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BACKWARD UNIQUENESS FOR HEAT EQUATIONS WITH COEFFICIENTS OF BOUNDED VARIATION IN TIME

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Abstract. Uniqueness of solutions to the backward Cauchy problem for heat equations with coefficients of bounded variation in time is shown through the Carleman estimate.

1. Introduction

We consider a heat operator in the time backward form

$$Lu = \partial_t + \sum_{j,k=1}^{d} \partial_{x_j}(a_{jk}(x,t)\partial_{x_j} u), \tag{1.1}$$

with real bounded and measurable coefficients $a_{jk}(x,t)$ on $\mathbb{R}^d \times [0,T]$, for some $T > 0$, satisfying $a_{jk}(x,t) = a_{kj}(x,t)$ $(j,k = 1,2,\dots,d)$ and

$$\sum_{j,k=1}^{d} a_{jk}(x,t)\xi_j\xi_k \geq D_0|\xi|^2 \tag{1.2}$$

for any $\xi \in \mathbb{R}^d$ with some positive D_0.

The Cauchy problem for $Lu = f$ on $\mathbb{R}^d \times [0,T]$ with Cauchy data on $t = 0$ is not well-posed. But the uniqueness of solutions to the Cauchy problem is valid under some conditions on the coefficients. Since the work of Mizohata [5], there are many works on this problem. See for example the survey paper of Vessella [6] and the papers cited therein. But it seems that the backward uniqueness for heat operators with discontinuous coefficients is not well studied.

We consider an operator whose coefficients $a_{jk}(x,t)$ $(j,k = 1,2,\dots,d)$ are of bounded variation in t uniformly with respect to $x \in \mathbb{R}^d$. That is, there exists a constant $M \geq 0$ such that we have

$$\sum_{l=1}^{L} \sup_{x\in\mathbb{R}^d} |a_{jk}(x,t_l) - a_{jk}(x,t_{l-1})| \leq M \tag{1.3}$$

for any partition of $[0,T]$, $t_0 = 0 < t_1 < \dots < t_L = T$, which means that $a_{jk}(x,t)$ is a $C_b^0(\mathbb{R}^d)$-valued function on $[0,T]$ with bounded variation. Here $C_b^0(\mathbb{R}^d)$ is a space of bounded and continuous functions on $\mathbb{R}^d$. While we assume that $a_{jk}(x,t)$ are

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Lipschitz continuous in x uniformly with respect to t, that is, we assume that we have, with some $L \geq 0$,

$$\sum_{j,k=1}^{d} |a_{jk}(x,t) - a_{jk}(y,t)| \leq L|x-y| \tag{1.4}$$

for any $x, y \in \mathbb{R}^d$ and any $t \in [0.T]$. Under these conditions we show the following.

Theorem 1.1. *Assume that the coefficients $a_{jk}(x,t)$ ($j,k = 1,2,\dots,d$) of the operator* (1.1) *are real, bounded and symmetric and satisfy* (1.2), (1.3) *and* (1.4)*. Let $u(x,t) \in L^2([0,T], H^1(\mathbb{R}^d_x)) \cap C^0([0,T], L^2(\mathbb{R}^d_x))$ satisfy $Lu \in L^2([0,T], L^2(\mathbb{R}^d_x))$,*

$$\|Lu(\cdot,t)\| \leq C\|u(\cdot,t)\|_1 \quad \textit{almost all } t \in [0,T],$$

and $u(x,0) = 0$. Then we have $u(x,t) = 0$ on $\mathbb{R}^d \times [0,T]$.

Here the spaces L^2 and H^1 and their norm $\|\cdot\|$ and $\|\cdot\|_1$ are standard ones whose definitions are given below. For a Banach space X, we denote by $L^2([0,T], X)$ and $C^0([0,T], X)$ the space of X-valued square integrable functions and the space of X-valued continuous functions respectively.

Remark 1.2. We note that $Lu \in L^2([0,T], L^2(\mathbb{R}^d_x))$ implies $\partial_t u(x,t)$ being in $L^2([0,T], H^{-1}(\mathbb{R}^d_x))$. While $\partial_t u(x,t)$ being in $L^2([0,T], H^{-1}(\mathbb{R}^d_x))$ and $u(x,t)$ in $L^2([0,T], H^1(\mathbb{R}^d_x))$ imply $u(x,t) \in C^0([0,T], L^2(\mathbb{R}^d_x))$ (see for example [3, Theorem 1. §1.1 Ch. XVIII]). Then the assumption $u(x,t) \in C^0([0,T], L^2(\mathbb{R}^d_x))$ follows from the other assumptions.

Theorem 1.1 is shown by using the Carleman estimate. Here, in order to indicate the principal idea of proof, we show how to obtain the Carleman estimate for a simple operator $\partial_t u + a(t)\partial_x^2 u$. We assume that the coefficient $a(t)$ satisfies that $C_1 \leq a(t) \leq C_2$ with positive C_1, C_2, and that for any positive ε there exists $T_\varepsilon \in (0, \varepsilon]$ such that we have, for any $h \in [0, T_\varepsilon]$,

$$\int_0^{T_\varepsilon} |a(t+h) - a(t)|\, dt \leq \varepsilon h.$$

We remark that the second assumption is satisfied if $a(t)$ is of bounded variation and continuous at $t = 0$.

We define the weight function $\psi_{1,\gamma}(t)$ by $\psi_{1,\gamma}(t) = \gamma \int_t^{T_\varepsilon} e^{\psi_\gamma(t)}\, d\tau$, where

$$\psi_\gamma(t) = \int_t^{T_\varepsilon} \frac{1}{\varepsilon}(1 + \gamma|a(\tau + 1/\gamma) - a(\tau)| + \gamma \int_0^1 |a(\tau + s/\gamma) - a(\tau)|\, ds)\, d\tau.$$

We note that $0 \leq \psi_\gamma(t) \leq 5/2$ when $\gamma \geq 1/T_\varepsilon$. Under these conditions, we show that there exist positive ε, γ_0 and C such that we have the estimate

$$\int_0^{T_\varepsilon} (\gamma\|e^{2\psi_{1,\gamma}(t)}u\|^2 + \|e^{2\psi_{1,\gamma}(t)}\partial_x u\|^2)\, dt \leq C \int_0^{T_\varepsilon} \|e^{2\psi_{1,\gamma}(t)}(\partial_t u + a(t)\partial_x^2 u)\|^2\, dt$$

for any $\gamma \geq \gamma_0$ and $u(x,t)$ satisfying $u(x,0) = 0$ and $u(x, T_\varepsilon) = 0$. Plancherel's theorem implies that we have only to show the estimate

$$(\gamma + \xi^2) \int_0^{T_\varepsilon} |e^{2\psi_{1,\gamma}(t)}u(t)|^2\, dt \leq C \int_0^{T_\varepsilon} |e^{2\psi_{1,\gamma}(t)}\hat{L}u(t)|^2\, dt \tag{1.5}$$

for any $\xi \in \mathbb{R}$, $\gamma \ge \gamma_0$ and $u(t)$ satisfying $u(0) = 0$ and $u(T_\varepsilon) = 0$. Here $\hat{L}u = \frac{d}{dt}u - a(t)\xi^2 u$. By setting $u(t) = e^{-\psi_{1,\gamma}(t)}v(t)$, we see that the estimate above is equivalent to

$$(\gamma + \xi^2)\int_0^{T_\varepsilon} |v(t)|^2\,dt \le C\int_0^{T_\varepsilon} |\tilde{L}v(t)|^2\,dt \tag{1.6}$$

where $\tilde{L} = \frac{d}{dt} + \gamma e^{\psi_\gamma(t)} - a(t)\xi^2$. In the following, we show (1.6) for a real valued $v(t)$ satisfying $v(0) = 0$ and $v(T_\varepsilon) = 0$.

We see from $v(0) = v(T_\varepsilon) = 0$ that

$$-\int_0^{T_\varepsilon} v(t)\tilde{L}v(t)\,dt = \int_0^{T_\varepsilon} (a(t)\xi^2 - \gamma e^{\psi_\gamma(t)})(v(t))^2\,dt.$$

Then, when $C_1\xi^2 \ge 2e^{5/2}\gamma$, we have

$$\int_0^{T_\varepsilon} |v(t)||\tilde{L}v(t)|\,dt \ge (C_1\xi^2/2)\int_0^{T_\varepsilon} |v(t)|^2\,dt.$$

Hence we have

$$C_3\int_0^{T_\varepsilon} |\tilde{L}v(t)|^2\,dt \ge (\xi^2 + \gamma)\int_0^{T_\varepsilon} |v(t)|^2\,dt$$

with some C_3, when $C_1\xi^2 e^{-5/2}/2 \ge \gamma \ge 1$.

For the case where $C_1\xi^2 e^{-5/2}/2 \le \gamma$, we first remark that

$$\int_0^{T_\varepsilon} (\tilde{L}v(t))^2\,dt = \int_0^{T_\varepsilon} \Big((v'(t))^2 + (\gamma e^{\psi_\gamma(t)} - a(t)\xi^2)^2(v(t))^2\Big)\,dt + I$$

where

$$I = 2\int_0^{T_\varepsilon} v'(t)(\gamma e^{\psi_\gamma(t)} - a(t)\xi^2)v(t)\,dt.$$

To estimate I, we regularize $a(t)$ by $a_\gamma(t) = \int_0^1 a(t + s/\gamma)\,ds$. Since $a_\gamma(t) = \gamma\int_t^{t+1/\gamma} a(s)\,ds$, we see that

$$|a_\gamma(t) - a(t)| \le \int_0^1 |a(t + s/\gamma) - a(t)|\,ds, \quad |a'_\gamma(t)| \le \gamma|a(t + 1/\gamma) - a(t)|.$$

Note that $|a_\gamma(t)| \le C_2$. We set $I = I_1 + I_2$, where

$$I_1 = 2\int_0^{T_\varepsilon} v'(t)(\gamma e^{\psi_\gamma(t)} - a_\gamma(t)\xi^2)(v(t))\,dt,$$
$$I_2 = 2\int_0^{T_\varepsilon} v'(t)(a_\gamma(t) - a(t))\xi^2 v(t))\,dt.$$

From $|a_\gamma(t) - a(t)|^2 \le 2C_2|a_\gamma(t) - a(t)|$ and Schwarz's inequality, we obtain

$$|I_2| \le \int_0^{T_\varepsilon} |v'(t)|^2\,dt + 2C_2\int_0^{T_\varepsilon}\int_0^1 |a(t + s/\gamma) - a(t)|\,ds\xi^4|v(t)|^2\,dt.$$

Note

$$I_1 = \int_0^{T_\varepsilon} (\gamma e^{\psi_\gamma(t)} - a_\gamma(t)\xi^2)\frac{d}{dt}(v(t))^2\,dt,$$

$v(0) = 0$, $v(T_\varepsilon) = 0$, and

$$-\frac{d}{dt}e^{\psi_\gamma(t)} = \frac{e^{\psi_\gamma(t)}}{\varepsilon}(1 + \gamma|a(t + 1/\gamma) - a(t)| + \gamma\int_0^1 |a(t + s/\gamma) - a(t)|\,ds).$$

By integrating by parts and $e^{\psi_\gamma(t)} \geq 1$,

$$I_1 \geq \int_0^{T_\varepsilon} \Big(\frac{\gamma}{\varepsilon}\big(1+\gamma|a(t+1/\gamma)-a(t)|+\gamma\int_0^1 |a(t+s/\gamma)-a(t)|\,ds\big)+a'_\gamma(t)\xi^2\Big)(v(t))^2\,dt.$$

Hence, noting $|a'_\gamma(t)| \leq \gamma|a(t+1/\gamma)-a(t)|$, we see that, if $\gamma/\varepsilon \geq \xi^2$,

$$I_1 \geq \int_0^{T_\varepsilon} (\frac{\gamma}{\varepsilon} + \frac{\gamma^2}{\varepsilon}\int_0^1 |a(\tau+s/\gamma)-a(\tau)|\,ds)(v(t))^2\,dt.$$

Then from $I \geq I_1 - |I_2|$ it follows that, if $\gamma/\varepsilon \geq \xi^2$,

$$I \geq \int_0^{T_\varepsilon} \Big((\frac{\gamma}{\varepsilon} + (\frac{\gamma^2}{\varepsilon} - 2C_2\xi^4)\int_0^1 |a(t+s/\gamma)-a(t)|\,ds\Big)(v(t))^2\,dt - \int_0^{T_\varepsilon} |v'(t)|^2\,dt,$$

from which we see that

$$I \geq \int_0^{T_\varepsilon} \frac{\gamma}{\varepsilon}(v(t))^2\,dt - \int_0^{T_\varepsilon} |v'(t)|^2\,dt,$$

when $\gamma/\varepsilon \geq \xi^2$ and $\gamma^2/\varepsilon \geq 2C_2\xi^4$.

We have some positive ε not depending on ξ or on γ such that, if $C_1\xi^2e^{-5/2}/2 \leq \gamma$, we have $\gamma/\varepsilon \geq \xi^2$ and $\gamma^2/\varepsilon \geq 2C_2\xi^4$. Therefore, with such ε, we see that

$$\int_0^{T_\varepsilon} (\tilde{L}v(t))^2\,dt \geq \int_0^{T_\varepsilon} \frac{\gamma}{\varepsilon}(v(t))^2\,dt$$

if $\gamma \geq 1/T_\varepsilon$ and $\gamma \geq C_1\xi^2e^{-5/2}/2$. Then we have with some positive C_4,

$$C_4\int_0^{T_\varepsilon} (\tilde{L}v(t))^2\,dt \geq (\gamma+\xi^2)\int_0^{T_\varepsilon} (v(t))^2\,dt$$

if $\gamma \geq 1/T_\varepsilon$ and $\gamma \geq C_1\xi^2e^{-5/2}/2$. Hence we obtain the estimate (1.6).

We remark that we need a more precise estimate than the estimate above for the proof of Theorem 1.1.

In the next section we recall the properties of the Hardy-Littlewood decomposition and the properties of functions of bounded variation for the preliminaries. We draw the Carleman estimate in the section 3. Finally we give the proof of Theorem 1.1 in the section 4. In this study the author is inspired by the paper of Del Santo and Pruzzi [2].

We denote the space of square integrable functions on $\mathbb{R}^d$ by $L^2(\mathbb{R}^d)$. The inner product in $L^2(\mathbb{R}^d)$ is given by

$$(u,v) = \int_{\mathbb{R}^d} u(x)\overline{v(x)}\,dx$$

and the norm by $\|v(\cdot)\| = \Big(\int_{\mathbb{R}^d} |v(x)|^2\,dx\Big)^{1/2}$.

The space $H^1(\mathbb{R}^d)$ consists of $u(x) \in L^2(\mathbb{R}^d)$ whose derivatives $\partial_{x_j}u(x)$ ($j = 1,2,\dots,d$) belong also to $L^2(\mathbb{R}^d)$. The norm $\|\cdot\|_1$ of $H^1(\mathbb{R}^d)$ is given by $\|u(\cdot)\|_1 = \sqrt{\|u(\cdot)\|^2 + \sum_{j=1}^d \|\partial_{x_j}u(\cdot)\|^2}$. We set $\|\nabla u\|^2 = \sum_{j=1}^d \|\partial_{x_j}u\|^2$.

Let $C^\infty(\Omega)$ be the space of infinitely differentiable functions on Ω, $W^{1,\infty}(\mathbb{R}^d)$ the space of bounded and Lipschitz continuous functions on $\mathbb{R}^d$ with the norm

$$\|u(\cdot)\|_{W^{1,\infty}} = \|u(\cdot)\|_{L^\infty} + \sum_{j=1}^d \|\partial_{x_j}u(\cdot)\|_{L^\infty}.$$

Here we denote by $\|u(\cdot)\|_{L^\infty}$ the essential supremum of $|u(x)|$ on $\mathbb{R}^d$.

We denote by $\hat{v}(\xi)$ the Fourier transform of $v(x)$ given by

$$\int_{\mathbb{R}^d} e^{-ix\xi} v(x)\, dx,$$

while the inverse Fourier transform of $w(\xi)$ is defined by

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi} w(\xi)\, d\xi.$$

In the following, we use C or C with some suffix in order to denote positive constants that may be different line by line.

2. Preliminaries

2.1. **The Littlewood-Payley decomposition.** We recall some properties of the Littlewood-Payley decomposition and the related results referring to [4].

Let $\phi_0(\xi) \in C^\infty(\mathbb{R}^d)$ satisfy $0 \le \phi_0(\xi) \le 1$, $\phi_0(\xi) = 1$ for $|\xi| \le 11/10$ and $\phi_0(\xi) = 0$ for $|\xi| \ge 19/10$. We define $\phi_n(\xi)$ with $n = 1, 2, 3, \dots$ by

$$\phi_n(\xi) = \phi_0(\frac{\xi}{2^n}) - \phi_0(\frac{\xi}{2^{n-1}}).$$

For a function $\phi(\xi)$, we denote the Fourier multiplier with $\phi(\xi)$ by ϕ; that is, ϕv is the inverse Fourier transform of $\phi(\xi)\hat{v}(\xi)$. We remark that

$$C^{-1}\|u\|^2 \le \sum_{n=0}^\infty \|\phi_n u\|^2 \le C\|u\|^2. \tag{2.1}$$

Lemma 2.1. *For $a(x) \in W^{1,\infty}(\mathbb{R}^d)$, we have*

$$\sum_{n=0}^\infty \|\phi_n a u - a\phi_n u\|_1^2 \le C(\|a\|_{W^{1,\infty}}\|u\|)^2. \tag{2.2}$$

Proof. We define the paraproduct $T_a u$ by $\sum_{l=0}^\infty a_{l-3}\phi_l u$. Here a_l is the inverse Fourier transform of $\phi_0(2^{-l}\xi)\hat{a}(\xi)$. It is well known that we have

$$\|au - T_a u\|_1 \le C\|a\|_{W^{1,\infty}}\|u\|.$$

See for example [4, Theorem 5.2.8]. This estimate and (2.1) imply

$$\sum_{n=0}^\infty \|(a\phi_n u - T_a\phi_n u)\|_1^2 \le C(\|a\|_{W^{1,\infty}}\|u\|)^2,$$

$$\sum_{n=0}^\infty \|\phi_n(au - T_a u)\|_1^2 \le C(\|a\|_{W^{1,\infty}}\|u\|)^2.$$

Then we have to show only that

$$\sum_{n=0}^\infty \|\phi_n T_a u - T_a\phi_n u)\|_1^2 \le C(\|a\|_{W^{1,\infty}}\|u\|)^2.$$

In the following, we assume that l and n are non-negative integers. Note that the spectrum of $a_{l-3}\phi_l u$, that is, the support of the Fourier transform of $a_{l-3}\phi_l u$ is contained in $2^{l-2} < |\xi| < 2^{l+2}$ if $l \ge 1$, while the spectrum of $a_{-3}\phi_0 u$ is contained in $|\xi| < 2$. Then we see that

$$\phi_n a_{l-3}\phi_l u = 0 \quad |l - n| \ge 3.$$

We remark also that $\phi_l(\xi)\phi_n(\xi) = 0$ if $|l-n| > 1$. Then $\phi_n T_a u - T_a \phi_n u$ is equal to

$$\sum_{l;|l-n|\le 2} \phi_n a_{l-3}\phi_l u - a_{l-3}\phi_l\phi_n u,$$

which, using the symbol of commutator, is equal to

$$\sum_{l;|l-n|\le 2} [\phi_n, a_{l-3}]\phi_l u.$$

Since $a_l(x) = \int_{\mathbb{R}^d} a(y)2^{ld}P(2^l(x-y))\,dy$ where $P(x)$ is the inverse Fourier transform of $\phi_0(\xi)$, then we have $|a_l(x) - a_l(y)| \le C\|a\|_{W^{1,\infty}}|x-y|$. Note that $[\phi_n, a_{l-3}]u$ is equal to

$$2^{nd}\int_{\mathbb{R}^d} Q(2^n(x-y))(a_l(y) - a_l(x))u(y)\,dy$$

where $Q(x) = P(x) - 2^{-d}P(x/2)$ if $n \ge 1$ and $Q(x) = P(x)$ if $n = 0$. Then

$$|[\phi_n, a_{l-3}]\phi_l u(x)| \le C\|a\|_{W^{1,\infty}}2^{-n}2^{nd}\int_{\mathbb{R}^d} P_1(2^n(x-y))|u(y)|\,dy$$

where $P_1(x) = |P(x)||x|$. Since $P_1(x)$ is integrable, we have

$$\|[\phi_n, a_{l-3}]\phi_l u\| \le C\|a\|_{W^{1,\infty}}2^{-n}\|u\|.$$

If $|l-n| \le 2$, the spectrum of $[\phi_n, a_{l-3}]\phi_l u$ is contained in $|\xi| \le 2^{n+2}$, then

$$\|[\phi_n, a_{l-3}]\phi_l u\|_1 \le (2^{n+4}+1)\|[\phi_n, a_{l-3}]\phi_l u\|.$$

Hence we obtain

$$\|\sum_{l;|l-n|\le 2}[\phi_n, a_{l-3}]\phi_l u\|_1 \le \sum_{l;|l-n|\le 2} C\|a\|_{W^{1,\infty}}\|\phi_l u\|.$$

Since $\phi_n T_a u - T_a\phi_n u = \sum_{l;|l-n|\le 2}[\phi_n, a_{l-3}]\phi_l u$, we have

$$\|\phi_n T_a u - T_a \phi_n u\|_1 \le \sum_{l;|l-n|\le 2} C\|a\|_{W^{1,\infty}}\|\phi_l u\|,$$

from which we obtain (2.2). $\square$

2.2. **Bounded variation.** Next we recall the properties of functions with bounded variation. (See, for example, the appendix of [1] for the detail.) Let X be a Banach space with a norm $\|\cdot\|_X$ and let $f(t)$ be a X-valued function on $[0,T]$ with bounded variation; that is, whose total variation $V(f,[0,T])$ given by

$$V(f,[0,T]) = \sup_{\substack{\text{any partition of } [0,T]\\ t_0=0<t_1<\dots<t_L=T}} \sum_{l=1}^{L}\|f(t_l) - f(t_{l-1})\|_X$$

is finite. Then, setting $V_f(t) = V(f,[0,t])$, we have

$$\|f(t) - f(s)\|_X \le V_f(t) - V_f(s)$$

for any $0 \le s \le t \le T$, which implies that $f(t)$ has at most countably many discontinuous points and there exists $f(t+0) = \lim_{h\searrow 0} f(t+h)$ for any $t \in [0,T)$ and that we have

$$\int_0^{T-h}\|f(t+h) - f(t)\|_X\,dt \le hV_f(T) \tag{2.3}$$

for any $0 \le h \le T$.

We see that (2.3) implies

$$\int_0^{T/2} \|f(t+h) - f(t)\|_X \, dt \le h V_f(T) \tag{2.4}$$

for any $0 \le h \le T/2$.

Furthermore we have $\|f(t+0) - f(t)\|_X = V_f(t+0) - V_f(t)$. Then we see from (2.4) that, when $f(t)$ is right continuous at $t = 0$, that is $f(0+0) = f(0)$, for any $\varepsilon > 0$ there exists a positive T_ε such that we have

$$\int_0^{T_\varepsilon} \|f(t+h) - f(t)\|_X \, dt \le h\varepsilon$$

for any $0 \le h \le T_\varepsilon$.

Remark 2.2. It follows from the argument above and assumption (1.3) that the right limit $\lim_{h\searrow 0} a_{jk}(x, t+h)$ converges uniformly on $\mathbb{R}^d_x$ for any $t \in [0, T)$. Then we see that (1.2) and (1.4) still hold for $a_{jk}(x, t+0) = \lim_{h\searrow 0} a_{jk}(x, t+h)$. Furthermore, we have $a_{jk}(x, t+0) = a_{jk}(x, t)$ except for at most countably many t. Then, in Theorem 1.1, we may assume that $a_{jk}(x, t+0) = a_{jk}(x, t)$ on $[0, T]$ uniformly with respect to $x \in \mathbb{R}^d$.

3. Carleman estimate

Noting Remark 2.2, we may assume that for any positive ε, there exist $T_\varepsilon > 0$ such that we have

$$\int_0^{T_\varepsilon} \sum_{j,k=1}^{d} \|a_{jk}(\cdot, t+h) - a_{jk}(\cdot, t)\|_{L^\infty} \, dt \le \varepsilon h \tag{3.1}$$

for any $h \in [0, T_\varepsilon]$. Here we may assume that $T_\varepsilon \le \varepsilon$.

We define $\psi_\gamma(t)$ and $\psi_{1,\gamma}(t)$ with $\gamma \ge 1/T_\varepsilon$ by

$$\psi_\gamma(t) = \int_t^{T_\varepsilon} \Big(\frac{1}{\varepsilon} + \frac{1}{\varepsilon} \sum_{j,k=1}^{d} \gamma \int_0^1 \|a_{jk}(x, \tau + \frac{s}{\gamma}) - a_{jk}(x, \tau)\|_{L^\infty} \, ds \Big) \, d\tau,$$

$$\psi_{1,\gamma}(t) = \gamma \int_t^{T_\varepsilon} e^{\psi_\gamma(\tau)} \, d\tau.$$

We note that, since

$$\sum_{j,k=1}^{d} \int_0^{T\varepsilon} \|a_{jk}(x, t + \frac{s}{\gamma}) - a_{jk}(x, t)\|_{L^\infty} \, dt \le \frac{\varepsilon s}{\gamma}$$

for $s \in [0, 1]$ and $\gamma \ge 1/T_\varepsilon$, we have, on $[0, T_\varepsilon]$,

$$0 \le \psi_\gamma(t) \le \frac{3}{2}. \tag{3.2}$$

In this section we show the following Carleman estimate.

Proposition 3.1. *There exists a positive constant ε_0 so that, for any $\varepsilon \in (0, \varepsilon_0)$ we have, with a positive γ_ε,*

$$\begin{aligned} &\frac{\gamma}{\varepsilon}\int_0^{T_\varepsilon} e^{2\psi_{1,\gamma}(t)}\|u(\cdot,t)\|^2\,dt + \frac{1}{\varepsilon}\int_0^{T_\varepsilon} e^{2\psi_{1,\gamma}(t)}\|\nabla u(\cdot,t)\|^2\,dt \\ &\le C\int_0^{T_\varepsilon} e^{2\psi_{1,\gamma}(t)}\|Lu(\cdot,t)\|^2\,dt \end{aligned} \tag{3.3}$$

for any $\gamma \ge \gamma_\varepsilon$ and any $u(x,t) \in L^2(\mathbb{R}^d \times [0,T_\varepsilon])$ satisfying $\partial_{x_j} u(x,t) \in L^2(\mathbb{R}^d \times [0,T_\varepsilon])$ $(j = 1,2,\dots,d)$ and $Lu \in L^2(\mathbb{R}^d \times [0,T_\varepsilon])$, $u(x,0) = 0$ and $u(x,T_\varepsilon) = 0$. Here the constant C is independent of ε and of γ.

We define the operator L_γ by $L_\gamma = e^{\psi_{1,\gamma}(t)} L e^{-\psi_{1,\gamma}(t)}$; that is,

$$L_\gamma u = \partial_t u + \gamma e^{\psi_\gamma(t)} u + \sum_{j,k=1}^{d} \partial_{x_j}(a_{jk}(x,t)\partial_{x_j} u).$$

Then, by replacing u by $e^{\psi_{1,\gamma}(t)}u$, (3.3) is equivalent to

$$\frac{\gamma}{\varepsilon}\int_0^{T_\varepsilon}\|u(\cdot,t)\|^2\,dt + \frac{1}{\varepsilon}\int_0^{T_\varepsilon}\|\nabla u(\cdot,t)\|^2\,dt \le C\int_0^{T_\varepsilon}\|L_\gamma u(\cdot,t)\|^2\,dt. \tag{3.4}$$

We remark that from (2.2) it follows that

$$\sum_{n=0}^{\infty}\|\phi_n\partial_{x_j}(a_{jk}\partial_k u) - \partial_{x_j}(a_{jk}\partial_k\phi_n u)\|^2 \le C\|\partial_{x_k} u\|^2,$$

from which we obtain

$$\sum_{n=0}^{\infty}\|\phi_n L_\gamma u - L_\gamma\phi_n u\|^2 \le C\|u\|_1^2.$$

Then, by (2.1) we get

$$\sum_{n=0}^{\infty}\|L_\gamma\phi_n u\|^2 \le C(\|L_\gamma u\|^2 + \|u\|_1^2). \tag{3.5}$$

Therefore, we consider the estimate of $\|L_\gamma\phi_n u\|$. Note that (1.2) implies

$$\sum_{j,k=1}^{d}(a_{jk}(x,t)\partial_{x_k} v, \partial_{x_j} v) \ge D_0\|\nabla v\|^2, \tag{3.6}$$

from which and from (3.2) we obtain the following: for $u(x,t)$ satisfying $u(x,0) = 0$ and $u(x,T_{\varepsilon_0}) = 0$,

$$-\int_0^{T_\varepsilon}(L_\gamma\phi_n u, \phi_n u) \ge D_0\int_0^{T_\varepsilon}\|\nabla(\phi_n u)\|^2\,dt - \gamma e^{3/2}\int_0^{T_\varepsilon}\|\phi_n u\|^2\,dt.$$

When $\frac{D_0}{4}2^{2(n-1)} \ge \gamma e^{3/2}$ and $n \ge 1$, we see, noting $\|\nabla(\phi_n u)\|^2 \ge 2^{2(n-1)}\|\phi_n u\|^2$, that

$$-\int_0^{T_\varepsilon}(L_\gamma\phi_n u, \phi_n u) \ge \frac{D_0}{2}\int_0^{T_\varepsilon}\|\nabla(\phi_n u)\|^2\,dt + \gamma e^{3/2}\int_0^{T_\varepsilon}\|\phi_n u\|^2\,dt.$$

Hence, by $|(L_\gamma\phi_n u, \phi_n u)| \le \frac{\varepsilon}{2}\|L_\gamma u\|^2 + \frac{1}{2\varepsilon}\|u\|^2$ we get

$$\varepsilon\int_0^{T_\varepsilon}\|L_\gamma u\|^2\,dt \ge D_0\int_0^{T_\varepsilon}\|\nabla(\phi_n u)\|^2\,dt + \int_0^{T_\varepsilon}(2\gamma - \frac{1}{\varepsilon})\|\phi_n u\|^2\,dt. \tag{3.7}$$

For the case where $\frac{D_0}{4}2^{2(n-1)} \le \gamma e^{3/2}$ with $\gamma \ge 1/T_\varepsilon$, we have the following lemma.

Lemma 3.2. *There exists a positive ε_0 such that under the condition that (3.1) is valid for $0 < \varepsilon < \varepsilon_0$, we have the following estimates. When $0 < \varepsilon < \varepsilon_0$ and $\frac{D_0}{4}2^{2(n-1)} \le \gamma e^{3/2}$ with $\gamma \ge 1/T_\varepsilon$, we have*

$$C\int_0^{T_\varepsilon} \|L_\gamma \phi_n u\|^2\,dt \ge \frac{1}{\varepsilon}\int_0^{T_\varepsilon} \|\nabla \phi_n u\|^2\,dt + \frac{\gamma}{\varepsilon}\int_0^{T_\varepsilon} \|\phi_n u\|^2\,dt \tag{3.8}$$

for any $u(x,t)$ satisfying $u(x,0) = 0$ and $u(x, T_{\varepsilon_0}) = 0$

Proof. Note that

$$\begin{aligned}
\|L_\gamma \phi_n u\|^2 &= \|\partial_t(\phi_n u)\|^2 + \|\gamma e^{\psi_\gamma}\phi_n u + \sum_{j,k=1}^d \partial_{x_j}(a_{jk}\partial_{x_k}\phi_n u)\|^2 \\
&\quad + 2\Re(\partial_t(\phi_n u), \gamma e^{\psi_\gamma}\phi_n u) + \sum_{j,k=1}^d 2\Re(\partial_t(\phi_n u), \partial_{x_j}(a_{jk}\partial_{x_k}\phi_n u)).
\end{aligned}$$

Let $\chi(s) \in C^\infty(\mathbb{R})$ satisfy $\chi(s) \ge 0$ on $\mathbb{R}$, $\chi(s) = 0$ on $(-\infty, 0] \cup [1, \infty)$ and $\int_{-\infty}^\infty \chi(s)\,ds = 1$. Set $D_1 = \sup_{s\in\mathbb{R}} |\chi(s)| + |\chi'(s)|$. We define the regularization of a_{jk}, $a^\gamma_{jk}(x,t)$, by

$$a^\gamma_{jk}(x,t) = \gamma \int_{-\infty}^\infty \chi(\gamma(s-t))a_{jk}(x,s)\,ds.$$

We see that

$$a^\gamma_{jk}(x,t) = \int_{-\infty}^\infty \chi(s)a_{jk}(x, t + s/\gamma)\,ds$$

from which and from $\int_{-\infty}^\infty \chi(s)\,ds = 1$ we see

$$a^\gamma_{jk}(x,t) - a_{jk}(x,t) = \int_{-\infty}^\infty \chi(s)(a_{jk}(x, t+s/\gamma) - a_{jk}(x,t))\,ds,$$

while from $\partial_t a^\gamma(x,t) = -\gamma\int_{-\infty}^\infty \chi'(s)a_{jk}(x, t + s/\gamma)\,ds$ and $\int_{-\infty}^\infty \chi'(s)\,ds = 0$, it follows that

$$\partial_t a^\gamma_{jk}(x,t) = -\gamma\int_{-\infty}^\infty \chi'(s)(a_{jk}(x, t+s/\gamma) - a_{jk}(x,t))\,ds.$$

Then we have

$$\begin{gathered}
|a^\gamma_{jk}(x,t) - a_{jk}(x,t)| \le D_1\int_0^1 |a_{jk}(x, t+s/\gamma) - a_{jk}(x,t)|\,ds, \\
|\partial_t a^\gamma_{jk}(x,t)| \le D_1\gamma\int_0^1 |a_{jk}(x, t+s/\gamma) - a_{jk}(x,t)|\,ds.
\end{gathered}$$

Furthermore we note that

$$|a^\gamma_{jk}(x,t)| \le \|a_{jk}(\cdot,t)\|_{L^\infty}$$

implies

$$|a^\gamma_{jk}(x,t) - a_{jk}(x,t)| \le 2\|a_{jk}(\cdot,t)\|_{L^\infty}.$$

Then we have

$$|a^\gamma_{jk}(x,t) - a_{jk}(x,t)| \le \sqrt{2\|a_{jk}(\cdot,t)\|_{L^\infty}}\Big(D_1\gamma\int_0^1 |a_{jk}(x, t+s/\gamma) - a_{jk}(x,t)|\,ds\Big)^{1/2}.$$

Using the estimates above, we estimate the term $(\partial_t(\phi_n u), \partial_{x_j}(a_{jk}\partial_{x_k}\phi_n u))$. Note that

$$\begin{aligned}&(\partial_t(\phi_n u), \partial_{x_j}(a_{jk}\partial_{x_k}\phi_n u))\\&= (\partial_t(\phi_n u), \partial_{x_j}(a^\gamma_{jk}\partial_{x_k}\phi_n u)) + (\partial_t(\phi_n u), \partial_{x_j}((a_{jk} - a^\gamma_{jk})\partial_{x_k}\phi_n u)).\end{aligned} \tag{3.9}$$

Note that $|(\partial_t(\phi_n u), \partial_{x_j}((a_{jk} - a^\gamma_{jk})\partial_{x_k}\phi_n u))| = |(\partial_t\partial_{x_j}(\phi_n u), ((a_{jk} - a^\gamma_{jk})\partial_{x_k}\phi_n u))|$ which is dominated by

$$2^{n+1}\|\partial_t(\phi_n u)\|\sqrt{2\|a_{jk}(\cdot,t)\|_{L^\infty}}\Big(D_1\int_0^1 \|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds\Big)^{1/2}\|\nabla\phi_n u\|.$$

Here we used $\phi_n(\xi) = 0$ for $|\xi| \ge 2^{n+1}$. Setting $K = \sum_{j,k=1}^d \sup_{t\in[0,T_\varepsilon]} \|a_{jk}(\cdot,t)\|_{L^\infty}$, we obtain, from Schwarz's inequality,

$$\begin{aligned}&\sum_{j,k=1}^d |(\partial_t(\phi_n u), \partial_{x_j}((a_{jk} - a^\gamma_{jk})\partial_{x_k}\phi_n u))|\\&\le 2^{n+1}\|\partial_t(\phi_n u)\|\sqrt{2K}\Big(D_1\sum_{j,k=1}^d\int_0^1 \|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds\Big)^{1/2}\|\nabla\phi_n u\|.\end{aligned}$$

Then we get

$$\begin{aligned}&2\sum_{j,k=1}^d |(\partial_t(\phi_n u), \partial_{x_j}((a_{jk} - a^\gamma_{jk})\partial_{x_k}\phi_n u))|\\&\le \|\partial_t(\phi_n u)\|^2 + 2^{2(n+1)+1}KD_1\sum_{j,k=1}^d\int_0^1 \|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds\|\nabla\phi_n u\|^2.\end{aligned}$$

Hence

$$\begin{aligned}&\|\partial_t(\phi_n u)\|^2 + 2\sum_{j,k=1}^d \Re(\partial_t(\phi_n u), \partial_{x_j}((a_{jk} - a^\gamma_{jk})\partial_{x_k}\phi_n u))\\&\ge -2^{2(n+1)+1}KD_1\sum_{j,k=1}^d\int_0^1 \|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds\|\nabla\phi_n u\|^2.\end{aligned}$$

On the other hand, noting that

$$\begin{aligned}&\sum_{j,k=1}^d 2\Re(\partial_t(\phi_n u), \partial_{x_j}(a^\gamma_{jk}\partial_{x_k}\phi_n u))\\&= \sum_{j,k=1}^d (\partial_{x_j}\phi_n u, (\partial_t a^\gamma_{jk})\partial_{x_k}\phi_n u) - \sum_{j,k=1}^d \partial_t(\partial_{x_j}\phi_n u, a^\gamma_{jk}\partial_{x_k}\phi_n u),\end{aligned}$$

we see that, when $u(x,0) = 0$ and $u(x,T_{\varepsilon_0}) = 0$ are satisfied,

$$\begin{aligned}&\Big|\int_0^{T_\varepsilon} 2\Re(\partial_t(\phi_n u), \partial_{x_j}(a^\gamma_{jk}\partial_{x_k}\phi_n u))\,dt\Big|\\&\le \int_0^{T_\varepsilon}\Big(\sum_{j,k=1}^d D_1\gamma\int_0^1 \|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds\Big)\|\nabla\phi_n u\|^2\,dt.\end{aligned}$$

Therefore, we see that, when $u(x,0)=0$ and $u(x,T_{\varepsilon_0})=0$ are satisfied,

$$\begin{aligned}
&\int_0^{T_\varepsilon}\Big(\|\partial_t(\phi_n u)\|^2+2\Re(\partial_t(\phi_n u),\sum_{j,k=1}^d \partial_{x_j}(a_{jk}\partial_{x_k}\phi_n u))\Big)\,dt\\
&\geq -(2^{2(n+1)+1}K+\gamma)D_1\sum_{j,k=1}^d\int_0^{T_\varepsilon}\int_0^1\|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds\|\nabla\phi_n u\|^2\,dt.
\end{aligned}$$

Similarly, we have

$$2\Re(\partial_t(\phi_n u),\gamma e^{\psi_\gamma}\phi_n u)=\partial_t(\phi_n u,\gamma e^{\psi_\gamma}\phi_n u)-\gamma(\frac{d}{dt}e^{\psi_\gamma})\|\phi_n u\|^2.$$

We note that

$$-\gamma(\frac{d}{dt}e^{\psi_\gamma})=\gamma e^{\psi_\gamma}\times(\frac{1}{\varepsilon}+\frac{\gamma}{\varepsilon}\sum_{j,k=1}^d\int_0^1\|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds)$$

from which and from $e^{\psi_\gamma(t)}\geq 1$, we obtain

$$-\gamma(\frac{d}{dt}e^{\psi_\gamma})\geq\frac{\gamma}{\varepsilon}(1+\gamma\sum_{j,k=1}^d\int_0^1\|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds)\,.$$

Then we see that, when $u(x,0)=0$ and $u(x,T_\varepsilon)=0$ are satisfied,

$$\begin{aligned}
&\int_0^{T_\varepsilon}2\Re(\partial_t(\phi_n u),\gamma e^{\psi_\gamma}\phi_n u)\,dt\\
&\geq\int_0^{T_\varepsilon}\frac{\gamma}{\varepsilon}(1+\gamma\sum_{j,k=1}^d\int_0^1\|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds)\|\phi_n u\|^2\,dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^{T_\varepsilon}\|L_\gamma\phi_n u\|^2\,dt&\geq\int_0^{T_\varepsilon}\frac{\gamma}{\varepsilon}\|\phi_n u\|^2\,dt\\
&\quad+\int_0^{T_\varepsilon}(\frac{\gamma^2}{\varepsilon}\|\phi_n u\|^2-(2^{2(n+1)+1}KD_1+D_1\gamma)\|\nabla\phi_n u\|^2)\\
&\quad\times\Big(\sum_{j,k=1}^d\int_0^1\|a_{jk}(\cdot,t+s/\gamma)-a_{jk}(\cdot,t)\|_{L^\infty}\,ds\Big)\,dt.
\end{aligned}$$

Since $\|\nabla\phi_n u\|^2\leq 2^{2(n+1)}\|\phi_n u\|^2$, when $\frac{D_0}{4}2^{2(n-1)}\leq\gamma e^{3/2}$, we have

$$\begin{gathered}
(2^{2(n+1)+1}KD_1+D_1\gamma)\|\nabla\phi_n u\|^2\leq C\gamma^2\|\phi_n u\|^2,\\
\|\nabla\phi_n u\|^2\leq C\gamma\|\phi_n u\|^2.
\end{gathered}$$

Choosing ε_0 small, we have, for $0<\varepsilon<\varepsilon_0$,

$$(\frac{\gamma^2}{\varepsilon}\|\phi_n u\|^2-(2^{2(n+1)+1}KD_1+D_1\gamma)\|\nabla\phi_n u\|^2)\geq 0.$$

Hence

$$\int_0^{T_\varepsilon}\|L_\gamma\phi_n u\|^2\,dt\geq\int_0^{T_\varepsilon}\frac{\gamma}{\varepsilon}\|\phi_n u\|^2\,dt.$$

Using $\gamma\|\phi_n u\|^2 \ge \frac{1}{C}\|\nabla\phi_n u\|^2$, we obtain

$$\int_0^{T_\varepsilon} \|L_\gamma \phi_n u\|^2\,dt \ge \int_0^{T_\varepsilon} (\frac{\gamma}{2\varepsilon}\|\phi_n u\|^2 + \frac{1}{2C\varepsilon}\|\nabla\phi_n u\|^2)\,dt.$$

□

Now we complete the proof of Proposition 3.1. We choose ε_0 and γ_ε so that the assertion of Lemma 3.2 is valid. Furthermore we choose γ_ε so large that we have $\gamma_\varepsilon > 2/\varepsilon$. Then we obtain from (3.7)

$$C\int_0^{T_\varepsilon} \|L_\gamma \phi_n u\|^2\,dt \ge \frac{1}{\varepsilon}\int_0^{T_\varepsilon} \|\nabla\phi_n u\|^2\,dt + \frac{\gamma}{\varepsilon}\int_0^{T_\varepsilon} \|\phi_n u\|^2\,dt$$

for n satisfying $\frac{D_0}{4}2^{2(n-1)} \ge \gamma e^{3/2}$ with $\gamma \ge \gamma_\varepsilon$. Hence it follows from the estimate above, (3.8) and (2.1) that

$$C\sum_{n=0}^{\infty}\int_0^{T_\varepsilon} \|L_\gamma \phi_n u\|^2\,dt \ge \frac{1}{\varepsilon}\int_0^{T_\varepsilon} \|\nabla u\|^2\,dt + \frac{\gamma}{\varepsilon}\int_0^{T_\varepsilon} \|u\|^2\,dt.$$

Noting (3.5), we have

$$C\int_0^{T_\varepsilon} (\|L_\gamma u\|^2 + \|u\|_1^2)\,dt \ge \frac{1}{\varepsilon}\int_0^{T_\varepsilon} \|\nabla u\|^2\,dt + \frac{\gamma}{\varepsilon}\int_0^{T_\varepsilon} \|u\|^2\,dt.$$

Then, by choosing ε_0 so small, we obtain the desired estimate (3.4). The proof of Proposition 3.1 is complete.

4. Proof of Theorem 1.1

First, we show time local uniqueness under the assumptions of Theorem 1.1. Then, using the well-known continuity argument, we show that the assertion of Theorem 1.1 is valid.

Proposition 4.1. *Under the assumptions of Theorem 1.1, there exists $t_0 \in (0, T]$ such that we have $u(x,t) = 0$ for $t \in [0, t_0]$.*

Proof. Set $f = Lu$. Then we see from the assumptions of Theorem 1.1, that $f \in L^2([0,T], L^2(\mathbb{R}^d))$ and $\|f(\cdot,t)\|^2 \le C_0(\|\nabla u(\cdot,t)\|^2 + \|u(\cdot,t)\|^2)$ for almost all $t \in [0,T]$. Let the non-negative function $\chi_0(t) \in C^\infty(\mathbb{R})$ satisfy

$$\chi_0(t) = \begin{cases} 1 & t < 3/4 \\ 0 & t > 7/8\,. \end{cases}$$

Set $u_\varepsilon(x,t) = \chi_0(t/T_\varepsilon)u(x,t)$. Here we use the notation of Proposition 3.1. Then we see that $u_\varepsilon(x,0) = 0$, $u_\varepsilon(x,T_\varepsilon) = 0$ and that

$$Lu_\varepsilon = \chi_0(t/T_\varepsilon)f + \frac{\chi_0'(t/T_\varepsilon)}{T_\varepsilon}u.$$

Then from (3.3) we obtain

$$\begin{aligned}
&\int_0^{T_\varepsilon} (\frac{1}{\varepsilon}\|\nabla u_\varepsilon\|^2 + \frac{\gamma}{\varepsilon}\|u_\varepsilon\|^2)e^{2\psi_{1,\gamma}(t)}\,dt \\
&\le C\int_0^{T_\varepsilon} (\|\chi_0(t/T_\varepsilon)f\|^2 + (\frac{\chi_0'(t/T_\varepsilon)}{T_\varepsilon})^2\|u\|^2)e^{2\psi_{1,\gamma}(t)}\,dt.
\end{aligned}$$

Since
$$\|\chi_0(t/T_\varepsilon)f\|^2 \le C_0(\|\nabla u_\varepsilon\|^2 + \|u_\varepsilon\|^2),$$
by choosing ε small, we have
$$\int_0^{T_\varepsilon} (\frac{1}{\varepsilon}\|\nabla u_\varepsilon\|^2 + \frac{\gamma}{\varepsilon}\|u_\varepsilon\|^2)e^{2\psi_{1,\gamma}(t)}\,dt \le C\int_0^{T_\varepsilon} (\frac{\chi_0'(t/T_\varepsilon)}{T_\varepsilon})^2\|u_\varepsilon\|^2 e^{2\psi_{1,\gamma}(t)}\,dt \quad (4.1)$$
for any $\gamma \ge \gamma_\varepsilon$. Since $\chi_0'(t/T_\varepsilon) = 0$ for $t \le 3T_\varepsilon/4$ and $\psi_{1,\gamma}(t)$ is decreasing, we note that the right-hand side of (4.1) can be dominated by
$$C_\varepsilon e^{2\psi_{1,\gamma}(3T_\varepsilon/4)}.$$
Since $\psi_{\gamma,1} = \gamma\int_t^{T_\varepsilon} e^{\psi_\gamma(\tau)}\,d\tau$ and $e^{\psi_\gamma(\tau)} \ge 1$ for $\tau \ge 0$, we see that for $t \in [0, T_\varepsilon/4]$,
$$\psi_{\gamma,1}(t) \ge \psi_{\gamma,1}(T_\varepsilon/4) \ge \psi_{\gamma,1}(3T_\varepsilon/4) + \gamma T_\varepsilon/2.$$
Then, noting that $u_\varepsilon(x,t) = u(x,t)$ on $[0, 3T_\varepsilon/4]$, we see that
$$\frac{\gamma}{\varepsilon}e^{2\psi_{\gamma,1}(3T_\varepsilon/4)+\gamma T_\varepsilon}\int_0^{T_\varepsilon/4}\|u\|^2\,dt$$
is not greater than the left hand side of (4.1). Then we have
$$\int_0^{T_\varepsilon/4}\|u\|^2\,dt \le \frac{\varepsilon C_\varepsilon}{\gamma}e^{-\gamma T_\varepsilon}.$$

As γ tends to infinity, the right hand side converges to zero. Then we see that $\int_0^{T_\varepsilon/4}\|u\|^2\,dt = 0$, which implies $u(x,t) = 0$ on $[0, T_\varepsilon/4]$. □

Using the same argument we have the following proposition.

Proposition 4.2. *We assume that the assumptions of Theorem 1.1 except for $u(x,0) = 0$ are satisfied. For any $t_0 \in [0,T)$ there exists $t_1 \in (t_0, T]$ such that, if $u(x,t_0) = 0$, then we have $u(x,t) = 0$ for $t \in [t_0, t_1]$.*

Now we prove Theorem 1.1. Let S be the subset of $(0,T)$ that consists of $t_0 \in (0,T)$ satisfying $u(x,t) = 0$ on $[0,t_0]$. From Proposition 4.1 and Proposition 4.2 we see that the set S is not empty and open set. Since $u(x,t) \in C^0([0,T], L^2(\mathbb{R}^d))$, we see that S is closed subset of $(0,T)$. Then the connectedness of $(0,T)$ implies $S = (0,T)$. Then we see that $u(x,t) = 0$ on $[0,T)$. Hence $u(x,t) = 0$ on $[0,T]$. The proof of Theorem 1.1 is complete.

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