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ENERGY QUANTIZATION FOR APPROXIMATE H-SURFACES AND APPLICATIONS

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Abstract. We consider weakly convergent sequences of approximate H-surface maps defined in the plane with their tension fields bounded in L^p for $p > 4/3$, and establish an energy quantization that accounts for the loss of their energies by the sum of energies over finitely many nontrivial bubbles maps on $\mathbb{R}^2$. As a direct consequence, we establish the energy identity at finite singular time to their H-surface flows.

1. Introduction

The main aim of this study is to discuss the energy quantization of weakly convergent sequences for the weak solutions of approximate H-surface maps. Similar to approximate harmonic or biharmonic maps with the controlled tension or bi-tension fields [9, 17, 15, 21, 22], we consider energy quantization of approximate H-surface maps not only its own interest but also an important application to H-surface flows. In fact, As a direct consequence we will show energy identity to so called H-surface flows.

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, and $H : \mathbb{R}^3 \to \mathbb{R}$ be a given bounded measurable function; i.e., $H(\cdot) \in L^\infty(\mathbb{R}^3)$. First we recall the notion of approximate H-surface maps.

Definition 1.1. A map $u \in W^{1,2}(\Omega, \mathbb{R}^3)$ is called an approximate H-surfaces, if there exists a tension field $\tau \in L^p_{\rm loc}(\Omega, \mathbb{R}^3), p \geq 1$ such that

$$\tau(u) = \Delta u - 2H(u)u_x \wedge u_y, \quad \text{in } \Omega. \tag{1.1}$$

In particular, if $\tau \equiv 0$, then the map u satisfies

$$\Delta u = 2H(u)u_x \wedge u_y, \quad \text{in } \Omega \tag{1.2}$$

which is called a H-surface.

It is well-known that if u is a conformal representation of a surface $\mathcal{S} = u(\Omega)$; i. e.,

$$\|u_x\|^2 - \|u_y\|^2 = u_x \cdot u_y = 0,$$

then $H(u)$ is the mean curvature of the surface $\mathcal{S}$ at the point u.

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Notice that H-surface is a critical point of the following energy functional in $W^{1,2}(\Omega,\mathbb{R}^3)$.

$$J_H(u) := \int_\Omega \Big(|\nabla u|^2 + \frac{4}{3}Q(u)u_x \wedge u_y\Big) \tag{1.3}$$

with

$$Q(u) = \Big(\int_0^{u_1} H(s,u_2,u_3)ds, \int_0^{u_2} H(u_1,s,u_3)ds, \int_0^{u_3} H(u_1,u_2,s)ds\Big).$$

From the view of geometrical significance, the system (1.2) can be regarded to be the minimization problem of the standard energy $E(u,\Omega) := \int_\Omega |\nabla v|^2$ with a constraint of the prescribed volume $V(v) := \frac{1}{3}\int_\Omega v \cdot v_x \wedge v_y =$ Constant; that is,

$$\min_{v\in W^{1,2}(\Omega,R^3)} \Big\{ \int_\Omega |\nabla v|^2 : v = \phi \text{ on } \partial\Omega, V(v) = C\Big\}, \tag{1.4}$$

for any given $\phi \in W^{1,2}(\Omega)$, and here H is so-called Lagrangian multiplier.

Wente [20] and Hildebrandt [11] made fundamental contributions on the existence of solutions to the planar Plateau problem or surfaces with constant mean curvature, respectively (see also Helein's monograph [10]). Later, Brezis-Coron [3] and Struwe [19] showed existence of multiple solutions of H-surface maps in a bounded domain of $\mathbb{R}^2$ for given boundary data. As we knew, for variable H there were many significant works by Rey [18], Bethuel-Rey [4], Caldiroli-Musina [7] and Chen-Levine [6]. Meanwhile, the regularity and bubbling phenomena analysis to so-called H-surface flows in $W^{1,2}(\Omega,\mathbb{R}^3)$ has been shown in various cases such as H is a constant, H depends only on two variables, or $H(u)$ is uniformly Lipschizt continuous (see Brezis-Coron[2] and Hong-Hsu [12]). In addition, for the high dimensional case ($n > 2$), Mou-Yang [16] introduced H-systems in a bounded domain of $\mathbb{R}^n$ and established the existence of multiple solutions of H-system for a constant H and given boundary data. Furthermore, Duzaar-Grotowski [8] studied the existence of solutions of the H-system with a variable function H from a domain into a higher dimensional compact Riemannian manifold. All in all, it is an important observation that H-surface maps are invariant under the dilation transformations in $\mathbb{R}^2$. Such a property leads to non-compactness of sequences of H-surfaces in $\mathbb{R}^2$, which prompts studies by Brezis-Coron [3] concerning the failure of strong convergence for weakly convergent H-surfaces. Roughly speaking, the results in [3] assert that the failure of strong convergence occurs at finitely many concentration points of its energy, where finitely many bubbles (i. e. any nontrivial solutions in $\mathbb{R}^2$) are generated, and the total energies from these bubbles account for the total loss of its energies during the process of convergence.

Based on the above observation, our main purpose is to extend the results from [3, 18, 12] to the context of suitable approximate H-surface maps due to its more flexible applications. More precisely, we have

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. Suppose that $\{u^k\}_{k=1}^\infty \subset W^{1,2}(\Omega,\mathbb{R}^3) \cap L^\infty(\Omega,\mathbb{R}^3)$ is a sequence of approximate H-surface satisfying*

$$\Delta u^k = 2H(u^k)u^k_x \wedge u^k_y + f^k \tag{1.5}$$

with $H(\cdot) \in L^\infty(\mathbb{R}^3)$ and $f^k(x) \in L^p(\Omega)$ with $p > 4/3$. Let

$$\sup_{k\in\mathbb{N}} \Big(\|\nabla u^k\|_{L^2(\Omega)} + \|H(u^k)\|_{L^\infty(\mathbb{R}^3)} + \|f^k\|_{L^p(\Omega)}\Big) \le M < +\infty. \tag{1.6}$$

Then there exist finite points $\{x_1, x_2, \dots, x_L\}$ *and an approximate H-surface* $u \in W^{1,2}(\Omega, \mathbb{R}^3)$ *so that*

$$\Delta u = 2H(u)u_x \wedge u_y + f, \quad \text{in } \Omega, \tag{1.7}$$

where $u^k \rightharpoonup u$ *in* $W^{1,2}$ *and* $f^k \rightharpoonup f$ *in* L^p. *Moreover, we have:*

(1) $u^k \to u$ *strongly in* $W^{1,2}_{\mathrm{loc}} \cap C^0_{\mathrm{loc}}(\Omega \setminus \{x_1, x_2, \dots, x_L\}, \mathbb{R}^3)$.

(2) *There exist* $L_i \in \mathbb{N}$ *and bubbles* $\{\omega_{ij}\}_{j=1}^{L_i}$, *which are nontrivial H-surface systems from* $\mathbb{R}^2$ *to* S^2, *such that*

$$\lim_{k\to\infty} \int_{B_{r_i}(x_i)} |\nabla u^k|^2 dx = \int_{B_{r_i}(x_i)} |\nabla u|^2 dx + \sum_{j=1}^{L_i} \int_{\mathbb{R}^2} |\nabla \omega_{ij}|^2 \tag{1.8}$$

where

$$r_i = \frac{1}{2} \min_{1\le j\le L, j\ne i} \{|x_i - x_j|, d(x_i, \partial\Omega)\}.$$

Similar to initially Lin-Wang's argument to deal with harmonic maps [15], our main idea is to show no concentration of energy in the neck. More precisely, that is done while establishing there is no concentration of angular energy in the neck region; then controlling the radial energy in the neck region by angular energy and L^p-norm of its tension field with $p > 4/3$ through so called Pohozaev argument [15] [21]. During using control the radial energy by the angular hessian energy and L^p-norm of its tension fields by Pohozaev argument in the neck region, the assumption $p > 4/3$ seems to be necessary to validate the Pohozaev argument, since we need $\Delta u^k \cdot (x \cdot \nabla u^k) \in L^1$ and $f^k \cdot (x \cdot \nabla u^k) \in L^1$.

A typical application of Theorem 1.2 is to study asymptotic behavior at finite time for H-surface flows in the plane. We can directly obtain identity energy at finite time to H-surface flows with initial data u_0 as follows.

$$\begin{gathered} u_t = \Delta u - 2H(u)u_x \wedge u_y, \quad (x,t) \in \Omega \times (0, +\infty) \\ u|_{t=0} = u_0, \qquad x \in \Omega \\ u|_{\partial\Omega} = u_0|_{\partial\Omega}, \quad t > 0, x \in \partial\Omega \end{gathered} \tag{1.9}$$

where $u_0 \in W^{1,2}(\Omega)$ and $H \in L^\infty(\mathbb{R}^3)$. In particular, note that any t-independent solution $u : \Omega \to \mathbb{R}^3$ of (1.9) is a H-surface system.

We are inspired by Hong-Hsu's energy inequality in [12, Theorem 3.7]: for arbitrary $u_0 \in C^2(\Omega, \mathbb{R}^3)$ satisfying $\|u_0\|_{L^\infty} \|H\|_{L^\infty} < 1$, there exists a time $T_0 > 0$ such that

$$\|u_t\|_{L^2(\Omega\times(0,T_0))} \le J_H[u_0], \tag{1.10}$$

where J_H is represented by (1.3). Then we also consider that, for a finite singular time $T_0 < +\infty$, energy identity accounting for the δ mass by finite many bubbles. This observation can be proved by applying the rescaled maps to conformal invariance of H-surface flows. Then, from the energy inequality (1.10) there exists a sequence $t_k \uparrow T_0$ such that $u^k := u(\cdot, t_k) \in W^{1,2}(\Omega, \mathbb{R}^3)$ satisfies

(i) $\tau_2(u^k) := \|u_t(t_k)\|_{L^2} \to 0$; and

(ii) u^k satisfies in the distribution sense

$$\Delta u^k = 2H(u^k)u^k_x \wedge u^k_y + \tau_2(u^k). \tag{1.11}$$

Therefore, from Theorem 1.2 we derive that an energy identity of the weak limit of H-surface flows are connected together without any neck region. In particular, the image of u_n converges pointwise to the image of the limit bubble tree maps, which is similar to harmonic map flows ([9] [17] [15]). More precisely, we have the following theorem.

Theorem 1.3. *For some $T_0 < +\infty$, let $u \in C^{2+\alpha,1+\alpha}_{\rm loc}(\Omega \times (0,T_0))$ be a solution to (1.9) with $\|u_0\|_{L^\infty(\Omega)}\|H\|_{L^\infty(\Omega)} < 1$, where T_0 is a singular time. Then there exist a finite many bubbles $\omega_i, i = 1,\dots,L$ such that*

$$\lim_{t\to T_0} E(u(\cdot,t),\Omega) = E(u(\cdot,T_0),\Omega) + \sum_{j=1}^{L} E(\omega_i,\mathbb{S}^2), \tag{1.12}$$

where $E(\cdot,\mathbb{S}^2)$ is the energy of finite many bubbles on the unit sphere $\mathbb{S}^2$.

The article is organized as follows. In §2, we establish a locally Höler continuity of weak solutions and the higher integrability of their first and second order derivatives, strong convergence and blow-up analysis to any approximate H-surface maps with the smallness energy condition and its tension field in L^p for some $p > 1$. In §3, we prove main Theorem 1.2 by establishing that there is no concentration of angular energy in the neck region; and then controlling the radial energy in the neck region by angular energy and L^p-norm of its tension field with $p > 4/3$ by the Pohozaev argument. In §4, As a consequence of the main Theorem, we set up the energy identity at finite singular time $T_0 < +\infty$ to sequences of H-surface flows.

2. A priori estimates of approximate H-surfaces

This section is mainly devoted to a locally Hölder continuity of weak solutions and the higher integrality of the first and second-order derivatives under the smallness energy. To this end, we need to use Riesz potential estimates in the Morrey spaces due to Adams [1]. For an open set $U \subset \mathbb{R}^n$, $1 \le p < +\infty$, $0 < \lambda \le n$, the Morrey space $M^{p,\lambda}(U)$ is defined by

$$M^{p,\lambda}(U) = \Big\{ f \in L^p(U) : \|f\|^p_{M^{p,\lambda}} = \sup_{B_r\subset U} r^{\lambda-n}\int_{B_r}|f|^p < +\infty\Big\}. \tag{2.1}$$

Note that the weak L^p space is denote by $L^{p,*}(\Omega)$: for any $t > 0$, which satisfies

$$\|f\|^p_{L^{p,*}(\Omega)} := \sup_{t>0} t^p|\{x \in \Omega||f(x)| > t\}| < \infty.$$

Therefore, the weak Morrey space $M^{p,\lambda}_*(\Omega)$ is defined to be the set of functions $f \in L^p_*(\Omega)$ satisfying

$$\|f\|^p_{M^{p,\lambda}_*(\Omega)} := \sup_{x\in\Omega, 0<\rho\le d}\{\rho^{\lambda-n}\|f\|^p_{L^p_*(\Omega\cap B_\rho(x))}\} < \infty \tag{2.2}$$

with $0 \le \lambda \le n$ and $d = diam(\Omega)$. Let $I_\beta(f)$ be the Riesz potential of order β $(0 < \beta \le n)$ defined by

$$I_\beta(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}}\,dy, \quad x \in \mathbb{R}^n. \tag{2.3}$$

Then we have the following Riesz potential estimates between Morrey spaces due to Adams [1].

Lemma 2.1. (1) *For any $\beta > 0, 0 < \lambda \le n, 1 < p < \frac{\lambda}{\beta}$, if $f \in M^{p,\lambda}(\mathbb{R}^n)$, then $I_\beta(f) \in M^{\tilde p,\lambda}(\mathbb{R}^n)$, where $\tilde p = \frac{\lambda p}{\lambda - p\beta}$. Moreover,*

$$\|I_\beta(f)\|_{M^{\tilde p,\lambda}(\mathbb{R}^n)} \le C\|f\|_{M^{p,\lambda}(\mathbb{R}^n)}. \tag{2.4}$$

(2) *For* $0<\beta<\lambda\le n$, *if* $f\in M^{1,\lambda}(\mathbb{R}^n)$, *then* $I_\beta(f)\in M_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^n)$. *Moreover,*
$$\|I_\beta(f)\|_{M_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^n)}\le C\|f\|_{M^{1,\lambda}(\mathbb{R}^n)}. \tag{2.5}$$

In terms of conformal invariance of the H-surface maps, we can do it in the unit disc B_1 in order to obtain so called ε_0-strong convergence for uniformly bounded sequences. First, we establish the higher integrability of the first and second-order derivatives to approximate H-surface maps (1.5).

Lemma 2.2. *Suppose* $u^k\in W^{1,2}(B_1,\mathbb{R}^3)$ *is a sequence of approximate H-surface maps; i.e.,*
$$\Delta u^k=2H(u^k)u_x^k\wedge u_y^k+f^k,\quad x\in B_1, \tag{2.6}$$
where $H\in L^\infty(R^3)$ *and* $f^k\in L^p(B_1,R^3)$ *with any* $1<p<2$. *If, for some sufficiently small constant* ε_0, *such that*
$$\int_{B_1}|\nabla u^k|^2dx\le\varepsilon_0^2. \tag{2.7}$$
Then:

(1) *we have* $u\in C^\alpha(B_{1/2},\mathbb{R}^3)$ *for* $\alpha\in(0,1-\frac{1}{p})$ *and*
$$[u^k]_{C^\alpha(B_{1/2})}\le C\Big(\varepsilon_0+\|f^k\|_{L^p(B_1)}\Big). \tag{2.8}$$

(2) *For any* $p'\in(2,\frac{2p}{2-p})$, *we have* $u\in W^{1,p'}(B_{1/2})$ *and*
$$\|\nabla u^k\|_{L^{p'}(B_{1/2})}\le C\left(\varepsilon_0+\|f^k\|_{L^p(B_1)}\right). \tag{2.9}$$

(3) *Furthermore,* $u^k\in W^{2,p}(B_{1/2})$ *and*
$$\|\nabla^2u^k\|_{L^p(B_{1/2})}\le C\Big(\varepsilon_0^2+\|f^k\|_{L^p(B_1)}\Big). \tag{2.10}$$

Proof. (1) Observe that
$$\begin{gathered}\Delta G_1^k=2H(u^k)u_x^k\wedge u_y^k,\quad \text{in } B_1\\ G_1^k=0,\quad \text{on } \partial B_1.\end{gathered} \tag{2.11}$$
and
$$\begin{gathered}\Delta G_2^k=f^k,\quad \text{in } B_1\\ G_2^k=0,\quad \text{on } \partial B_1.\end{gathered} \tag{2.12}$$
Set $h^k=u^k-G_1^k-G_2^k$, we obtain
$$\begin{gathered}\Delta h^k=0,\quad \text{in } B_1\\ h^k=u^k,\quad \text{on } \partial B_1.\end{gathered} \tag{2.13}$$

To estimate G_1^k, noting that $u_x^k\wedge u_y^k\in\mathcal{H}^1(B_1)$ (Hardy spaces), by [10, Theorem 3.2.9], for $0<\theta<1$, we have
$$\|\nabla G_1^k\|_{L^2(B_\theta)}\le C\|2H(u^k)u_x^k\wedge u_y^k\|_{\mathcal{H}^1(B_1)}. \tag{2.14}$$
Due to the boundedness of $H(u^k)$ and Wente's inequality (see [10, Theorem 3.1.2]),
$$\|\nabla G_1^k\|_{L^2(B_\theta)}\le C\|\nabla u^k\|_{L^2(B_1)}^2\le C\varepsilon_0\|\nabla u^k\|_{L^2(B_1)}. \tag{2.15}$$
For G_2^k, by L^p-theory of Laplace operator Δ we get
$$\|\nabla^2G_2^k\|_{L^p(B_\theta)}\le C\|f^k\|_{L^p(B_1)},$$

it follows, from the Sobolev's theorem, that

$$\begin{aligned}\|\nabla G_2^k\|_{L^2(B_\theta)} &\le C\|\nabla^2 G_2^k\|_{L^1(B_\theta)}\\ &\le C\theta^{2(1-\frac{1}{p})}\|\nabla^2 G_2^k\|_{L^p(B_\theta)} \le C\theta^{2(1-\frac{1}{p})}\|f^k\|_{L^p(B_1)}.\end{aligned} \tag{2.16}$$

Moreover, in accordance with the standard estimates of harmonic functions, one obtains

$$\|\nabla h^k\|_{L^2(B_\theta)} \le C\|\nabla u^k\|_{L^2(B_1)} \le C\varepsilon_0. \tag{2.17}$$

Now, we put all estimates (2.15),(2.16) and (2.17) together. It yields

$$\|\nabla u^k\|_{L^2(B_\theta)} \le C\varepsilon_0\|\nabla u^k\|_{L^2(B_1)} + C(\varepsilon_0 + \theta^{2(1-\frac{1}{p})})\|f^k\|_{L^p(B_1)}. \tag{2.18}$$

By iterating [13, Lemma 3.4], we obtain $u^k \in C^\alpha(B_\vartheta)$ with $0 < \alpha < 1-\frac{1}{p}, 0 < \vartheta < 1$, and

$$\int_{B_\vartheta} |\nabla u^k|^2 dx \le C\vartheta^{2\alpha}. \tag{2.19}$$

(2) To get a higher integrability, we assume that $\widetilde{u}^k$ and $\widetilde{f}^k$ defined in $\mathbb{R}^2$ are extensions of u^k and f^k from B_1 respectively, such that $\|\nabla\widetilde{u}^k\|_{L^2(\mathbb{R}^2)} \le C\|\nabla\widetilde{u}^k\|_{L^2(B_1)}$ and $\|\widetilde{f}^k\|_{L^p(\mathbb{R}^2)} \le C\|\widetilde{f}^k\|_{L^p(B_1)}$. Let $\Gamma(x)$ be the fundamental solution of Laplacian operator Δ in $\mathbb{R}^2$, then

$$\widetilde{u}^k(x) = \int_{\mathbb{R}^2} \Gamma(x-y)\Delta\widetilde{u}^k(y)dy.$$

Therefore, for $x \in B_{1/2}$, we have

$$\begin{aligned}\nabla u^k(x) &= \int_{\mathbb{R}^2} \nabla G(x-y)\left(2H(\widetilde{u}^k)\widetilde{u}_x^k \wedge \widetilde{u}_y^k + \widetilde{f}^k\right)(y)dy\\ &\le C\Big(\Big|\int_{\mathbb{R}^2} \nabla G(x-y)|\nabla\widetilde{u}^k|^2 dy\Big| + \Big|\int_{\mathbb{R}^2} \nabla G(x-y)|\widetilde{f}^k|dy\Big|\Big)\\ &= I_1(|\nabla\widetilde{u}^k|^2) + I_1(|\widetilde{f}^k|).\end{aligned} \tag{2.20}$$

Note that $|\nabla\widetilde{u}|^2 \in M^{1,2-2\alpha}(\mathbb{R}^2)$ and $\widetilde{f}^k \in L^p(\mathbb{R}^2)$, by Lemma 2.4 it implies $\nabla u^k \in L^{p_0,*}(B_{1/2})$ with $2 < p_0 = \min\{\frac{2-\alpha}{1-\alpha}, \frac{2p}{2-p}\}$ for any $1 < p < 2$; and

$$\begin{aligned}\|\nabla u^k\|_{L^{p_0,*}(B_{1/2})} &\le \|I_1(|\nabla\widetilde{u}^k|^2)\|_{L^{p_0,*}(B_{1/2})} + \|I_1(|\widetilde{f}^k|)\|_{L^{p_0,*}(B_{1/2})}\\ &\le C(\|\nabla\widetilde{u}^k\|_{L^2(\mathbb{R}^2)} + \|\widetilde{f}^k\|_{L^p(\mathbb{R}^2)})\\ &\le C(\|\nabla u^k\|_{L^2(B_1)} + \|f^k\|_{L^p(B_1)})\end{aligned} \tag{2.21}$$

Thanks to $L^{p_0,*}(B_1) \subset L^{p'}(B_1)$ with $2 < p' < p_0$, we obtain that $\nabla u^k \in L^{p'}(B_{1/2})$, and

$$\|\nabla u^k\|_{L^{p'}(B_{1/2})} \le C(\|\nabla u^k\|_{L^2(B_1)} + \|f^k\|_{L^p(B_1)}). \tag{2.22}$$

According to the assumption of smallness energy, it clearly yields (2.9).

(3) On the basis of Calderon-Zygmund's L^p-theory and (2.6), we have

$$\begin{aligned}\|\nabla^2 u^k\|_{L^p(B_{1/2})} &\le C\|\Delta u^k\|_{L^p(B_{\frac{3}{4}})}\\ &\le C\Big(\||\nabla u^k|^2\|_{L^p(B_{\frac{3}{4}})} + \|f^k\|_{L^p(B_{\frac{3}{4}})}\Big)\\ &\le C\Big(\|\nabla u^k\|_{L^2(B_{\frac{3}{4}})}\|\nabla u^k\|_{L^{p'}(B_{\frac{3}{4}})} + \|f^k\|_{L^p(B_1)}\Big),\end{aligned} \tag{2.23}$$

with $2 < p' < p_0$. Thanks to the smallness assumption (2.7) and the higher integrability of derivative (2.9), we obtain $u^k \in W^{2,p}(B_{1/2})$ and

$$\begin{aligned}\|\nabla^2 u^k\|_{L^p(B_{1/2})} &\le C\Big(\varepsilon_0\|\nabla u^k\|_{L^{p'}(B_{\frac{3}{4}})} + \|f^k\|_{L^p(B_1)}\Big)\\ &\le C\Big(\varepsilon_0^2 + \|f^k\|_{L^p(B_1)}\Big).\end{aligned} \tag{2.24}$$

This completes the proof. □

It is well known that the energy concentration leads to both the failure of strong convergence and the formation of singularity for sequences of approximate H-surface maps. With the help of the higher integrability of the first and second derivatives for weak solutions, we further consider the blow-up analysis and ε_0-strong convergence to a sequence of approximate H-surface maps (1.5).

Proof of Theorem 1.2 (1). Since $u^k \rightharpoonup u$ weakly in $W^{1,2}(\Omega, R^2)$, we have that $\mu_k = |\nabla u^k|^2 dx$ is a family of nonnegative Radon measures such that $N = \sup_k \mu_k(\Omega) < \infty$. Therefore, after taking possible subsequences, we may assume that there is a nonnegative Radon measure μ such that $\mu_k \to \mu$ as convergence of Radon measures. Moreover, by Fatou's Lemma, we have that there is a nonnegative Radon measure ν, called as the defect measure, such that $\mu = |\nabla u|^2 dx + \nu$. Denote by Σ the support of ν. Then we have

$$\Sigma = \cap_{r>0}\Big\{x \in \Omega : \liminf_k \int_{B_r(x)} |\nabla u^k|^2 dy \ge \varepsilon_0^2\Big\}. \tag{2.25}$$

Let $\Sigma_1 = \{x_1, x_2, \dots, \}$ be any discrete points of Σ, and $\{B_{\delta_0}(x_i)\}_{i=1}^{\infty}$ be mutually disjoint balls for small δ_0. Then we have

$$\liminf_k \int_{B_r(x_i)} |\nabla u^k|^2 dy \ge \varepsilon_0^2, \quad \forall\ 1 \le i \le \infty.$$

Therefore, there exists a natural number K such that for $k \ge K$ we have

$$\int_{B_r(x_i)} |\nabla u^k|^2 dy \ge \varepsilon_0^2, \quad \forall\ 1 \le i \le \infty.$$

Let $\mathcal{H}_0$ denote the 0-dimensional Hausdorff measure, then

$$\begin{aligned}\varepsilon_0^2 \mathcal{H}_0(\Sigma) &\le \sum_{i=0}^{\infty} \int_{B_r(x_i)} |\nabla u^k|^2 dy\\ &= H_0(\Sigma) \int_{\cup_{i=0}^{\infty} B_r(x_i)} |\nabla u^k|^2 dy\\ &\le \int_\Omega |\nabla u^k|^2 dy \le N < \infty;\end{aligned} \tag{2.26}$$

this implies $\mathcal{H}_0(\Sigma) \le L := \frac{N}{\varepsilon_0^2}$. By a compact embedding: $W^{1,p'}(B_1, \mathbb{R}^3) \hookrightarrow W^{1,2}(B_1, \mathbb{R}^3)(p' > 2)$ due to Lemma 2.2, therefore, for any compact subset $K \subset \Omega \backslash \Sigma$ it follows from a simple covering argument that $\nu(K) = 0$ and $u^k \to u$ strongly in $W^{1,2}(K, \mathbb{R}^3)$. Moreover, for any $x_0 \in K$ there is a $r_0 > 0$ such that

$$\lim_k \int_{B_{r_0}(x_0)} |\nabla u^k|^2 \le \varepsilon_0^2.$$

By the standard diagonal process we can extract a subsequence of u^k, still denoted as itself, such that $u^k \to u$ in $W^{1,2}(\Omega\setminus\{x_1,\dots,x_L\},\mathbb{R}^3)\cap C^0(\Omega\setminus\{x_1,\dots,x_L\}),\mathbb{R}^3)$. Hence, it is easy to see that the expression (1.8) holds with "=" replaced by "$\geq$".

To prove "$\leq$" of (1.8), we need to show that the L^2-norm of ∇u^k over any neck region is arbitrarily small. This will mainly be done in the next sections. Therefore, we will return to the proof of Theorem 1.2 (2) in the next section.

3. No concentration of energy in the neck region

In this section, we show that there is no concentration of $\|\nabla u^k\|_{L^2}$ in the neck region. This will be done in two steps: the first step is to show that there is no angular energy concentration in the neck region by comparing with radial harmonic functions over dyadic annulus. The second step is to control the radial component of energy by the angular component of energy by way of the Pohozaev argument.

Proof of Theorem 1.2 (2). Without loss of generality, we suppose that $\{u^k\}\subset W^{1,2}(B_1,\mathbb{R}^3)$ is a sequence of approximate H-surface maps with

$$\sup_{k\in\mathbb{N}}\Big(\|\nabla u^k\|_{L^2(B_1)}+\|H(u^k)\|_{L^\infty(\mathbb{R}^3)}+\|f^k\|_{L^p(B_1)}\Big)\leq M, \tag{3.1}$$

which satisfy $u^k\rightharpoonup u$ in $W^{1,2}(B_1)$, $f^k\rightharpoonup f$ in $L^p(B_1)$, and $u^k\to u$ in $W^{1,2}_{\rm loc}(B_1\setminus\{0\})$ but not in $W^{1,2}(B_1)$. In according of Ding and Tian [9], we may assume that the total number of bubbles generated at 0 is $L=1$. Then, for any $\epsilon>0$, there is $r_k\downarrow 0, R>1$ large enough and $0<\delta<\epsilon$ such that for k sufficiently large there holds

$$\int_{B_{2\rho}\setminus B_\rho}|\nabla u^k|^2\leq\epsilon^2,\quad \forall\frac12 Rr_k\leq\rho\leq 2\delta.$$

Let us consider it in two steps.

Step 1. Angular energy estimate in the neck region: From (1.6) and Lemma 2.2, it follows that for any $\alpha\in(0,1-\frac1p)$ and $p'\in(2,\frac{2p}{2-p})$ with $1<p<2$, $u^k\in C^\alpha\cap W^{1,p'}(B_{2\rho}\setminus B_\rho)$ and

$$[u^k]_{C^\alpha(B_{2\rho}\setminus B_\rho)}+\|\nabla u^k\|_{L^{p'}(B_{2\rho}\setminus B_\rho)}\leq C\epsilon,\quad \forall\frac12 Rr_k\leq\rho\leq 2\delta.$$

In the sequel, to deal with the boundary terms we need the following property by Fubini's theorem:

$$r\int_{\partial B_r}|\nabla u^k|^2\leq 8\sup_k\int_{B_{2r}\setminus B_r}|\nabla u^k|^2\leq C\epsilon^2,$$

which holds for $r=Rr_k,\delta$. For convenience's sake, we assume the above inequality holds for all $k\geq 1$.

For simplicity, we may assume $\frac{\delta}{Rr_k}$ is a positive integer. We make a dyadic decomposition to the annulus $\frac12 Rr_k\leq|x|\leq 2\delta$. Let $N_k\in\mathbb{N}$ be such that $2^{N_k}=[\frac{\delta}{Rr_k}]$, and set

$$\mathcal{A}_k^i:=B_{2^{i+1}Rr_k}\setminus B_{2^iRr_k},\quad \mathcal{B}_k^i:=B_{2^{i+2}Rr_k}\setminus B_{2^{i-1}Rr_k},\quad 1\leq i\leq N_k-1. \tag{3.2}$$

Then, $B_\delta\setminus B_{Rr_k}=\cup_{i=0}^{N_k-1}\mathcal{A}_k^i$ and $B_{2\delta}\setminus B_{\frac12 Rr_k}=\cup_{i=0}^{N_k-1}\mathcal{B}_k^i$. We now introduce a radial harmonic function v^k on the annulus $B_{2\delta}\setminus B_{Rr_k}$ as follows. For $0\leq i\leq N_k-1$,

$v^k(x) = v^k(|x|)$ satisfies

$$\begin{gathered} \Delta v^k = 0 \quad \text{in } \mathcal{A}_k^i, \\ v^k(r) = \fint_{\partial B_{2^{i+1}Rr_k}} u^k, \quad \text{if } r = 2^{i+1}Rr_k, \\ v^k(r) = \fint_{\partial B_{2^iRr_k}} u^k, \quad \text{if } r = 2^iRr_k; \end{gathered} \tag{3.3}$$

where $\fint$ denotes the average integral. By the standard estimate of harmonic functions, we have $v^k \in C^\alpha(\mathcal{A}_k^i) \cap W^{1,p'}(\mathcal{A}_k^i)$ for all $0 \le i \le N_k - 1$; and

$$\big[v_k\big]_{C^\alpha(\mathcal{A}_k^i)} \le C\big[u_k\big]_{C^\alpha(\mathcal{A}_k^i)} \le C\epsilon;$$

in particular,

$$\operatorname{osc}_{\mathcal{A}_k^i}(u^k - v^k) \le C\varepsilon, \quad \forall 0 \le i \le N_k - 1. \tag{3.4}$$

Now we perform the estimate similar to Ding-Tian's argument by [9] or Lin-Wang [15] on harmonic maps. Applying the Green's identity due to $u_k - v_k \in W^{2,p}(\mathcal{A}_k^i)$ we get that for $0 \le i \le N_k - 1$,

$$\int_{\mathcal{A}_k^i} \Delta(u^k - v^k)(u^k - v^k) = -\int_{\mathcal{A}_k^i} |\nabla(u^k - v^k)|^2 + \int_{\partial\mathcal{A}_k^i} \frac{\partial(u^k - v^k)}{\partial\nu}(u^k - v^k). \tag{3.5}$$

By summing over $0 \le i \le N_k - 1$, we derive that

$$\begin{aligned} &\int_{B_\delta \setminus B_{Rr_k}} |\nabla(u^k - v^k)|^2 \\ &= -\sum_{i=0}^{N_k-1} \int_{\mathcal{A}_k^i} \Delta(u^k - v^k)(u^k - v^k) + \Big(\int_{\partial B_\delta} - \int_{\partial B_{Rr_k}}\Big) \frac{\partial(u^k - v^k)}{\partial\nu}(u^k - v^k) \\ &= -\sum_{i=0}^{N_k-1} \int_{\mathcal{A}_k^i} \Delta u^k(u^k - v^k) + \Big(\int_{\partial B_\delta} - \int_{\partial B_{Rr_k}}\Big) \frac{\partial u^k}{\partial\nu}(u^k - v^k), \end{aligned} \tag{3.6}$$

where we used that $\Delta v^k = 0$ in $\mathcal{A}_k^i$ and $\int_{\partial B_\rho} \frac{\partial v^k}{\partial\nu}(u^k - v^k) = 0$ for $\rho = \delta$ and Rr_k, which is due to the radial form of v^k and the boundary conditions: $v^k = \fint_{\partial B_\rho} u^k$.

Let us first check the estimates of the integral on the boundary in the right hand side of (3.6). It follows from Hölder inequality and Fubini's Theorem that

$$\begin{aligned} \Big|\int_{\partial B_\delta} \frac{\partial u^k}{\partial\nu}(u^k - v^k)\Big| &\le \int_{\partial B_\delta} |\nabla u^k||u^k - v^k| \\ &\le C\max_{\partial B_\delta} |u^k - v^k|\Big(\delta\int_{\partial B_\delta} |\nabla u^k|^2\Big)^{1/2} \\ &\le C\epsilon\Big(\int_{B_{2\delta}\setminus B_{\frac{1}{2}\delta}} |\nabla u^k|^2\Big)^{1/2} \le C\epsilon^2. \end{aligned} \tag{3.7}$$

Similarly,

$$
\begin{aligned}
\Big|\int_{\partial B_{Rr_k}} \frac{\partial u^k}{\partial \nu}(u^k - v^k)\Big| &\le \max_{\partial B_{Rr_k}} |u^k - v^k| \int_{\partial B_{Rr_k}} |\nabla u_k| \\
&\le C\epsilon\Big(Rr_k \int_{\partial B_{Rr_k}} |\nabla u^k|^2\Big)^{1/2} \\
&\le C\epsilon\Big(\int_{B_{2Rr_k}\setminus B_{\frac{1}{2}Rr_k}} |\nabla u^k|^2\Big)^{1/2} \le C\epsilon^2.
\end{aligned} \tag{3.8}
$$

Next, we estimate the first term in the right hand side of (3.6). It yields

$$
\begin{aligned}
\int_{\mathcal{A}_k^i} \Delta u^k (u^k - v^k) &= \int_{\mathcal{A}_k^i} (H(u^k)u_x^k \wedge u_y^k + f^k)(u^k - v^k) \\
&\le C \sup_{0\le i\le N_k - 1} \operatorname{osc}_{\mathcal{A}_k^i}(u_k - v_k) \int_{\mathcal{A}_k^i} (|u^k|^2 + |f^k|) \\
&\le C\epsilon \int_{\mathcal{A}_k^i} (|u^k|^2 + |f^k|)
\end{aligned} \tag{3.9}
$$

Putting theses estimates of (3.7) (3.8) (3.9) into the inequality (3.6), then we conclude that

$$
\int_{B_\delta\setminus B_{Rr_k}} |\nabla(u^k - v^k)|^2 \le C\epsilon \int_{B_\delta\setminus B_{Rr_k}} (|u^k|^2 + |f^k|) + C\epsilon^2 \le C\epsilon. \tag{3.10}
$$

Since v^k is radial form and $|\nabla u^k|^2 = |\frac{\partial u^k}{\partial r}|^2 + \frac{1}{r^2}|\frac{\partial u^k}{\partial \theta}|^2$, it follows that for $rR_K \le R \le \delta$,

$$
\begin{aligned}
\int_{B_\delta\setminus B_{Rr_k}} \frac{1}{r^2}\Big|\frac{\partial u^k}{\partial \theta}\Big|^2 &= \int_{B_\delta\setminus B_{Rr_k}} \frac{1}{r^2}\Big|\frac{\partial (u^k - v^k)}{\partial \theta}\Big|^2 \\
&\le \int_{B_\delta\setminus B_{Rr_k}} |\nabla(u^k - v^k)|^2 \le C\epsilon,
\end{aligned} \tag{3.11}
$$

where $\frac{\partial u^k}{\partial \theta}$ and $\frac{\partial u^k}{\partial r}$ denote the tangential component and the radial component of ∇u^k, respectively.

Step 2. Radial component of energy in the neck region: Now we are in a position to employ the Pohozaev argument to control $\int_{B_\delta\setminus B_{Rr_k}} |\frac{\partial u^k}{\partial r}|^2$ by $\int_{B_\delta\setminus B_{Rr_k}} \frac{1}{r^2}|\frac{\partial u^k}{\partial \theta}|^2$ and $\|f^k\|_{L^p(B_\delta)}$. Observe that $x\cdot\nabla u^k \in L^{p'}(B_\delta), \Delta u^k \in L^p(B_\delta)$ and $f^k \in L^p(B_\delta)$ with $2 < p' < \frac{2p}{2-p}$ for $1 < p < 2$, we have that $f^k\cdot(x\cdot\nabla u^k) \in L^1(B_\delta)$ and $\Delta u^k \cdot (x\cdot\nabla u^k) \in L^1(B_\delta)$ only if $p > \frac{4}{3}$. That is due to $\frac{2p}{2-p} > \frac{p}{p-1}$ at this point. On the other hand, thanks to Equ.(1.5) it implies that $(\Delta u^k - f^k) = H(u^k)u_x^k \wedge u_y^k \perp T_{u^k(x)}\mathcal{S}$, a. e. $x \in B_\delta$ with $\mathcal{S} = u^k(B_\delta)$. Therefore, by multiplying (1.5) by $x\cdot\nabla u^k$ and integrating it over B_r, for $0 < r < \delta$, yields

$$
\int_{B_r} \Delta u^k \cdot (x\cdot\nabla u^k) = \int_{B_r} f^k\cdot(x\cdot\nabla u^k).
$$

In terms of Green's identity, we have

$$
\begin{aligned}
\int_{B_r} \Delta u^k \cdot (x \cdot \nabla u^k) &= -\int_{B_r} \nabla u^k \cdot \nabla (x \cdot \nabla u^k) + \int_{\partial B_r} (x \cdot \nabla u^k)(\nabla u^k \cdot \frac{x}{|x|}) \\
&= \frac{1}{2} \int_{B_r} x \cdot \nabla (|\nabla u^k|^2) - \int_{B_r} |\nabla u^k|^2 + r \int_{\partial B_r} \Big|\frac{\partial u^k}{\partial r}\Big|^2 \\
&= -\frac{1}{2} r \int_{\partial B_r} |\nabla u^k|^2 + r \int_{\partial B_r} \Big|\frac{\partial u^k}{\partial r}\Big|^2 .
\end{aligned} \tag{3.12}
$$

Therefore,

$$
r \int_{\partial B_r} \Big|\frac{\partial u^k}{\partial r}\Big|^2 - \frac{1}{2} r \int_{\partial B_r} |\nabla u^k|^2 = \int_{B_r} f^k \cdot (x \cdot \nabla u^k).
$$

It follows from $|\nabla u^k|^2 = |\frac{\partial u^k}{\partial r}|^2 + \frac{1}{r^2} |\frac{\partial u^k}{\partial \theta}|^2$ that

$$
\int_{\partial B_r} \Big|\frac{\partial u^k}{\partial r}\Big|^2 \le \int_{\partial B_r} r^{-2} \Big|\frac{\partial u^k}{\partial \theta}\Big|^2 + 2 \int_{B_r} \Big|f^k\Big| \Big|\nabla u^k\Big|. \tag{3.13}
$$

Integrating it over the interval $[Rr_k, \delta]$, it follows from Hölder inequality and the tangential estimate of (3.11) that for $0 < \delta < \epsilon$ there hold

$$
\begin{aligned}
\int_{B_\delta \setminus B_{Rr_k}} \Big|\frac{\partial u^k}{\partial r}\Big|^2 &\le \int_{B_\delta \setminus B_{Rr_k}} \frac{1}{r^2} \Big|\frac{\partial u^k}{\partial \theta}\Big|^2 + 2\delta \Big\|f^k\Big\|_{L^p(B_\delta)} \Big\|\nabla u^k\Big\|_{L^{p'}(B_\delta)} \\
&\le C(\epsilon + \delta) \le C\epsilon .
\end{aligned} \tag{3.14}
$$

Putting (3.11) and (3.14) together, it yields

$$
\int_{B_\delta \setminus B_{Rr_k}} \Big|\nabla u^k\Big|^2 = \int_{B_\delta \setminus B_{Rr_k}} \Big|\frac{\partial u^k}{\partial r}\Big|^2 + \int_{B_\delta \setminus B_{Rr_k}} \frac{1}{r^2} \Big|\frac{\partial u^k}{\partial \theta}\Big|^2 \le C\epsilon. \tag{3.15}
$$

This implies that there is no neck formation between any two bubbles. Hence the proof of Theorem 1.2 is complete.

4. Application to H-surface flows

As an application of Theorem 1.2, we will establish the energy identity at a finite time singular point for the sequences of H-surface flows.

4.1. **Proof of Theorem 1.3.** Without loss of generality, suppose that $\Omega = B_1$ and (x_0, T_0) is the only singular point at $t = T_0$. Let $u^k(x,t) = u(\lambda_k x; t_k + \lambda_k^2 t)$, then $u^k(x,t)$ still satisfies the same equation as (1.9), and by (1.10) we have

$$
\int_{-2}^{2} \int_{B_{\lambda_k^{-1}}} \Big|\frac{\partial u^k}{\partial t}\Big|^2 = \int_{t_k - 2\lambda_k^2}^{t_k + 2\lambda_k^2} \int_{B_1} \Big|\frac{\partial u}{\partial t}\Big|^2 \to 0, \quad \text{as } k \to \infty. \tag{4.1}
$$

By Fubini's theorem, there exists $\eta_k \in (-1, -1/2)$ such that

$$
\int_{B_{\lambda_k^{-1}}} \Big|\frac{\partial u^k(\cdot, \eta_k)}{\partial t}\Big|^2 \to 0, \quad \int_{B_{\lambda_k^{-1}} \times (-2,2)} \Big|\frac{\partial u^k}{\partial t}\Big|^2 \to 0. \tag{4.2}
$$

Here, just similar to [12], then we have the following energy inequality: for any $0 < s \le \tau < T$ and $B_{2R}(x) \subset \Omega, x \in \Omega$, there holds

$$
E(u(\tau), B_R(x)) \le 5E(u(s), B_{2R}(x)) + C\frac{\tau - s}{R^2} J_H(u_0). \tag{4.3}
$$

With the help of the energy inequality (4.3), it is known from [15, Lemma 4.1] that there exists unique positive m such that, in the sense of Radon measure, we have

$$|\nabla u|^2(x,t)dx \to m\delta_{x_0} + |\nabla u|^2(x,T_0)dx, \quad \text{as } t \to T_0, \tag{4.4}$$

at the only singular point at $t = T_0$, where δ_{x_0} denotes Dirac δ-mass at x_0. Also from (4.3) it follows that

$$\int_{B_R} \Big|\nabla u^k(\cdot,\eta_k)\Big|^2 \geq \int_{B_1} \Big|\nabla u^k(\cdot,T_0)\Big|^2 - CR^{-2}J_0,$$

let $R \to \infty$, by (4.4) which implies

$$\lim_{R\to\infty} \int_{B_R} \Big|\nabla u^k(\cdot,\eta_k)\Big|^2 = m. \tag{4.5}$$

Therefore, for each $R > 0$, we know from (4.5) that $u^k(\cdot,\eta_k) \rightharpoonup v$ weakly in $W^{1,2}(B_R,\mathbb{R}^3)$. We claim that v is a constant map. Indeed, let $|t_k| \leq 2\lambda_k^2$ we observe that

$$\int_{B_R} \Big|u^k(\cdot,\eta_k) - u^k(\cdot,-t_k\lambda_k^{-2})\Big|^2 \leq 4\int_{-2}^{2}\int_{B_{\lambda_k^{-1}}} \Big|\frac{\partial u^k}{\partial t}\Big|^2 \to 0,$$

and

$$\int_{B_R} \Big|\nabla u^k(\cdot,-t_k\lambda_k^{-2})\Big|^2 = \int_{B_{\lambda_k R}} \Big|\nabla u\Big|^2(\cdot,T_0) \to 0.$$

For each $R > 0$, now we apply Theorem 1.2 of approximate H-surfaces to $u^k(\cdot,\eta_k)$ on the ball B_R to conclude that there exist finite number bubbles $\{\omega_{i,R}\}_{i=1}^{L_R}$ such that

$$\lim_{k\to\infty} \int_{B_R} \Big|\nabla u^k\Big|^2(\cdot,\eta_k) = \sum_{i=1}^{L_R} E(\omega_{i,R},\mathbb{S}^2). \tag{4.6}$$

Further, we know that $1 \leq L_R \leq \Big[\frac{m}{\varepsilon_0}\Big]$ because there is a ε_0 such that any bubble $\omega : \mathbb{S}^2 \to \mathbb{R}^3$ satisfying $E(\omega,\mathbb{S}^2) \geq \varepsilon_0$. Hence, there exists a $d \in \Big[1, \frac{m}{\varepsilon_0}\Big]$ such that, after possible a subsequence, $L_R = d$ and

$$m = \lim_{R\uparrow\infty}\lim_{k\to\infty} \int_{B_R} \Big|\nabla u^k\Big|^2(\cdot,\eta_k) = \lim_{R\uparrow\infty} \sum_{i=1}^{d} E(\omega_{i,R},\mathbb{S}^2). \tag{4.7}$$

Note that $\{\omega_{i,R}\}_{i=1}^{d}$ have uniformly boundedness of energies, from Brezis-Coron [3] one concludes that there exist $N_i \in \Big[1, \frac{m}{\varepsilon_0}\Big]$ and N_i bubbles $\{\omega_{i,j}\}_{j=1}^{N_i}$ such that

$$\lim_{R\uparrow\infty} E(\omega_{i,R},\mathbb{S}^2) = \sum_{j=1}^{N_i} E(\omega_{i,j},\mathbb{S}^2). \tag{4.8}$$

Now, putting all (4.6),(4.7) and (4.8) together, it follows that

$$m = \sum_{i=1}^{d}\sum_{j=1}^{N_i} E(\omega_{i,j},\mathbb{S}^2).$$

The proof of Theorem 1.3 is complete.

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