathbf{W}^{2,2}_1(\Omega)$. We set $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{\xi}$. Using [7, Theorem 3.1], we prove that $\boldsymbol{\omega}$ satisfies Problem (4.35). Using the regularity of Corollary 4.2, we prove that $\boldsymbol{\omega}$ belongs to $\mathbf{W}^{2,2}_1(\Omega)$ and satisfies

$$\|\boldsymbol{\omega}\|_{\mathbf{W}^{1,2}_{-k}(\Omega)} \leqslant C\Big(\|\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}\|_{\mathbf{W}^{0,2}_1(\Omega)} + \|\mathbf{v}\times\mathbf{n}\|_{\mathbf{H}^{3/2}(\Gamma)}\Big)$$

and then estimate (4.37) follows from (4.38). □

5. Generalized solutions for (1.1) and (1.2)

We start this sequel by introducing the space

$$\mathbf{E}^2(\Omega) = \{\mathbf{v}\in\mathbf{W}^{1,2}_0(\Omega) : \Delta\mathbf{v}\in[\mathring{\mathbf{H}}^2_{-1}(\operatorname{div},\Omega)]'\}.$$

This is a Banach space with the norm

$$\|\mathbf{v}\|_{\mathbf{E}^2(\Omega)} = \|\mathbf{v}\|_{\mathbf{W}^{1,2}_0(\Omega)} + \|\Delta\mathbf{v}\|_{[\mathring{\mathbf{H}}^2_{-1}(\operatorname{div},\Omega)]'}.$$

We have the following preliminary result.

Lemma 5.1. *The space* $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ *is dense in* $\mathbf{E}^2(\Omega)$.

Proof. Let P be a continuous linear mapping from $\mathbf{W}_0^{1,2}(\Omega)$ to $\mathbf{W}_0^{1,2}(\mathbb{R}^3)$, such that $P\mathbf{v}|_\Omega = \mathbf{v}$ and let $\boldsymbol{\ell} \in (\mathbf{E}^2(\Omega))'$, such that for any $\mathbf{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, we have $\langle \boldsymbol{\ell}, \mathbf{v}\rangle = 0$. We want to prove that $\boldsymbol{\ell} = \mathbf{0}$ on $\mathbf{E}^2(\Omega)$. Then there exists $(\mathbf{f}, \mathbf{g}) \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3) \times \mathring{\mathbf{H}}^2_{-1}(\mathrm{div}, \Omega)$ such that: for any $\mathbf{v} \in \mathbf{E}^2(\Omega)$,

$$\langle \boldsymbol{\ell}, \mathbf{v}\rangle = \langle \mathbf{f}, P\,\mathbf{v}\rangle_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)\times \mathbf{W}_0^{1,2}(\mathbb{R}^3)} + \langle \Delta \mathbf{v}, \mathbf{g}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)]'\times \mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)}.$$

Observe that we can easily extend by zero the function $\mathbf{g}$ in such a way that $\widetilde{\mathbf{g}} \in \mathbf{H}^2_{-1}(\mathrm{div}, \mathbb{R}^3)$. Now we take $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$. Then we have by assumption that

$$\langle \mathbf{f}, \boldsymbol{\varphi}\rangle_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)\times \mathbf{W}_0^{1,2}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \widetilde{\mathbf{g}} \cdot \Delta\boldsymbol{\varphi} dx = 0,$$

because $\langle \mathbf{f}, \boldsymbol{\varphi}\rangle = \langle \mathbf{f}, P\mathbf{v}\rangle$ where $\mathbf{v} = \boldsymbol{\varphi}|_\Omega$. Thus we have $\mathbf{f} + \Delta\widetilde{\mathbf{g}} = \mathbf{0}$ in $\boldsymbol{\mathcal{D}}'(\mathbb{R}^3)$. Then we can deduce that $\Delta\widetilde{\mathbf{g}} = -\mathbf{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3)$ and due to [2, Theorem 1.3], there exists a unique $\boldsymbol{\lambda} \in \mathbf{W}_0^{1,2}(\mathbb{R}^3)$ such that $\Delta\boldsymbol{\lambda} = \Delta\widetilde{\mathbf{g}}$. Thus the harmonic function $\boldsymbol{\lambda} - \widetilde{\mathbf{g}}$ belonging to $\mathbf{W}^{0,2}_{-1}(\mathbb{R}^3)$ is necessarily equal to zero. Since $\mathbf{g} \in \mathbf{W}_0^{1,2}(\Omega)$ and $\widetilde{\mathbf{g}} \in \mathbf{W}_0^{1,2}(\mathbb{R}^3)$, we deduce that $\mathbf{g} \in \mathring{\mathbf{W}}_0^{1,2}(\Omega)$. As $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\mathbf{W}}_0^{1,2}(\Omega)$, there exists a sequence $\mathbf{g}_k \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\mathbf{g}_k \to \mathbf{g}$ in $\mathbf{W}_0^{1,2}(\Omega)$, when $k \to \infty$. Then $\nabla \cdot \mathbf{g}_k \to \nabla \cdot \mathbf{g}$ in $L^2(\Omega)$. Since $\mathbf{W}_0^{1,2}(\Omega)$ is imbedded in $\mathbf{W}^{0,2}_{-1}(\Omega)$, we deduce that $\mathbf{g}_k \to \mathbf{g}$ in $\mathbf{H}^2_{-1}(\mathrm{div}, \Omega)$. Now, we consider $\mathbf{v} \in \mathbf{E}^2(\Omega)$ and we want to prove that $\langle \boldsymbol{\ell}, \mathbf{v}\rangle = 0$. Observe that:

$$\begin{aligned}
\langle \boldsymbol{\ell}, \mathbf{v}\rangle &= -\langle \Delta\widetilde{\mathbf{g}}, P\mathbf{v}\rangle_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)\times \mathbf{W}_0^{1,2}(\mathbb{R}^3)} + \langle \Delta \mathbf{v}, \mathbf{g}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\Omega)]'\times \mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\Omega)}\\
&= \lim_{k\to\infty}(-\int_\Omega \Delta\mathbf{g}_k \cdot \mathbf{v} dx + \langle \Delta\mathbf{v}, \mathbf{g}_k\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)]'\times \mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)}\\
&= \lim_{k\to\infty}(-\int_\Omega \Delta\mathbf{g}_k \cdot \mathbf{v} dx + \int_\Omega \mathbf{v}\cdot \Delta\mathbf{g}_k dx) = 0.
\end{aligned}$$

□

As a consequence, we have the following result.

Corollary 5.2. *The linear mapping* $\gamma : \mathbf{v} \to \mathbf{curl}\,\mathbf{v}|_\Gamma \times \mathbf{n}$ *defined on* $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ *can be extended to a linear continuous mapping*

$$\gamma : \mathbf{E}^2(\Omega) \to \mathbf{H}^{-1/2}(\Gamma).$$

Moreover, we have the Green formula:for any $\mathbf{v} \in \mathbf{E}^2(\Omega)$ *and any* $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega)$ *such that* $\mathrm{div}\,\boldsymbol{\varphi} = 0$ *in* Ω *and* $\boldsymbol{\varphi}\cdot\mathbf{n} = 0$ *on* Γ,

$$-\langle \Delta\mathbf{v},\, \boldsymbol{\varphi}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\Omega)]'\times \mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)} = \int_\Omega \mathbf{curl}\,\mathbf{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\, d\mathbf{x} - \langle \mathbf{curl}\,\mathbf{v}\times\mathbf{n},\, \boldsymbol{\varphi}\rangle_\Gamma, \quad (5.1)$$

where the duality on Γ *is defined by* $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^{1/2}(\Gamma)}$.

Proof. Let $\mathbf{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$. Observe that if $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega)$ such that $\boldsymbol{\varphi}\cdot\mathbf{n} = 0$ on Γ we deduce that $\boldsymbol{\varphi} \in \mathbf{X}^2_{-1,T}(\Omega)$, then (5.1) holds for such $\boldsymbol{\varphi}$. Now, let $\boldsymbol{\mu} \in \mathbf{H}^{1/2}(\Gamma)$, then there exists $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$ on Γ and that $\mathrm{div}\,\boldsymbol{\varphi} = 0$ with

$$\|\boldsymbol{\varphi}\|_{\mathbf{W}_0^{1,2}(\Omega)} \leqslant C\|\boldsymbol{\mu}_t\|_{\mathbf{H}^{1/2}(\Gamma)} \leqslant C\|\boldsymbol{\mu}\|_{\mathbf{H}^{1/2}(\Gamma)}. \quad (5.2)$$

As a consequence, using (5.1), we have

$$|\langle \mathbf{curl}\,\mathbf{v}\times\mathbf{n},\boldsymbol{\mu}\rangle_\Gamma| \leqslant C\|\mathbf{v}\|_{\mathbf{E}^2(\Omega)}\|\boldsymbol{\mu}\|_{\mathbf{H}^{1/2}(\Gamma)}.$$

Thus,

$$\|\,\mathbf{curl}\,\mathbf{v}\times\mathbf{n}\|_{\mathbf{H}^{-1/2}(\Gamma)}| \leqslant C\|\mathbf{v}\|_{\mathbf{E}^2(\Omega)}.$$

We deduce that the linear mapping γ is continuous for the norm $\mathbf{E}^2(\Omega)$. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{E}^2(\Omega)$, γ can be extended to by continuity to $\gamma\in\mathcal{L}(\mathbf{E}^2(\Omega),\mathbf{H}^{-1/2}(\Gamma))$ and formula (5.1) holds for all $\mathbf{v}\in\mathbf{E}^2(\Omega)$ and $\boldsymbol{\varphi}\in\mathbf{W}_0^{1,2}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi}=0$ in Ω and $\boldsymbol{\varphi}\cdot\mathbf{n}=0$ on Γ. □

Proposition 5.3. *Let* $\mathbf{f}$ *belongs to* $\mathbf{W}_1^{0,2}(\Omega)$ *with* $\operatorname{div}\mathbf{f}=0$ *in* Ω, $g\in H^{1/2}(\Gamma)$ *and* $\mathbf{h}\in\mathbf{H}^{-1/2}(\Gamma)$ *verify the compatibility conditions: for any* $\mathbf{v}\in\mathbf{Y}_{1,T}^2(\Omega)$,

$$\int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}+\langle\mathbf{h}\times\mathbf{n},\,\mathbf{v}\rangle_{\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^{1/2}(\Gamma)}=0, \tag{5.3}$$

$$\mathbf{f}\cdot\mathbf{n}+\operatorname{div}_\Gamma(\mathbf{h}\times\mathbf{n})=0 \quad \text{on } \Gamma, \tag{5.4}$$

where $\operatorname{div}_\Gamma$ *is the surface divergence on* Γ. *Then, the problem*

$$\begin{gathered} -\Delta\mathbf{z}=\mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{z}=0 \quad \text{in } \Omega,\\ \mathbf{z}\cdot\mathbf{n}=g \quad \text{and} \quad \mathbf{curl}\,\mathbf{z}\times\mathbf{n}=\mathbf{h}\times\mathbf{n} \quad \text{on } \Gamma, \end{gathered} \tag{5.5}$$

has a unique solution $\mathbf{z}$ *in* $\mathbf{W}_0^{1,2}(\Omega)$ *satisfying the estimate*

$$\|\mathbf{z}\|_{\mathbf{W}_0^{1,2}(\Omega)} \le C\Big(\|\mathbf{f}\|_{\mathbf{W}_1^{0,2}(\Omega)}+\|g\|_{H^{1/2}(\Gamma)}+\|\mathbf{h}\times\mathbf{n}\|_{\mathbf{H}^{-1/2}(\Gamma)}\Big). \tag{5.6}$$

Moreover, if $\mathbf{h}$ *in* $\mathbf{H}^{1/2}(\Gamma)$, g *in* $H^{3/2}(\Gamma)$ *and* Ω' *is of class* $C^{2,1}$, *then the solution* $\mathbf{z}$ *is in* $\mathbf{W}_1^{2,2}(\Omega)$ *and satisfies the estimate*

$$\|\mathbf{z}\|_{\mathbf{W}_1^{2,2}(\Omega)} \le C\Big(\|\mathbf{f}\|_{\mathbf{W}_1^{0,2}(\Omega)}+\|g\|_{H^{3/2}(\Gamma)}+\|\mathbf{h}\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big). \tag{5.7}$$

Proof. First, note that if $\mathbf{h}\in\mathbf{H}^{-1/2}(\Gamma)$, then $\mathbf{h}\times\mathbf{n}$ also belongs to $\mathbf{H}^{-1/2}(\Gamma)$. On the other hand, let us consider the Neumann problem:

$$\Delta\theta=0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial\,\theta}{\partial\,\mathbf{n}}=g \quad \text{on } \Gamma. \tag{5.8}$$

It is shown in [7, Theorem 3.9], that this problem has a unique solution $\theta\in W_0^{2,2}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\|\theta\|_{W_0^{2,2}(\Omega)} \le C\|g\|_{H^{1/2}(\Gamma)}. \tag{5.9}$$

Setting $\boldsymbol{\xi}=\mathbf{z}-\nabla\theta$, then problem (5.5) becomes: find $\boldsymbol{\xi}\in\mathbf{W}_0^{1,2}(\Omega)$ such that

$$\begin{gathered} -\Delta\boldsymbol{\xi}=\mathbf{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{\xi}=0 \quad \text{in } \Omega,\\ \boldsymbol{\xi}\cdot\mathbf{n}=0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{\xi}\times\mathbf{n}=\mathbf{h}\times\mathbf{n} \quad \text{on } \Gamma. \end{gathered} \tag{5.10}$$

Now, observe that problem (5.10) is reduced to the variational problem: Find $\boldsymbol{\xi}\in\mathbf{V}_{-1,T}^2(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\mathbf{X}_{-1,T}^2(\Omega) \quad \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}=\int_\Omega \mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}+\langle\mathbf{h}\times\mathbf{n},\boldsymbol{\varphi}\rangle_\Gamma. \tag{5.11}$$

Indeed, every solution of (5.10) also solves (5.11). Conversely, let $\boldsymbol{\xi}$ a solution of the problem (5.11). Then,

$$\forall\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\Omega), \quad \langle\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}-\mathbf{f},\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}=0.$$

So $-\Delta\boldsymbol{\xi} = \mathbf{f}$ in Ω. Moreover, by the fact that $\boldsymbol{\xi}$ belongs to the space $\mathbf{V}^2_{-1,T}(\Omega)$, we have $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi}\cdot\mathbf{n} = 0$ on Γ. Then, it remains to verify the boundary condition $\mathbf{curl}\,\boldsymbol{\xi}\times\mathbf{n} = \mathbf{h}\times\mathbf{n}$ on Γ. Observe that $\boldsymbol{\xi}$ belongs to $\mathbf{E}^2(\Omega)$ so by (5.1) and comparing with (5.11) we deduce that for any $\boldsymbol{\varphi}\in\mathbf{X}^2_{-1,T}(\Omega)$, we have

$$\langle\mathbf{curl}\,\boldsymbol{\xi}\times\mathbf{n},\boldsymbol{\varphi}\rangle_\Gamma = \langle\mathbf{h}\times\mathbf{n},\,\boldsymbol{\varphi}\rangle_\Gamma.$$

Proceeding as in the proof of Theorem 4.3, we prove that $\mathbf{curl}\,\boldsymbol{\xi}\times\mathbf{n} = \mathbf{h}\times\mathbf{n}$ on Γ.

On the other hand, let us introduce the following problem: Find $\boldsymbol{\xi}\in\mathbf{V}^2_{-1,T}(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\mathbf{V}^2_{-1,T}(\Omega)\quad \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} + \langle\mathbf{h}\times\mathbf{n},\boldsymbol{\varphi}\rangle_\Gamma. \tag{5.12}$$

As in the proof of Theorem 4.3, we use Lax-Milgram lemma to prove the existence of a unique solution $\boldsymbol{\xi}$ in $\mathbf{V}^2_{-1,T}(\Omega)$ of Problem (5.12). Using Theorem 2.5, we prove that this solution $\boldsymbol{\xi}$ belongs to $\mathbf{W}^{1,2}_0(\Omega)$ and the following estimate follows immediately

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,2}_0(\Omega)} \leqslant C\Big(\|\mathbf{f}\|_{\mathbf{W}^{0,2}_1(\Omega)} + \|\mathbf{h}\times\mathbf{n}\|_{\mathbf{H}^{-1/2}(\Gamma)}\Big). \tag{5.13}$$

Next, we extend (5.12) to any test function $\boldsymbol{\varphi}$ in $\mathbf{X}^2_{-1,T}(\Omega)$. Let $\widetilde{\boldsymbol{\varphi}}\in\mathbf{X}^2_{-1,T}(\Omega)$ and let us solve the exterior Neumann problem:

$$\Delta\chi = \operatorname{div}\widetilde{\boldsymbol{\varphi}}\quad\text{in }\Omega\quad\text{and}\quad \frac{\partial\chi}{\partial\mathbf{n}} = 0\quad\text{on }\Gamma. \tag{5.14}$$

It is shown in [10, Theorem 4.12] that this problem has a unique solution χ in $W^{2,2}_0(\Omega)$ up to an additive constant. Then, we set

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \nabla\chi. \tag{5.15}$$

Since $W^{2,2}_0(\Omega)$ is imbedded in $W^{1,2}_{-1}(\Omega)$, then $\boldsymbol{\varphi}$ belongs to $\mathbf{V}^2_{-1,T}(\Omega)$. Now, if (5.12) holds, we have

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\mathbf{x} + \langle\mathbf{h}\times\mathbf{n},\widetilde{\boldsymbol{\varphi}}\rangle_\Gamma - \int_\Omega \mathbf{f}\cdot\,\nabla\chi\mathrm{d}\mathbf{x} - \langle\mathbf{h}\times\mathbf{n},\,\nabla\chi\rangle_\Gamma.$$

Using (1.8) and (5.4), we obtain

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\mathbf{x} + \langle\mathbf{h}\times\mathbf{n},\widetilde{\boldsymbol{\varphi}}\rangle_\Gamma.$$

This implies that problem (5.11) and problem (5.12) are equivalent and thus problem (5.10) has a unique solution $\boldsymbol{\xi}$ in $\mathbf{W}^{1,2}_0(\Omega)$. Finally, we set $\mathbf{z} = \boldsymbol{\xi}+\nabla\theta\in\mathbf{W}^{1,2}_0(\Omega)$ the unique solution of (5.5). Finally, (5.6) follows immediately from (5.13) and (5.9).

Regularity of the solution. We suppose in addition that $\mathbf{h}$ is in $\mathbf{H}^{1/2}(\Gamma)$, g in $H^{3/2}(\Gamma)$ and Ω' is of class $C^{2,1}$ and let $\mathbf{z}$ in $\mathbf{W}^{1,2}_0(\Omega)$ be the weak solution of Problem (5.5). Setting $\boldsymbol{\omega} = \mathbf{curl}\,\mathbf{z}$, then $\boldsymbol{\omega}$ satisfies

$$\boldsymbol{\omega}\in\mathbf{L}^2(\Omega),\quad \operatorname{div}\boldsymbol{\omega} = 0\in W^{0,2}_1(\Omega),$$
$$\mathbf{curl}\,\boldsymbol{\omega} = \mathbf{f}\in\mathbf{W}^{0,2}_1(\Omega),\quad \boldsymbol{\omega}\times\mathbf{n} = \mathbf{curl}\,\mathbf{z}\times\mathbf{n}\in\mathbf{H}^{1/2}(\Gamma).$$

Applying Theorem 4.4 (with $k = 0$), we prove that $\boldsymbol{\omega}$ belongs to $\mathbf{W}^{1,2}_1(\Omega)$. This implies that $\mathbf{z}$ satisfies

$$\mathbf{z}\in\mathbf{W}^{1,2}_0(\Omega),\quad \operatorname{div}\mathbf{z} = 0\in W^{1,2}_1(\Omega),\quad \mathbf{curl}\,\mathbf{z}\in\mathbf{W}^{1,2}_1(\Omega)\quad \mathbf{z}\cdot\mathbf{n}\in H^{3/2}(\Gamma).$$

Applying Proposition 3.2, we prove that $\mathbf{z}$ belongs to $\mathbf{W}^{2,2}_1(\Omega)$ and we have the estimate (5.7). $\square$

Next, we solve the Stokes problem (1.1).

Theorem 5.4 (Weak solutions for (1.1)). *Suppose that $g = 0$ and $\chi = 0$. For $\mathbf{f}$ given in $[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\Omega)]'$ and $\mathbf{h}$ given in $\mathbf{H}^{-1/2}(\Gamma)$ satisfying (5.3). The Stokes problem (1.1) has a unique solution $(\mathbf{u},\pi) \in \mathbf{W}^{1,2}_0(\Omega) \times L^2(\Omega)$ and we have the estimate*

$$\|\mathbf{u}\|_{\mathbf{W}^{1,2}_0(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C\big(\|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{-1}\,\mathrm{div},\Omega)]'} + \|\mathbf{h}\times\mathbf{n}\|_{\mathbf{H}^{-1/2}(\Gamma)}\big). \tag{5.16}$$

Proof. At first, observe that problem (1.1) is reduced to the variational problem: Find $\mathbf{u} \in \mathbf{V}^2_{-1,T}(\Omega)$ such that

$$\begin{gathered}\forall \boldsymbol{\varphi} \in \mathbf{V}^2_{-1,T}(\Omega),\\ \int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = \langle \mathbf{f},\boldsymbol{\varphi}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)]'\times\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)} + \langle \mathbf{h}\times\mathbf{n},\,\boldsymbol{\varphi}\rangle_\Gamma.\end{gathered} \tag{5.17}$$

Indeed, every solution of (1.1) also solves (5.17). Conversely, let $\mathbf{u}$ a solution of problem (5.17). Then

$$\text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega) \text{ such that } \mathrm{div}\,\boldsymbol{\varphi} = 0, \quad \langle -\Delta\mathbf{u} - \mathbf{f},\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

By De Rham theorem, there exists $q \in \mathcal{D}'(\Omega)$ such that

$$-\Delta\mathbf{u} - \mathbf{f} = \nabla q \quad \text{in } \Omega.$$

Note that $[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\Omega)]'$ is imbedded in $\mathbf{W}^{-1,2}_0(\Omega)$ and thus $-\Delta\mathbf{u}-\mathbf{f} \in \mathbf{W}^{-1,2}_0(\Omega)$. It follows from [7, Theorem 2.7], that there exists a unique real constant C and a unique $\pi \in L^2(\Omega)$ such that π has the decomposition $q = \pi + C$.

Observe that since $\mathbf{f}$ and $\nabla\pi$ are two elements of $[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)]'$, it is the same for $\Delta\mathbf{u}$. Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)$, we obtain for any $\boldsymbol{\varphi} \in \mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\Omega)$ such that $\mathrm{div}\,\boldsymbol{\varphi} = 0$:

$$\langle \nabla\pi,\boldsymbol{\varphi}\rangle_{[\mathring{\mathbf{H}}_{-1}(\mathrm{div},\,\Omega)]'\times\mathring{\mathbf{H}}_{-1}(\mathrm{div},\,\Omega)} = 0.$$

Moreover, if $\boldsymbol{\varphi} \in \mathbf{V}^2_{-1,T}(\Omega)$, using Corollary 5.2 we have

$$\begin{aligned}&\langle -\Delta\mathbf{u},\,\boldsymbol{\varphi}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\,\Omega)]'\times\mathbf{H}^2_{-1}(\mathrm{div},\,\Omega)}\\ &= \int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \langle \mathbf{curl}\,\mathbf{u}\times\mathbf{n},\boldsymbol{\varphi}\rangle_{\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^{1/2}(\Gamma)}.\end{aligned}$$

We deduce that for all $\boldsymbol{\varphi} \in \mathbf{V}^2_{-1,T}(\Omega)$

$$\langle \mathbf{curl}\,\mathbf{u}\times\mathbf{n},\boldsymbol{\varphi}\rangle_{\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^{1/2}(\Gamma)} = \langle \mathbf{h}\times\mathbf{n},\boldsymbol{\varphi}\rangle_{\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^{1/2}(\Gamma)}.$$

Let now $\boldsymbol{\mu}$ any element of the space $\mathbf{H}^{1/2}(\Gamma)$. So, there exists an element $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_0(\Omega)$ such that $\mathrm{div}\,\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$ on Γ. It is clear that $\boldsymbol{\varphi} \in \mathbf{V}^2_{-1,T}(\Omega)$ and

$$\begin{aligned}\langle \mathbf{curl}\,\mathbf{u}\times\mathbf{n},\boldsymbol{\mu}\rangle_\Gamma - \langle \mathbf{h}\times\mathbf{n},\boldsymbol{\mu}\rangle_\Gamma &= \langle \mathbf{curl}\,\mathbf{u}\times\mathbf{n},\boldsymbol{\mu}_t\rangle_\Gamma - \langle \mathbf{h}\times\mathbf{n},\,\boldsymbol{\mu}_t\rangle_\Gamma\\ &= \langle \mathbf{curl}\,\mathbf{u}\times\mathbf{n},\boldsymbol{\varphi}\rangle_\Gamma - \langle \mathbf{h}\times\mathbf{n},\boldsymbol{\varphi}\rangle_\Gamma = 0.\end{aligned}$$

This implies that $\mathbf{cur}\,\mathbf{u}\times\mathbf{n} = \mathbf{h}\times\mathbf{n}$ on Γ. As a consequence, Problem (5.17) and (1.1) are equivalent. As in the proof of Theorem 4.3, we use Lax-Milgram lemma to prove the existence of a unique solution $\mathbf{u}$ in $\mathbf{V}^2_{-1,T}(\Omega)$ of Problem (5.17). Using

Theorem 2.5, we prove that this solution $\mathbf{u}$ belongs to $\mathbf{W}_0^{1,2}(\Omega)$. Then the pair $(\mathbf{u}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times \mathbf{L}^2(\Omega)$ is the unique solution of the problem (1.1). The estimate (5.16) follows from (2.4). $\square$

Corollary 5.5. *Let* $\mathbf{f}$*,* χ*,* g*,* $\mathbf{h}$ *such that*

$$\mathbf{f} \in [\mathring{\mathbf{H}}^2_{-1}(\mathrm{div}, \Omega)]', \quad \chi \in L^2(\Omega), \quad g \in H^{1/2}(\Gamma), \quad \mathbf{h} \in \mathbf{H}^{-1/2}(\Gamma),$$

and that (5.3) *holds. Then, the Stokes problem* (1.1) *has a unique solution* $(\mathbf{u}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ *and we have:*

$$\begin{aligned} &\|\mathbf{u}\|_{\mathbf{W}_0^{1,2}(\Omega)} + \|\pi\|_{L^2(\Omega)} \\ &\leqslant C\Big(\|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div}, \Omega)]'} + \|\chi\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{H}^{-1/2}(\Gamma)}\Big). \end{aligned} \tag{5.18}$$

Proof. **First case:** We suppose that $\chi = 0$. Let $\theta \in W_0^{2,2}(\Omega)$ be a solution of the exterior Neumann problem (5.8). Setting $\mathbf{z} = \mathbf{u} - \nabla\theta$, then, problem (1.1) becomes: Find $(\mathbf{z}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ such that

$$\begin{gathered} -\Delta \mathbf{z} + \nabla \pi = \mathbf{f} \quad \text{and} \quad \mathrm{div}\, \mathbf{z} = 0 \quad \text{in } \Omega, \\ \mathbf{z} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \mathbf{z} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma. \end{gathered} \tag{5.19}$$

Due to Theorem 5.4, this problem has a unique solution $(\mathbf{z}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. Thus $\mathbf{u} = \mathbf{z} + \nabla\theta$ belongs to $\mathbf{W}_0^{1,2}(\Omega)$ and using (5.9) and (5.16), we deduce that

$$\|\mathbf{u}\|_{\mathbf{W}_0^{1,2}(\Omega)} + \|\pi\|_{L^2(\Omega)} \le C\big(\|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div}, \Omega)]'} + \|g\|_{H^{1/2}(\Gamma)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{H}^{-1/2}(\Gamma)}\big). \tag{5.20}$$

Second case: We suppose that $\chi \in L^2(\Omega)$. We solve the following Neumann problem in Ω:

$$\Delta\theta = \chi \quad \text{in } \Omega, \quad \frac{\partial\theta}{\partial\mathbf{n}} = g \quad \text{on } \Gamma. \tag{5.21}$$

It follows from [7, Theorem 3.9] that Problem (5.21) has a unique solution θ in $W_0^{2,2}(\Omega)/\mathbb{R}$ and we have

$$\|\theta\|_{W_0^{2,2}(\Omega)/\mathbb{R}} \leqslant C\Big(\|\chi\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\Big). \tag{5.22}$$

Setting $\mathbf{z} = \mathbf{u} - \nabla\theta$, then Problem problem (1.1) becomes: Find $(\mathbf{z}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ such that

$$\begin{gathered} -\Delta \mathbf{z} + \nabla \pi = \mathbf{f} + \nabla\chi \quad \text{and} \quad \mathrm{div}\, \mathbf{z} = 0 \quad \text{in } \Omega, \\ \mathbf{z} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \mathbf{z} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma. \end{gathered} \tag{5.23}$$

Observe that $\mathbf{f} + \nabla\chi$ belongs to $[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div}, \Omega)]'$ and $\langle \nabla\chi, \mathbf{v}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div}, \Omega)]' \times \mathring{\mathbf{H}}^2_{-1}(\mathrm{div}, \Omega)} = 0$ for all $\mathbf{v}$ in $\mathbf{Y}^2_{1,T}(\Omega)$. According to the first step, this problem has a unique solution $(\mathbf{z}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. Thus $\mathbf{u} = \mathbf{z} + \nabla\theta$ belongs to $\mathbf{W}_0^{1,2}(\Omega)$ and estimate (5.18) follows from (5.20) and (5.22). $\square$

Now, we study the problem (1.2).

Theorem 5.6 (Weak solutions for (1.2))**.** *Assume that* $\chi = 0$*,* $\mathbf{f}$ *is in the space* $[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)]'$*,* $\mathbf{g}$ *is in* $\mathbf{H}^{1/2}(\Gamma)$ *and* π_0 *is in* $H^{1/2}(\Gamma)$*, satisfying the compatibility condition*

$$\forall \boldsymbol{\lambda} \in \mathbf{Y}^2_{1,N}(\Omega), \quad \langle \mathbf{f}, \boldsymbol{\lambda}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)]' \times \mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)} = \langle \boldsymbol{\lambda} \cdot \mathbf{n}, \pi_0\rangle_\Gamma. \tag{5.24}$$

Then the Stokes problem (1.2) *has a unique solution* $(\mathbf{u}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times W_1^{1,2}(\Omega)$ *and we have*

$$\|\mathbf{u}\|_{\mathbf{W}_0^{1,2}(\Omega)} + \|\pi\|_{W_1^{1,2}(\Omega)} \leqslant C(\|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\,\Omega)]'} + \|\mathbf{g}\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)} + \|\pi_0\|_{H^{1/2}(\Gamma)}). \tag{5.25}$$

Proof. First, we consider the problem

$$\Delta\pi = \operatorname{div}\mathbf{f} \quad \text{in } \Omega, \quad \pi = \pi_0 \quad \text{on } \Gamma. \tag{5.26}$$

Since $\mathbf{f} \in [\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)]'$, we deduce from Proposition 1.4 that $\operatorname{div}\mathbf{f}$ belongs to $\mathbf{W}_1^{-1,2}(\Omega)$. Now, let $(v(1) - 1)$ an element of $\mathcal{A}_0^{\Delta}$, it is clear that $\nabla(v(1) - 1)$ belongs to $\mathbf{Y}^2_{1,N}(\Omega)$. Then using the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\mathring{\mathbf{W}}^{1,2}_{-1}(\Omega)$ and (5.24), we prove that

$$\begin{aligned}\langle \operatorname{div}\mathbf{f}, (v(1)-1)\rangle_{\mathbf{W}_1^{-1,2}(\Omega)\times\mathring{\mathbf{W}}^{1,2}_{-1}(\Omega)} &= -\langle \mathbf{f}, \nabla(v(1)-1)\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)]'\times\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)} \\ &= -\langle \nabla(v(1)-1)\cdot\mathbf{n}, \pi_0\rangle_\Gamma.\end{aligned} \tag{5.27}$$

Since (5.27) is satisfied, we apply [7, Theorem 3.6] to prove that Problem (5.26) has a unique solution $\pi \in W_1^{1,2}(\Omega)$ and we have the following estimate:

$$\|\pi\|_{W_1^{1,2}(\Omega)} \leqslant C\Big(\|\operatorname{div}\mathbf{f}\|_{\mathbf{W}_1^{-1,2}(\Omega)} + \|\pi_0\|_{H^{1/2}(\Gamma)}\Big). \tag{5.28}$$

Setting $\mathbf{F} = \mathbf{f} - \nabla\pi$, then $\mathbf{F}$ belongs to $[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\,\Omega)]'$. Thus Problem (1.2) becomes: Find $\mathbf{u} \in \mathbf{W}_0^{1,2}(\Omega)$ such that:

$$\begin{gathered} -\Delta\mathbf{u} = \mathbf{F} \quad \text{and} \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u}\times\mathbf{n} = \mathbf{g}\times\mathbf{n} \quad \text{on } \Gamma \quad \text{and} \quad \int_\Gamma \mathbf{u}\cdot\mathbf{n}\,d\sigma = 0. \end{gathered} \tag{5.29}$$

Using (5.24) and the fact that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)$, we prove that

$$\forall \boldsymbol{\lambda} \in \mathbf{Y}^2_{1,N}(\Omega), \quad \langle \mathbf{F}, \boldsymbol{\lambda}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)]'\times\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)} = 0. \tag{5.30}$$

Therefore, $\mathbf{F}$ satisfies the assumptions of Corollary 4.2 and thus Problem (5.29) has a unique solution $\mathbf{u} \in \mathbf{W}_0^{1,2}(\Omega)$ with

$$\|\mathbf{u}\|_{\mathbf{W}_0^{1,2}(\Omega)} \leqslant C\Big(\|\mathbf{F}\|_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)]'} + \|\mathbf{g}\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big). \tag{5.31}$$

Thus estimate (5.25) follows from (5.31) and from (5.28). □

Corollary 5.7. *Let* $\mathbf{f}$*,* χ*,* $\mathbf{g}$*,* π_0 *such that*

$$\mathbf{f} \in [\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)]', \quad \chi \in W_1^{1,2}(\Omega), \quad \mathbf{g} \in \mathbf{H}^{1/2}(\Gamma), \quad \pi_0 \in H^{1/2}(\Gamma),$$

and satisfying the compatibility condition:

$$\forall \boldsymbol{\lambda} \in \mathbf{Y}^2_{1,N}(\Omega), \quad \langle \mathbf{f}, \boldsymbol{\lambda}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)]'\times\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)} = \langle \boldsymbol{\lambda}\cdot\mathbf{n}, \pi_0 - \chi\rangle_\Gamma. \tag{5.32}$$

Then the Stokes problem (1.2) *has a unique solution* $(\mathbf{u}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times W_1^{1,2}(\Omega)$*. Moreover, we have the estimate*

$$\begin{aligned} &\|\mathbf{u}\|_{\mathbf{W}_0^{1,2}(\Omega)} + \|\pi\|_{W_1^{1,2}(\Omega)} \\ &\leqslant C(\|\mathbf{f}\|_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\,\Omega)]'} + \|\mathbf{g}\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)} + \|\pi_0\|_{H^{1/2}(\Gamma)} + \|\chi\|_{W_1^{1,2}(\Omega)}). \end{aligned} \tag{5.33}$$

Proof. First, we consider the problem

$$\Delta\pi = \operatorname{div}\mathbf{f} + \Delta\chi \quad \text{in } \Omega, \quad \pi = \pi_0 \quad \text{on } \Gamma. \tag{5.34}$$

Since $\mathbf{f} \in [\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)]'$, we deduce from Proposition 1.4 that $\operatorname{div}\mathbf{f} + \Delta\chi$ belongs to $\mathbf{W}_1^{-1,2}(\Omega)$. Proceeding as in the proof of Theorem 5.6, we prove that

$$\langle \operatorname{div}\mathbf{f} + \Delta\chi, (v(1) - 1)\rangle_{\mathbf{W}_1^{-1,2}(\Omega)\times\mathring{\mathbf{W}}^{1,2}_{-1}(\Omega)} = -\langle \nabla(v(1) - 1)\cdot\mathbf{n}, \pi_0\rangle_\Gamma$$

and then we apply [7, Theorem 3.6] to prove that Problem (5.34) has a unique solution $\pi \in W_1^{1,2}(\Omega)$ and we have the following estimate:

$$\|\pi\|_{W_1^{1,2}(\Omega)} \leqslant C\Big(\|\operatorname{div}\mathbf{f} + \Delta\chi\|_{\mathbf{W}_1^{-1,2}(\Omega)} + \|\pi_0\|_{H^{1/2}(\Gamma)}\Big). \tag{5.35}$$

Thus Problem (1.2) becomes: Find $\mathbf{u} \in \mathbf{W}_0^{1,2}(\Omega)$ such that

$$\begin{gathered} -\Delta\mathbf{u} = \mathbf{f} - \nabla\pi \quad \text{and} \quad \operatorname{div}\mathbf{u} = \chi \quad \text{in } \Omega, \\ \mathbf{u}\times\mathbf{n} = \mathbf{g}\times\mathbf{n} \quad \text{on } \Gamma \quad \text{and} \quad \int_\Gamma \mathbf{u}\cdot\mathbf{n}\, d\sigma = 0. \end{gathered} \tag{5.36}$$

On the other hand, let us solve the Dirichlet problem

$$\Delta\theta = \chi \quad \text{in } \Omega, \quad \theta = 0 \quad \text{on } \Gamma.$$

Since $W_1^{1,2}(\Omega)$ is imbedded in $L^2(\Omega)$, it follows from [7, Theorem 3.5], that this problem has a unique solution $\theta \in W_0^{2,2}(\Omega)$ (there is no compatibility condition) and we have the estimate

$$\|\theta\|_{W_0^{2,2}(\Omega)} \leqslant C\|\chi\|_{L^2(\Omega)}. \tag{5.37}$$

Setting

$$\mathbf{z} = \mathbf{u} - \Big(\nabla\theta - \frac{1}{C_1}\langle\nabla\theta\cdot\mathbf{n}, 1\rangle_\Gamma \nabla(v(1) - 1)\Big),$$

where $v(1)$ is the unique solution in $W_0^{1,2}(\Omega)$ of the Dirichlet problem (2.1) and $C_1 = \int_\Gamma \frac{\partial v(1)}{\partial\mathbf{n}}\, d\sigma$. We know from [7, Lemma 3.11] that $C_1 > 0$ and that $\nabla(v(1) - 1)$ belongs to $\mathbf{Y}^2_{1,N}(\Omega)$. Then Problem (5.36) becomes: Find $\mathbf{z} \in \mathbf{W}_0^{1,2}(\Omega)$ such that

$$\begin{gathered} -\Delta\mathbf{z} = \mathbf{f} - \nabla\pi + \nabla\chi \quad \text{and} \quad \operatorname{div}\mathbf{z} = 0 \quad \text{in } \Omega, \\ \mathbf{z}\times\mathbf{n} = \mathbf{g}\times\mathbf{n} \quad \text{on } \Gamma \quad \text{and} \quad \int_\Gamma \mathbf{z}\cdot\mathbf{n}\, d\sigma = 0. \end{gathered} \tag{5.38}$$

Now, we will solve the Problem (5.38). Setting $\mathbf{F} = \mathbf{f} - \nabla\pi + \nabla\chi$, then $\mathbf{F}$ belongs to $[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)]'$. Using (5.32) and the fact that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl}, \Omega)$, we prove that

$$\forall\boldsymbol{\lambda} \in \mathbf{Y}^2_{1,N}(\Omega), \quad \langle\mathbf{F}, \boldsymbol{\lambda}\rangle_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)]'\times\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)} = 0. \tag{5.39}$$

Therefore, $\mathbf{F}$ satisfies the assumptions of Corollary 4.2 and thus Problem (5.38) has a unique solution $\mathbf{z} \in \mathbf{W}_0^{1,2}(\Omega)$ with

$$\|\mathbf{z}\|_{\mathbf{W}_0^{1,2}(\Omega)} \leqslant C\Big(\|\mathbf{F}\|_{[\mathring{\mathbf{H}}^2_{-1}(\mathbf{curl},\Omega)]'} + \|\mathbf{g}\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big) \tag{5.40}$$

and estimate (5.33) holds. $\square$

6. Strong solutions for (1.1) and (1.2)

We prove in this sequel the existence and the uniqueness of strong solutions for Problem (1.1) and (1.2), we start with Problem (1.1).

Theorem 6.1. *Suppose that Ω' is of class $C^{2,1}$. Let $\mathbf{f}$, χ, g, $\mathbf{h}$ be such that*

$$\mathbf{f} \in \mathbf{W}_1^{0,2}(\Omega), \quad \chi \in W_1^{1,2}(\Omega), \; g \in H^{3/2}(\Gamma), \quad \mathbf{h} \in \mathbf{H}^{1/2}(\Gamma),$$

and that (5.3) *holds. Then, the Stokes problem* (1.1) *has a unique solution* $(\mathbf{u}, \pi) \in \mathbf{W}_1^{2,2}(\Omega) \times W_1^{1,2}(\Omega)$ *and we have*

$$\|\mathbf{u}\|_{\mathbf{W}_1^{2,2}(\Omega)} + \|\pi\|_{W_1^{1,2}(\Omega)} \leqslant \|\mathbf{f}\|_{\mathbf{W}_1^{0,2}(\Omega)} + \|\chi\|_{W_1^{1,2}(\Omega)} + \|g\|_{H^{3/2}(\Gamma)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma)}. \tag{6.1}$$

Proof. **First case:** $\chi = 0$. Since $\mathbf{W}_1^{0,2}(\Omega)$ is included in $[\mathring{\mathbf{H}}^2_{-1}(\mathrm{div},\Omega)]'$, we deduce that we are under the hypothesis of Corollary 5.5 and so Problem (1.1) has a unique solution $(\mathbf{u}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. Setting $\mathbf{z} = \mathbf{curl}\,\mathbf{u}$, then $\mathbf{z}$ satisfies

$$\begin{gathered} \mathbf{z} \in \mathbf{L}^2(\Omega), \quad \mathrm{div}\,\mathbf{z} = 0 \in W_1^{0,2}(\Omega), \\ -\Delta \mathbf{z} = \mathbf{curl}\,\mathbf{f}, \quad \mathbf{z} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \in \mathbf{H}^{1/2}(\Gamma). \end{gathered}$$

Applying Corollary 4.2 (with $k = -1$) and the uniqueness argument, we prove that $\mathbf{z}$ belongs to $\mathbf{W}_1^{1,2}(\Omega)$. This implies that $\mathbf{u}$ satisfies

$$\mathbf{u} \in \mathbf{W}_0^{1,2}(\Omega), \quad \mathrm{div}\,\mathbf{u} = 0 \in W_1^{1,2}(\Omega), \quad \mathbf{curl}\,\mathbf{u} \in \mathbf{W}_1^{1,2}(\Omega), \quad \mathbf{u} \cdot \mathbf{n} = g \in H^{3/2}(\Gamma).$$

Applying Proposition 3.2, we prove that $\mathbf{u}$ belongs to $\mathbf{W}_1^{2,2}(\Omega)$ and thus $\nabla \pi = \mathbf{f} + \Delta \mathbf{u} \in \mathbf{W}_1^{0,2}(\Omega)$. Since π is in $L^2(\Omega)$ then π is in $W_1^{1,2}(\Omega)$.

Second case: χ is in $W_1^{1,2}(\Omega)$. Since Ω' is of class $C^{2,1}$, it follows from [7, Theorem 3.9] that there exists a unique solution θ in $W_1^{3,2}(\Omega)/\mathbb{R}$ satisfies Problem (5.21) and

$$\|\theta\|_{W_1^{3,2}(\Omega)/\mathbb{R}} \leqslant C\Big(\|\chi\|_{W_1^{1,2}(\Omega)} + \|g\|_{H^{3/2}(\Gamma)}\Big). \tag{6.2}$$

The rest of the proof is similar to that Corollary 5.5. □

Remark 6.2. Assume that the hypothesis of Theorem 6.1 hold and suppose in addition that $\chi = 0$. Let $(\mathbf{u}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ the unique solution of Problem (1.1) then π satisfies the problem

$$\mathrm{div}(\nabla \pi - \mathbf{f}) = 0 \quad \text{in } \Omega \quad \text{and} \quad (\nabla \pi - \mathbf{f}) \cdot \mathbf{n} = -\,\mathrm{div}_\Gamma(\mathbf{h} \times \mathbf{n}) \quad \text{on } \Gamma. \tag{6.3}$$

It follows from [10] that Problem (6.3) has a solution π in $W_1^{1,2}(\Omega)$. Setting $\mathbf{F} = \nabla \pi - \mathbf{f} \in \mathbf{W}_1^{0,2}(\Omega)$. Then problem (1.1) becomes

$$\begin{gathered} -\Delta \mathbf{u} = \mathbf{F} \quad \text{and} \quad \mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g \quad \text{and} \quad \mathbf{curl}\,\mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma. \end{gathered}$$

Therefore, $\mathbf{F}$, g and $\mathbf{h}$ satisfy the assumptions of Proposition 5.3 and thus $\mathbf{u}$ belongs to $\mathbf{W}_1^{2,2}(\Omega)$.

Next, we study the regularity of the solution for Problem (1.2).

Theorem 6.3. *Suppose that Ω' is of class $C^{2,1}$. Let $\mathbf{f}$, χ, $\mathbf{g}$, π_0 be such that*

$$\mathbf{f} \in \mathbf{W}_1^{0,2}(\Omega), \quad \chi \in W_1^{1,2}(\Omega), \quad \mathbf{g} \in \mathbf{H}^{3/2}(\Gamma), \quad \pi_0 \in H^{1/2}(\Gamma),$$

and satisfying the compatibility condition (5.32)*. Then the Stokes problem* (1.2) *has a unique solution* $(\mathbf{u},\pi)\in \mathbf{W}_1^{2,2}(\Omega)\times W_1^{1,2}(\Omega)$*. Moreover, we have the estimate*

$$\|\mathbf{u}\|_{\mathbf{W}_1^{2,2}(\Omega)}+\|\pi\|_{W_1^{1,2}(\Omega)}\leqslant C(\|\mathbf{f}\|_{\mathbf{W}_1^{0,2}(\Omega)}+\|\mathbf{g}\times\mathbf{n}\|_{\mathbf{H}^{3/2}(\Gamma)}+\|\pi_0\|_{H^{1/2}(\Gamma)}+\|\chi\|_{W_1^{1,2}(\Omega)}). \tag{6.4}$$

Proof. **First case:** We suppose that $\chi=0$. Since $\mathbf{W}_1^{0,2}(\Omega)$ is in $[\mathring{\mathbf{H}}_{-1}^{2}(\mathbf{curl},\Omega)]'$, we deduce that we are under the hypothesis of Corollary 5.6 and so Problem (1.2) has a unique solution $(\mathbf{u},\pi)\in \mathbf{W}_0^{1,2}(\Omega)\times W_1^{1,2}(\Omega)$. Setting $\mathbf{z}=\mathbf{curl}\,\mathbf{u}$. Observe that $\mathbf{u}\times\mathbf{n}=\mathbf{g}\times\mathbf{n}$ belongs to $\mathbf{H}^{3/2}(\Gamma)$ and thus $\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}$ belongs to $H^{1/2}(\Gamma)$ and so $\mathbf{z}$ satisfies

$$\begin{gathered}\mathbf{z}\in\mathbf{L}^2(\Omega),\quad \operatorname{div}\mathbf{z}=0\in W_1^{0,2}(\Omega),\\ \mathbf{curl}\,\mathbf{z}=\mathbf{f}-\nabla\pi\in\mathbf{W}_1^{0,2}(\Omega),\quad \mathbf{z}\cdot\mathbf{n}=\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}\in\mathbf{H}^{1/2}(\Gamma).\end{gathered}$$

Applying Proposition 3.1 (with $k=0$), we prove that $\mathbf{z}$ belongs to $\mathbf{W}_1^{1,2}(\Omega)$. This implies that $\mathbf{u}$ satisfies

$$\begin{gathered}\mathbf{u}\in\mathbf{W}_0^{1,2}(\Omega),\quad \operatorname{div}\mathbf{u}=0\in W_1^{1,2}(\Omega),\\ \mathbf{curl}\,\mathbf{u}\in\mathbf{W}_1^{1,2}(\Omega),\quad \mathbf{u}\times\mathbf{n}=\mathbf{g}\times\mathbf{n}\in H^{3/2}(\Gamma).\end{gathered}$$

Applying Proposition 4.5, we prove that $\mathbf{u}$ belongs to $\mathbf{W}_1^{2,2}(\Omega)$.

Second case: χ is in $W_1^{1,2}(\Omega)$. The proof of this case is very similar to that Corollary 5.7. □

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