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HYERS-ULAM STABILITY OF LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS IN COMPLEX BANACH SPACES

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Abstract. We prove the Hyers-Ulam stability of linear second-order differential equations in complex Banach spaces. That is, if y is an approximate solution of the differential equation $y''+\alpha y'(t)+\beta y=0$ or $y''+\alpha y'(t)+\beta y=f(t)$, then there exists an exact solution of the differential equation near to y.

1. Introduction and preliminaries

In 1940, Ulam [21] gave a wide-ranging talk about a series of important unsolved problems. Among those was the question concerning the stability of group homomorphisms. Hyers [3] solved the problem for the case of approximately additive mappings between Banach spaces. Since then, the stability problems of functional equations have been extensively investigated by several mathematicians [4, 16, 17].

Assume that Y is a normed space and I is an open subset of $\mathbb{R}$. Suppose that $a_i : I \to \mathbb{K}$ and $h : I \to Y$ are continuous functions and $\mathbb{K}$ is either $\mathbb{R}$ of $\mathbb{C}$, for any function $f : I \to Y$ satisfying the differential inequality

$$\|a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x)\| \le \varepsilon$$

for all $x \in I$ and for some $\varepsilon \ge 0$. We say that

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0$$

satisfies the Hyers-Ulam stability, if there exists a solution $f_0 : I \to Y$ of the above differential equation and $\|f(x) - f_0(x)\| \le K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of ε only.

If the above statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(t)$ and $\Phi(\varepsilon)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the Hyers-Ulam-Rassias stability (or generalized Hyers-Ulam stability).

Obłoza may be the first author to investigate the Hyers-Ulam stability of differential equations (see [14, 15]). Then, Alsina and Ger prove the Hyers-Ulam stability of $y'(t) - y(t) = 0$ [1]. The above result of Alsina and Get has been generalized by Miura, Takahasi and Choda [13], by Miura [10], and also by Takahasi, Miura and Miyajima [19].

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While [12], Miura et al [5] also proved the Hyers-Ulam stability of linear differential equations of first order $y'(t) + g(t)y(t) = 0$ and $\varphi(t)y'(t) = y(t)$.

Furthermore, the result of Hyers-Ulam stability for first-order linear differential equations has been generalized in (see [6, 7, 12, 20, 22]).

In the meantime, Yongjin Li et al [9] do some work in linear differential equations of second order in the form of $y''(t)+\alpha y'(t)+\beta y(t) = 0$ and $y''(t)+\alpha y'(t)+\beta y(t) = f(t)$ under the assumption that the characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$ has two different positive roots. And the Hyers-Ulam stability for second-order linear differential equations in the form of $y''(t) + \beta(x)y = 0$ with boundary conditions was investigated in [2].

The aim of this article is to study the Hyers-Ulam-Rassias stability of the following linear differential equations of second order in complex Banach spaces:

$$y''(t) + \alpha y'(t) + \beta y(t) = 0 \tag{1.1}$$

and

$$y''(t) + \alpha y'(t) + \beta y(t) = f(t) \tag{1.2}$$

2. Main Results

In the following theorems, we will prove the Hyers-Ulam-Rassias stability of linear differential equations of second order.

Before stating the main theorem, we need the following lemma. For the sake of convenience, all the integrals and derivations will be viewed as existing and $\Re(\omega)$ denotes the real part of complex number ω.

Lemma 2.1. *Let X be a complex Banach space and let $I = (a, b)$ be an open interval, where $a, b \in R$ are arbitrarily given with $-\infty < a < b < +\infty$. Assume that g is an arbitrarily complex number, $h : I \to C$ is continuous and integrable on I. Moreover, suppose $\varphi : I \to [0, \infty)$ is an integrable function on I. If a continuously differentiable function $y : I \to X$ satisfies the differential inequality*

$$\|y'(t) + gy(t) + h(t)\| \le \varphi(t) \tag{2.1}$$

for all $t \in I$, then there exists a unique $x \in X$ such that

$$\|y(t) - e^{-\int_a^t g du}(x - \int_a^t e^{\int_a^v g du} h(v)dv)\| \le e^{-\Re(\int_a^t g du)} \int_t^b \varphi(v)e^{\Re(\int_a^v g du)}dv$$

Proof. For simplicity, we use the notation

$$z(t) := e^{\int_a^t g du} y(t) + \int_a^t e^{\int_a^v g du} h(v)dv$$

for each $t \in I$. By making use of this notation and by (2.1), we obtain

$$\begin{aligned}
\|z(t) - z(s)\| &= \|e^{\int_a^t g du} y(t) - e^{\int_a^s g du} y(s) + \int_s^t e^{\int_a^v g du} h(v)dv\| \\
&= \| \int_s^t \frac{d}{dv}(e^{\int_a^v g du} y(v))dv + \int_s^t e^{\int_a^v g du} h(v)dv\| \\
&= \| \int_s^t e^{\int_a^v g du}(y'(v) + gy(v) + h(v))dv\| \\
&\le | \int_s^t e^{\Re(\int_a^v g du)} \varphi(v)dv|
\end{aligned}$$

for any $s, t \in I$.

Since g is a constant number, we know that $e^{\Re(\int_a^v gdu)}$ is boundary and continuous. What is more, $\varphi(v)$ is integrable and hence $e^{\Re(\int_a^v gdu)}\varphi(v)$ is integrable. Since X is completed, there exists an $x \in X$ such that $z(s)$ converges to x as $s \to b$.

Thus, it follows from the above argument that for any $t \in I$,

$$\begin{aligned}
&\|y(t) - e^{-\int_a^t gdu}(x - \int_a^t e^{\int_a^v gdu}h(v)dv)\| \\
&= \|e^{-\int_a^t gdu}(z(t) - x)\| \\
&\le e^{-\Re(\int_a^t gdu)}\|z(t) - z(s)\| + e^{-\Re(\int_a^t gdu)}\|z(s) - x\| \\
&\le e^{-\Re(\int_a^t gdu)}|\int_s^t \varphi(v)e^{\Re(\int_a^v gdu)}dv| + e^{-\Re(\int_a^t gdu)}\|z(s) - x\| \\
&\to e^{-\Re(\int_a^t gdu)}\int_t^b \varphi(v)e^{\Re(\int_a^v gdu)}dv
\end{aligned}$$

as $s \to b$, since $z(s) \to x$ as $s \to b$. Obviously, $y_0(t) = e^{-\int_a^t gdu}(x - \int_a^t e^{\int_a^v gdu}h(v)dv)$ is a solution of $y'(t) + gy(t) + h(t) = 0$.

It now remains to prove the uniqueness of x. Assume that $x_1 \in X$ also satisfies (2.1) in place of x. Then, we have

$$\|e^{-\int_a^t gdu}(x - x_1)\| \le 2e^{-\Re(\int_a^t gdu)}\int_t^b \varphi(v)e^{\Re(\int_a^v gdu)}dv$$

for any $t \in I$. It follows from the integrability hypotheses that

$$\|x - x_1\| \le 2\int_t^b e^{\Re(\int_a^v gdu)}\varphi(v)dv \to 0$$

as $t \to b$. This implies the uniqueness of x. □

Corollary 2.2. *Let X be a complex Banach space and let $I = (a, b)$ be an open interval, where $a, b \in R$ are arbitrarily given with $-\infty < a < b < +\infty$. Assume that g is an arbitrarily complex number, $h : I \to C$ is continuous and integrable on I. Moreover, suppose $\varphi : I \to [0, \infty)$ is an integrable function on I. If a continuously differentiable function $y : I \to X$ satisfies the differential inequality*

$$\|y'(t) + gy(t) + h(t)\| \le \varphi(t) \tag{2.2}$$

for all $t \in I$, then there exists a unique $x \in X$ such that

$$\|y(t) - e^{-\int_b^t gdu}(x - \int_b^t e^{\int_b^v gdu}h(v)dv)\| \le e^{-\Re(\int_b^t gdu)}\int_a^t \varphi(v)e^{\Re(\int_b^v gdu)}dv$$

Proof. Let $J = (-b, -a)$ and define $h_1(t) = h(-t), y_1(t) = y(-t)$ and $\varphi_1(t) = \varphi(-t)$, respectively. Using these definitions, we may transform the inequality (2.2) into

$$\|y_1'(t) - gy_1(t) - h_1(t)\| \le \varphi_1(t)$$

for each $t \in J$.

Hence, we can now use Lemma 2.1 to conclude that there exists a unique $x \in X$ such that

$$\|y_1(t) - e^{\int_{-b}^t gdu}(x + \int_{-b}^t e^{-\int_{-b}^v gdu}h_1(v)dv)\|$$

$$\leq e^{\Re(\int_{-b}^{t} g du)} \int_{t}^{-a} \varphi_1(v) e^{-\Re(\int_{-b}^{v} g du)} dv$$

for any $t \in J$. Indeed, we can transform the above inequality into

$$\|y(t) - e^{-\int_b^t g du}(x - \int_b^t e^{\int_b^v g du} h(v) dv)\| \leq e^{-\Re(\int_b^t g du)} \int_a^t \varphi(v) e^{\Re(\int_b^v g du)} dv$$

by some tedious substitutions. □

In the following theorems, we investigate the Hyers-Ulam-Rassias of (1.1) and (1.2).

Theorem 2.3. *Let $\varphi : I \to [0, \infty)$ be an integrable function on I. Assume that α, β are complex numbers. If a twice continuously differentiable function $y(t)$ satisfies the inequality*

$$\|y''(t) + \alpha y'(t) + \beta y(t)\| \leq \varphi(t) \tag{2.3}$$

Then (1.1) *has the Hyers-Ulam-Rassias stability.*

Proof. Let λ_1 and λ_2 be the roots of the characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$. Define $g(t) = y'(t) - \lambda_1 y(t)$, thus

$$\begin{aligned}|g'(t) - \lambda_2 g(t)| &= |y''(t) - \lambda_1 y'(t) - \lambda_2(y'(t) - \lambda_1 y(t))| \\ &= |y''(t) - \alpha y'(t) + \beta y(t)|\end{aligned}$$

Hence, we have $|g'(t) - \lambda_2 g(t)| \leq \varphi(t)$. By using Lemma 2.1, there exists a unique $x_1 \in X$ such that

$$\|g(t) - x_1 e^{t\lambda_2 - a\lambda_2}\| \leq e^{\Re(t\lambda_2 - a\lambda_2)} \int_t^b e^{-\Re(\int_a^v \lambda_2 du)} \varphi(v) dv$$

where $x_1 = \lim_{t \to b} g(t) e^{-\lambda_2 t + \lambda_2 a}$ and $x_1 e^{t\lambda_2 - a\lambda_2}$ satisfies $g'(t) - \lambda_2 g(t) = 0$.

Since $g(t) = y'(t) - \lambda_1 y(t)$, we have

$$\|y'(t) - \lambda_1 y(t) - x_1 e^{t\lambda_2 - a\lambda_2}\| \leq e^{\Re(t\lambda_2 - a\lambda_2)} \int_t^b e^{-\Re(\int_a^v \lambda_2 du)} \varphi(v) dv$$

For simplicity, we define $\psi(t) = e^{\Re(t\lambda_2 - a\lambda_2)} \int_t^b e^{-\Re(\int_a^v \lambda_2 du)} \varphi(v) dv$, thus

$$\|y'(t) - \lambda_1 y(t) - x_1 e^{t\lambda_2 - a\lambda_2}\| \leq \psi(t)$$

By using Lemma 2.1 again, there exists a unique $x_2 \in X$ such that

$$\begin{aligned}&\|y(t) - e^{\int_a^t \lambda_1 du}(x_2 + \int_a^t e^{-\int_a^v \lambda_1 du} x_1 e^{v\lambda_2 - a\lambda_2} dv)\| \\ &\leq e^{\Re(\int_a^t \lambda_1 du)} \int_t^b \psi(v) e^{\Re(\int_a^v -\lambda_1 du)} dv\end{aligned}$$

where $x_2 = \lim_{t \to b}(e^{-\int_a^t \lambda_1 du} y(t) - \int_a^t e^{-\int_a^v \lambda_1 du} \cdot x_1 \cdot e^{\int_a^v \lambda_2 du} dv)$. Furthermore, it is easy to prove that $e^{\int_a^t \lambda_1 du}(x_2 + \int_a^t e^{-\int_a^v \lambda_1 du} x_1 e^{v\lambda_2 - a\lambda_2} dv)$ is a solution of (1.1). □

Theorem 2.4. *Let $\varphi : I \to [0, \infty)$ is an integrable function on I. Assume that α, β are complex numbers, and $f : I \to X$ is continuous and integrable on I. If a twice continuously differentiable function $y(t)$ satisfies the inequality*

$$\|y''(t) + \alpha y'(t) + \beta y(t) - f(t)\| \leq \varphi(t) \tag{2.4}$$

Then (1.2) *has the Hyers-Ulam-Rassias stability.*

Proof. Let λ_1 and λ_2 be the roots of the characteristic equation $\lambda^2+\alpha\lambda+\beta=0$. Define $g(t)=y'(t)-\lambda_1 y(t)$, thus

$$|g'(t)-\lambda_2 g(t)|=|y''(t)-\lambda_1 y'(t)-\lambda_2(y'(t)-\lambda_1 y(t))|=|y''(t)-\alpha y'(t)+\beta y(t)|.$$

Hence, we have $|g'(t)-\lambda_2 g(t)-f(t)|\le\varphi(t)$. By using Lemma 2.1, there exists a unique $x_1\in X$ such that

$$\|g(t)-e^{t\lambda_2-a\lambda_2}(x_1+\int_a^t e^{-\int_a^v\lambda_2du}f(v)dv)\|\le e^{\Re(t\lambda_2-a\lambda_2)}\int_t^b e^{-\Re(\int_a^v\lambda_2du)}\varphi(v)dv$$

where $x_1=\lim_{t\to b}(g(t)e^{-\lambda_2t+\lambda_2a}-\int_a^t e^{-\int_a^v\lambda_2du}f(v)dv)$.

For simplicity, we define

$$\phi(t)=e^{t\lambda_2-a\lambda_2}(x_1+\int_a^t e^{-\int_a^v\lambda_2du}f(v)dv),$$
$$\psi(t)=e^{\Re(t\lambda_2-a\lambda_2)}\int_t^b e^{-\Re(\int_a^v\lambda_2du)}\varphi(v)dv$$

thus,

$$\|y'(t)-\lambda_1 y(t)-\phi(t)\|\le\psi(t)$$

By using Lemma 2.1 again, there exists a unique $x_2\in X$ such that

$$\|y(t)-e^{\int_a^t\lambda_1du}(x_2+\int_a^t e^{-\int_a^v\lambda_1du}\cdot\phi(v)dv)\|\le e^{\Re(\int_a^t\lambda_1du)}\int_t^b\psi(v)e^{\Re(\int_a^v-\lambda_1du)}dv$$

where

$$x_2=\lim_{t\to b}(e^{-\int_a^t\lambda_1du}y(t)-\int_a^t e^{-\int_a^v\lambda_1du}\phi(v)dv).$$

Furthermore, it is easy to show that $e^{\int_a^t\lambda_1du}(x_2+\int_a^t e^{-\int_a^v\lambda_1du}\cdot\phi(v)dv)$ is a solution of (1.2). □

If α and β are real numbers, the approximating function will be a real function even if the roots of the characteristic equation are complex numbers.

Corollary 2.5. *Let $\varphi:I\to[0,\infty)$ be an integrable function on I. Assume that α,β are real numbers, $y(t)$ satisfies the inequality*

$$\|y''(t)+\alpha y'(t)+\beta y(t)\|\le\varphi(t)$$

where $y:I\to X$ is a twice continuously differentiable function, X is a real Banach space. Then (1.1) *has the Hyers-Ulam-Rassias stability. Moreover, the approximating function is a real function.*

Proof. What we have to do is just to verify that if the approximate function is real. It is easy to know that when the roots are real, the corollary holds. Therefore, we suppose that the roots of the characteristic equation are complex numbers. Let $r_1=p_1+ip_2$ and $r_2=p_1-ip_2$ be the roots of the characteristic equation, and $\lim_{t\to b}y(t)=d_1,\lim_{t\to b}y'(t)=d_2$, so $\lim_{t\to b}g(t)=d_2-r_1*d_1$ $(p_1,p_2,d_1,d_2$ are real numbers).

By some tedious calculations, we can know that the approximating function is

$$\frac{1}{p_2}[d_1p_2\cos(p_2(b-t))+(-d_2+d_1p_1)\sin(p_2(b-t))](\cosh(p_1(b-t))-\sinh(p_1(b-t))).$$

which is a real function. This completes the proof of our corollary. □

Corollary 2.6. *Let $\varphi : I \to [0, \infty)$ be an integrable function on I. Assume that α, β are real numbers, $y(t)$ satisfies the inequality*

$$\|y''(t) + \alpha y'(t) + \beta y(t) - f(t)\| \leq \varphi(t)$$

where $y : I \to X$ is a twice continuously differentiable function and $f : I \to X$ is continuous and integrable on I, X is a real Banach space. Then Eq(1.2) has the Hyers-Ulam-Rassais stability. Furthermore, the approximating function is a real function.

Remark 2.7. By using the corollary 2.2, we can get similar results with Theorem 2.3, Theorem 2.4, Corollary 2.5 and Corollary 2.6.

Example 2.8. Given $y \in C^2(1, 2)$, $\lim_{t\to 2} y(t) = 2$, $\lim_{t\to 2} y'(t) = 1$, and y satisfies the inequality $|y''(t) - 3y'(t) + 2y(t)| < t$. By using Theorem 2.3, we have

$$|y(t) - (3e^{t-2} - e^{2t-4})| \leq \frac{3}{4} - 3e^{t-2} + \frac{5}{4}e^{2t-4} + \frac{1}{2}t;$$

moreover, $y_0(t) = 3e^{t-2} - e^{2t-4}$ satisfies $\lim_{t\to 2} y_0(t) = 2, \lim_{t\to 2} y_0'(t) = 1$ and $y_0'' - 3y_0' + 2y_0 = 0$.

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