Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 171, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

GLOBAL PROPERTIES AND MULTIPLE SOLUTIONS FOR BOUNDARY-VALUE PROBLEMS OF IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. This article presents global properties and existence of multiple solutions for a class of boundary value problems of impulsive differential equations. We first show that the spectral properties of the linearization of these problems are similar to the well-know properties of the standard Sturm-Liouville problems. These spectral properties are then used to prove two Rabinowitz-type global bifurcation theorems. Finally, we use the global bifurcation theorems to obtain multiple solutions for the above problems having specified nodal properties.

1. Introduction

In this article, we study the global properties for the boundary-value problem

$$\begin{gathered} -x''(t) = \lambda f(t,x), \quad t \in (0,1), \quad t \neq \frac{1}{2}, \\ \Delta x|_{t=\frac{1}{2}} = \lambda\beta x(\frac{1}{2}), \\ \Delta x'|_{t=\frac{1}{2}} = -\lambda\beta x'(\frac{1}{2}-0), \\ x(0) = x(1) = 0, \end{gathered} \tag{1.1}$$

where $\lambda \neq 0$, $\beta \neq 0$, $\Delta x|_{t=\frac{1}{2}} = x(\frac{1}{2}+0) - x(\frac{1}{2})$, $\Delta x'|_{t=\frac{1}{2}} = x'(\frac{1}{2}+0) - x'(\frac{1}{2}-0)$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

The theory of impulsive differential equations has been a significant development in recent years and played a very important role in modern applied mathematical models of real processes rising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics (see [4, 6, 7, 11, 26, 27]). There have appeared numerous papers on impulsive differential equations and many of them are committed to study the existence of solutions for boundary value problems of second order impulsive differential equations. The general methods to solve these problems include topological degree theory (see [12, 23, 30]), the

2000 *Mathematics Subject Classification.* 34B09, 34B15, 34B37.
Key words and phrases. Eigenvalues; bifurcation technique; global behavior; multiple solutions.

Submitted June 4, 2013. Published July 26, 2013.

method of upper and lower solutions (see [9, 10, 13]) and variational method (see [3, 20, 21, 23, 27, 28, 31]).

In natural sciences, there are various concrete problems involving bifurcation phenomena, for example, Taylor vortices [2] and catastrophic shifts in ecosystems [25]. Rabinowitz [21] established the global bifurcation theorem from the trivial solution and in [22] the bifurcation from infinity was studied. In recent years, this theory has been successfully applied to Sturm-Liouville problems for ordinary differential equations, integral equations and partial differential equations (see [15, 16, 17, 20]). In [14], using bifurcation techniques, Liu and O'Regan studied the existence of multiple solutions for the second order impulsive differential equation

$$\begin{gathered} x''(t)+ra(t)f(t,x(t))=0, \quad t\in(0,1),\ t\neq t_i, \\ \Delta x|_{t=t_i}=\alpha_i x(t_i-0), \quad i=1,2,\dots,k, \\ x(0)=x(1)=0, \end{gathered} \tag{1.2}$$

in which they convert (1.2) into

$$\begin{gathered} y''(t)+\frac{r}{\prod_{0<t_i<t}(1+\alpha_i)}a(t)f(t,\prod_{0<t_i<t}(1+\alpha_i)y(t))=0, \quad t\in(0,1), \\ y(0)=y(1)=0. \end{gathered} \tag{1.3}$$

The aim of this conversion is to prove the existence of multiple solutions for above problems by using the properties of the eigenvalues and eigenfunctions of the linear equations corresponding to (1.3). In this article, without using this conversion, in section 2, we study the properties of the eigenvalues and eigenfunctions of the linear equations corresponding to (1.1), in section 3, we investigate the global bifurcation results and the existence of multiple solutions for BVP (1.1) and in section 4, we give an example as the applications.

Let $\mathcal{F}:\mathcal{E}\longrightarrow\mathcal{E}_1$ where $\mathcal{E}$ and $\mathcal{E}_1$ are real Banach spaces and $\mathcal{F}$ is continuous. Suppose the equation $\mathcal{F}(U)=0$ possesses a simple curve of solutions Ψ given by $\{U(t)|t\in[a,b]\}$. If for some $\tau\in(a,b),\mathcal{F}$ possesses zeros not lying on Ψ in every neighborhood of $U(\tau)$, then $U(\tau)$ is said to be a bifurcation point for $\mathcal{F}$ with respect to the curve Ψ (see [21]).

A special family of such equations has the form

$$u=G(\lambda,u) \tag{1.4}$$

where $\lambda\in\mathbb{R},u\in E$, a real Banach space with the norm $\|\cdot\|$ and $G:\mathcal{E}\equiv\mathbb{R}\times E\to E$ is compact and continuous. In addition, $G(\lambda,u)=\lambda Lu+H(\lambda,u)$, where $H(\lambda,u)$ is $o(\|u\|)$ for u near 0 uniformly on bounded λ intervals and L is a compact linear map on E. A solution of (1.4) is a pair $(\lambda,u)\in\mathcal{E}$. The known curve of solutions $\Theta=\{(\lambda,0)|\lambda\in\mathbb{R}\}$ will henceforth be referred to as the trivial solutions. The closure of the set of nontrivial solutions of (1.4) will be denoted by Σ. A component of Σ is a maximal closed connected subset.

If there exists $\mu\in\mathbb{R}$ and $0\neq v\in E$ such that $v=\mu Lv,\mu$ is said to be a real characteristic value of L. The set of real characteristic values of L will be denoted by $\sigma(L)$. The multiplicity of $\mu\in\sigma(L)$ is the dimension of $\cup_{j=1}^{\infty}N((\mu L-I)^j)$ where I is the identity map on E and $N(P)$ denotes the null space of P. Since L is compact, μ is of finite multiplicity. It is well known that if $\mu\in\mathbb{R}$, a necessary condition for $(\mu,0)$ to be a bifurcation point of (1.4) with respect to Θ is that $\mu\in\sigma(L)$.

Lemma 1.1 ([21]). *If $\mu \in \sigma(L)$ is simple, then Σ contains a component C_μ which can be decomposed into two subcontinua C_μ^+, C_μ^- such that for some neighborhood B of $(\mu, 0)$,*

$$(\lambda, u) \in C_\mu^+ (C_\mu^-) \cap B, \quad \text{and} \quad (\lambda, u) \neq (\mu, 0)$$

implies $(\lambda, u) = (\lambda, \alpha v + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \mu| = o(1), \|w\| = o(|\alpha|)$, at $\alpha = 0$. Moreover, each of C_μ^+, C_μ^- either

1. *meets infinity in Σ , or*
2. *meets $(\hat{\mu}, 0)$ where $\mu \neq \hat{\mu} \in \sigma(L)$, or*
3. *contains a pair of points $(\lambda, u), (\lambda, -\mu), u \neq 0$.*

Lemma 1.2 ([22]). *Assume that L is compact and linear, $H(\lambda, u)$ is continuous on $\mathbb{R} \times E$, $H(\lambda, u) = o(\|u\|)$ at $u = \infty$ uniformly on bounded λ intervals, and $\|u\|^2 H(\lambda, \frac{u}{\|u\|^2})$ is compact. If $\mu \in \sigma(L)$ is of odd multiplicity, then Σ possesses an unbounded component D_μ which meets (μ, ∞). Moreover if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap \sigma(L) = \mu$ and $\mathcal{M}$ is a neighborhood of (μ, ∞) whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*

1. *$D_\mu \backslash \mathcal{M}$ is bounded in $\mathbb{R} \times E$ in which case $D_\mu \backslash \mathcal{M}$ meets $\Theta = \{(\lambda, 0) | \lambda \in \mathbb{R}\}$ or*
2. *$D_\mu \setminus \mathcal{M}$ is unbounded.*

If (2) occurs and $D_\mu \backslash \mathcal{M}$ has a bounded projection on $\mathbb{R}$, then $D_\mu \backslash \mathcal{M}$ meets (μ, ∞) where $\mu \neq \hat{\mu} \in \sigma(L)$.

Lemma 1.3 ([22]). *Suppose the assumptions of Lemma 1.2 hold. If $\mu \in \sigma(L)$ is simple, then D_μ can be decomposed into two subcontinua D_μ^+, D_μ^- and there exists a neighborhood $\mathcal{O} \subset \mathcal{M}$ of (μ, ∞) such that $(\lambda, u) \in D_\mu^+ (D_\mu^-) \cap \mathcal{O}$, and $(\lambda, u) \neq (\mu, \infty)$ implies $(\lambda, u) = (\lambda, \alpha v + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \mu| = o(1)$, $\|w\| = o(|\alpha|)$, at $|\alpha| = \infty$.*

2. Preliminaries

Let $PC[0,1] = \{x : [0,1] \to \mathbb{R} : x(t)$ is continuous at $t \neq \frac{1}{2}$, and left continuous at $t = \frac{1}{2}$, and $x(\frac{1}{2} + 0) = \lim_{t \to \frac{1}{2}^+} x(t)$ exists $\}$ with the norm

$$\|x\| = \sup_{t \in [0,1]} |x(t)|.$$

Let $PC'[0,1] = \{x \in PC[0,1] : x'(t)$ is continuous at $t \neq \frac{1}{2}$, and $x'(\frac{1}{2} - 0) = \lim_{t \to \frac{1}{2}^-} x'(t)$, and $x'(\frac{1}{2} + 0) = \lim_{t \to \frac{1}{2}^+} x'(t)$ exist $\}$ with the norm

$$\|x\|_1 = \max\{\sup_{t \in [0,1]} |x(t)|, \sup_{t \in [0,1]} |x'(t)|\}.$$

Let $E = \{x \in PC'[0,1] : x(0) = x(1) = 0\}$. It is well known that E is a Banach space with the norm $\| \cdot \|_1$.

Let S_k^+ denote the set of functions in E which have exactly $k-1$ simple nodal zeros in $(0,1)$ and are positive near $t = 0$. (By a nodal zero we mean the function changes sign at the zeros and at at a simple nodal zero, the derivative of the function is nonzero.) And set $S_k^- = -S_k^+$, $S_k = S_k^+ \cup S_k^-$. They are disjoint in E. Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$.

For the rest of the paper, we always assume the initial-value problem

$$-x''(t) = \lambda f(t, x), \quad t \in (0,1), \ t \neq \frac{1}{2},$$

$$
\begin{gathered}
\Delta x|_{t=\frac{1}{2}} = \lambda\beta x(\frac{1}{2}), \\
\Delta x'|_{t=\frac{1}{2}} = -\lambda\beta x'(\frac{1}{2}-0), \\
x(t_0) = x'(t_0) = 0,
\end{gathered}
$$

has the unique trivial solution $x \equiv 0$ on $[0,1]$ for any $t_0 \in [0,1]$.

Lemma 2.1 ([8]). *$x(t) \in PC[J,\mathbb{R}] \cap C^2[J',\mathbb{R}]$ is the solution of (1.1) equivalent to $x(t) \in PC'[J,\mathbb{R}]$ is the solution of the integral equation*

$$
x(t) = \begin{cases} \lambda \int_0^1 G(t,s) f(s,x(s))ds - \lambda\beta t[x(\frac{1}{2}) - \frac{1}{2}x'(\frac{1}{2}-0)], & t \in [0,\frac{1}{2}], \\ \lambda \int_0^1 G(t,s) f(s,x(s))ds + \lambda\beta(1-t)[x(\frac{1}{2}) + \frac{1}{2}x'(\frac{1}{2}-0)], & t \in (\frac{1}{2},1], \end{cases}
$$

where $J = [0,1], J' = J \setminus \{\frac{1}{2}\}$,

$$
G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}
$$

Lemma 2.2. *If $a > 0$, then the linear boundary-value problem*

$$
\begin{gathered}
-u''(t) = \lambda a u(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \\
\Delta u|_{t=\frac{1}{2}} = \lambda\beta u(\frac{1}{2}), \\
\Delta u'|_{t=\frac{1}{2}} = -\lambda\beta u'(\frac{1}{2}-0), \\
u(0) = u(1) = 0,
\end{gathered} \tag{2.1}
$$

possess an increasing sequence of eigenvalues

$$
0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots, \quad \lim_{k\to+\infty} \lambda_k = +\infty.
$$

And eigenfunction u_k corresponding to λ_k has exactly $k-1$ nodal zeros on $(0,1)$.

Proof. Let $u(t)$ be the solution of (2.1). We consider the three following cases:
Case (i) $\lambda = 0$. Then $u(t)$ can be written as

$$
u(t) = \begin{cases} c_1 t + c_2, & t \in [0,\frac{1}{2}], \\ c_3 t + c_4, & t \in (\frac{1}{2},1]. \end{cases}
$$

From (2.1), we have

$$
\begin{gathered}
c_2 = 0, \\
-\frac{1}{2}c_1 - c_2 + \frac{1}{2}c_3 + c_4 = 0, \\
-c_1 + c_3 = 0, \\
c_3 + c_4 = 0,
\end{gathered}
$$

which implies $c_1 = c_2 = c_3 = c_4 = 0$. Then $u(t) \equiv 0$. Thus $\lambda = 0$ is not the eigenvalue of (2.1).
Case (ii) $\lambda < 0$. Then $u(t)$ can be written as

$$
u(t) = \begin{cases} c_1 e^{\sqrt{-\lambda a} t} + c_2 e^{-\sqrt{-\lambda a} t}, & t \in [0,\frac{1}{2}], \\ c_3 e^{\sqrt{-\lambda a} t} + c_4 e^{-\sqrt{-\lambda a} t}, & t \in (\frac{1}{2},1]. \end{cases}
$$

From (2.1), we have

$$
\begin{gathered}
c_1 + c_2 = 0,\\
(1+\lambda\beta)e^{\frac{\sqrt{-\lambda a}}{2}}c_1 + (1+\lambda\beta)e^{-\frac{\sqrt{-\lambda a}}{2}}c_2 - e^{\frac{\sqrt{-\lambda a}}{2}}c_3 - e^{-\frac{\sqrt{-\lambda a}}{2}}c_4 = 0,\\
(1-\lambda\beta)e^{\frac{\sqrt{-\lambda a}}{2}}c_1 - (1-\lambda\beta)e^{-\frac{\sqrt{-\lambda a}}{2}}c_2 - e^{\frac{\sqrt{-\lambda a}}{2}}c_3 + e^{-\frac{\sqrt{-\lambda a}}{2}}c_4 = 0,\\
e^{\sqrt{-\lambda a}}c_3 + e^{-\sqrt{-\lambda a}}c_4 = 0,
\end{gathered}
$$

which implies $c_1 = c_2 = c_3 = c_4 = 0$. Then $u(t) \equiv 0$. Thus $\lambda < 0$ is not the eigenvalue of (2.1).

Case (iii) $\lambda > 0$. Then $u(t)$ can be written as

$$
u(t) = \begin{cases} c_1\cos(\sqrt{\lambda a}t) + c_2\sin(\sqrt{\lambda a}t), & t\in[0,\frac12],\\ c_3\cos(\sqrt{\lambda a}t) + c_4\sin(\sqrt{\lambda a}t), & t\in(\frac12,1]. \end{cases}
$$

From (2.1) we know $c_1 = 0$ and

$$
\begin{gathered}
\cos(\frac{\sqrt{\lambda a}}{2})c_3 + \sin(\frac{\sqrt{\lambda a}}{2})c_4 - (1+\lambda\beta)\sin(\frac{\sqrt{\lambda a}}{2})c_2 = 0,\\
-\sin(\frac{\sqrt{\lambda a}}{2})c_3 + \cos(\frac{\sqrt{\lambda a}}{2})c_4 - (1-\lambda\beta)\cos(\frac{\sqrt{\lambda a}}{2})c_2 = 0,\\
\cos(\sqrt{\lambda a})c_3 + \sin(\sqrt{\lambda a})c_4 = 0.
\end{gathered}
$$

The determinant of the coefficient matrix $\det = \sin\sqrt{\lambda a}$. Letting $\det = 0$, we have $\lambda_k = \frac{k^2\pi^2}{a}$, $k = 1,2,\dots$. Then

$$
c_3 = 0,\quad c_4 = (1+\beta(-1)^{k+1}\frac{k^2\pi^2}{a})c_2,
$$

which implies $\lambda_k = \frac{k^2\pi^2}{a}$ is the eigenvalues of (2.1), and

$$
0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots,\quad \lim_{k\to+\infty}\lambda_k = +\infty.
$$

The eigenfunction corresponding to λ_k is

$$
u_k(t) = \begin{cases} \sin(k\pi t), & t\in[0,\frac12],\\ (1+\beta(-1)^{k+1}\frac{k^2\pi^2}{a})\sin(k\pi t), & t\in(\frac12,1]. \end{cases}
$$

Now we prove that u_k has exactly $k-1$ nodal zeros on $(0,1)$. Actually, if k is odd, then $u_k(t)$ has $\frac{k-1}{2}$ zeros on $(0,\frac12]$, $u_k(t)$ has $\frac{k-1}{2}$ zeros on $(\frac12,1)$, and if k is even, then $u_k(t)$ has $\frac{k}{2}$ zeros on $(0,\frac12]$, $u_k(t)$ has $\frac{k-2}{2}$ zeros on $(\frac12,1)$. Thus $u_k(t) = 0$ has $k-1$ zeros on $(0,1)$. Using the fact

$$
u_k'(t) = \begin{cases} k\pi\cos(k\pi t), & t\in[0,\frac12),\\ (1+\beta(-1)^{k+1}\frac{k^2\pi^2}{a})k\pi\cos(k\pi t), & t\in(\frac12,1], \end{cases}
$$

we see that $u_k(t)$ has exactly $k-1$ nodal zeros on $(0,1)$. The proof is complete. □

Lemma 2.3. *For each $k \ge 1$ the algebraic multiplicity of eigenvalue λ_k is equal to 1.*

Proof. Define the operator $K : PC'[0,1] \to PC'[0,1]$ as follows:

$$
(Kx)(t) = \begin{cases} \int_0^1 G(t,s)ax(s)ds - \beta t[x(\frac12) - \frac12 x'(\frac12 - 0)], & t\in[0,\frac12],\\ \int_0^1 G(t,s)ax(s)ds + \beta(1-t)[x(\frac12) + \frac12 x'(\frac12 - 0)], & t\in(\frac12,1]. \end{cases}
$$

We need to prove only that $\ker(I-\lambda_k K)^2 \subset \ker(I-\lambda_k K)$. For any $y \in ker(I-\lambda_k K)^2$, we have $(I-\lambda_k K)^2 y = 0$, so $(I-\lambda_k K)y \in ker(I-\lambda_k K)$.

Let $\bar{\beta}_k = 1+\beta(-1)^{k+1}\lambda_k$. From Lemma 2.2, there exists a γ satisfying

$$(I-\lambda_k K)y = \gamma \sin(\sqrt{\lambda_k a}t), \quad t \in [0, \frac{1}{2}],$$
$$(I-\lambda_k K)y = \gamma\bar{\beta}_k \sin(\sqrt{\lambda_k a}t), \quad t \in (\frac{1}{2}, 1];$$

that is, y satisfies

$$y'' + \lambda_k a y = -\gamma\lambda_k a \sin(\sqrt{\lambda_k a}t), \quad t \in [0, \frac{1}{2}),$$
$$y'' + \lambda_k a y = -\gamma\lambda_k a\bar{\beta}_k \sin(\sqrt{\lambda_k a}t), \quad t \in (\frac{1}{2}, 1],$$

and $y(0) = y(1) = 0$. Now we prove $\gamma = 0$. Actually, the general solution of the above differential equation is of the form

$$\begin{aligned} y(t) &= c_1 \cos(\sqrt{\lambda_k a}t) + c_2 \sin(\sqrt{\lambda_k a}t) - \frac{\gamma}{2}\sin(\sqrt{\lambda_k a}t) \\ &\quad + \frac{\gamma\sqrt{\lambda_k a}t}{2}\cos(\sqrt{\lambda_k a}t), \quad t \in [0, \frac{1}{2}], \end{aligned}$$
$$\begin{aligned} y(t) &= c_1 \cos(\sqrt{\lambda_k a}t) + c_2 \sin(\sqrt{\lambda_k a}t) - \frac{\gamma\bar{\beta}_k}{4}\sin(\sqrt{\lambda_k a}t) + \frac{\gamma\bar{\beta}_k}{4}\sin(\sqrt{\lambda_k a} \\ &\quad - \sqrt{\lambda_k a}t) + \frac{\gamma\sqrt{\lambda_k a}\bar{\beta}_k t}{2}\cos(\sqrt{\lambda_k a}t) - \frac{\gamma\sqrt{\lambda_k a}\bar{\beta}_k}{4}\cos(\sqrt{\lambda_k a}t), \quad t \in (\frac{1}{2}, 1]. \end{aligned}$$

From $y(0) = y(1) = 0$, we have $c_1 = 0$ and

$$c_2 \sin(\sqrt{\lambda_k a}) - \frac{\gamma\bar{\beta}_k}{4}\sin(\sqrt{\lambda_k a}) + \frac{\gamma\sqrt{\lambda_k a}\bar{\beta}_k}{2}\cos(\sqrt{\lambda_k a}) - \frac{\gamma\sqrt{\lambda_k a}\bar{\beta}_k}{4}\cos(\sqrt{\lambda_k a}) = 0.$$

By Lemma 2.2, $\lambda_k = k^2\pi^2/a$ satisfies $\sin(\sqrt{\lambda_k a}) = 0, \cos(\sqrt{\lambda_k a}) \neq 0$ which implies $\gamma = 0$.

Thus $y \in ker(I-\lambda_k K)$. Therefore $\ker(I-\lambda_k K)^2 \subset \ker(I-\lambda_k K)$. The proof is complete. ☐

Lemma 2.4. *For any positive integer k, S_k, S_k^+ and S_k^- are open in E.*

Proof. We prove only that S_k is open in E. Let $t_1, t_2, \dots, t_{k-1} \in (0,1)$ are $k-1$ simple nodal zeros on $(0,1)$ of $u(t) \in S_k$.

Suppose $u'(t_j) = c_j > 0$. Then there exists $\delta_j > 0$ which satisfies for any $t \in [t_j - \delta_j, t_j + \delta_j], u'(t)$ is continuous, $u'(t) > \frac{c_j}{2}$, $u(t_j - \delta_j) < 0$ and $u(t_j + \delta_j) > 0$.

Letting $\varphi \in E$ and $\|\varphi - u\|_1 \le \sigma$, where

$$\begin{aligned} \sigma = \min\Big\{&\frac{c_j}{2}, \frac{u(t_j+\delta_j)}{2}, -\frac{u(t_j-\delta_j)}{2}, -\frac{1}{2}\max_{t\in[t_{j-1}+\delta_{j-1}, t_j-\delta_j]} u(t), \\ &\frac{1}{2}\max_{t\in[t_j+\delta_j, t_{j+1}-\delta_{j+1}]} u(t)\Big\}, \end{aligned}$$

we have

$$\begin{gathered} \varphi'(t) = \varphi'(t) - u'(t) + u'(t) \ge -\sigma + \frac{c_j}{2} > 0, t \in [t_j - \delta_j, t_j + \delta_j], \\ \varphi(t_j + \delta_j) > u(t_j + \delta_j) - \sigma > 0, \\ \varphi(t_j - \delta_j) < u(t_j - \delta_j) + \sigma < 0. \end{gathered}$$

Since $\varphi(t)$ is continuous on $[t_j-\delta_j,t_j+\delta_j]$, there exists unique $t_j^*\in[t_j-\delta_j,t_j+\delta_j]$ which satisfies $\varphi(t_j^*)=0,\varphi'(t_j^*)>0$.

Since for any $t\in[t_{j-1}+\delta_{j-1},t_j-\delta_j],u(t)<0$, and for any $t\in[t_j+\delta_j,t_{j+1}-\delta_{j+1}],u(t)>0$, one has $\varphi(t)<u(t)+\sigma<0$ for any $t\in[t_{j-1}+\delta_{j-1},t_j-\delta_j]$ and $\varphi(t)>u(t)+\sigma>0$ for any $t\in[t_j+\delta_j,t_{j+1}-\delta_{j+1}]$. Therefore for any $t\in[t_{j-1}+\delta_{j-1},t_{j+1}-\delta_{j+1}],\varphi(t)$ has the unique simple zero t_j^*.

If $u'(t_j)=c_j<0$, we can get for any $t\in[t_{j-1}+\delta_{j-1},t_{j+1}-\delta_{j+1}],\varphi(t)$ has the unique simple zero t_j^* by the same method as above also. Because $u(t)$ has exactly $k-1$ simple zeros on $(0,1)$, $\varphi(t)$ has exactly $k-1$ simple zeros on $(0,1)$. Therefore $\varphi(t)\in S_k$, which implies S_k are open in E.

By the same argument, we can prove S_k^+ and S_k^- are open in E. The proof is complete. □

3. Main results

In this section, we assume that

(H1) There exist two positive numbers f_0 and f_∞ such that $f_0=\lim_{|x|\to0}\frac{f(t,x)}{x}$ and $f_\infty=\lim_{|x|\to\infty}\frac{f(t,x)}{x}$ both uniformly with respect to $t\in[0,1]$.

Let $\zeta,\xi\in C([0,1]\times\mathbb{R},\mathbb{R})$ be such that

$$f(t,x)=f_0x+\zeta(t,x),\quad f(t,x)=f_\infty x+\xi(t,x). \tag{3.1}$$

Clearly, if (H1) holds, we have

$$\lim_{|x|\to0}\frac{\zeta(t,x)}{x}=0,\quad \lim_{|x|\to\infty}\frac{\xi(t,x)}{x}=0 \tag{3.2}$$

both uniformly with respect to $t\in[0,1]$.

Now we define three operators L_0,L_∞, and A as follows:

$$(L_0x)(t)=\begin{cases}\int_0^1G(t,s)f_0x(s)ds-\beta t[x(\frac12)-\frac12x'(\frac12-0)], & t\in[0,\frac12],\\ \int_0^1G(t,s)f_0x(s)ds+\beta(1-t)[x(\frac12)+\frac12x'(\frac12-0)], & t\in(\frac12,1],\end{cases} \tag{3.3}$$

$$(L_\infty x)(t)=\begin{cases}\int_0^1G(t,s)f_\infty x(s)ds-\beta t[x(\frac12)-\frac12x'(\frac12-0)], & t\in[0,\frac12],\\ \int_0^1G(t,s)f_\infty x(s)ds+\beta(1-t)[x(\frac12)+\frac12x'(\frac12-0)], & t\in(\frac12,1],\end{cases} \tag{3.4}$$

and

$$(Ax)(t)=\begin{cases}\int_0^1G(t,s)f(s,x(s))ds-\beta t[x(\frac12)-\frac12x'(\frac12-0)], & t\in[0,\frac12],\\ \int_0^1G(t,s)f(s,x(s))ds+\beta(1-t)[x(\frac12)+\frac12x'(\frac12-0)], & t\in(\frac12,1].\end{cases} \tag{3.5}$$

Obviously, L_0 and L_∞ are compact and linear.

From Lemma 2.1, we know that $x(t)$ is the solution of (1.1) if and only if $x(t)$ is the solution of

$$x=\lambda Ax. \tag{3.6}$$

Let $\Gamma=\mathbb{R}\times E$. A solution of (3.6) is a pair $(\lambda,x)\in\Gamma$. The known curve of solutions $\{(\lambda,0)|\lambda\in\mathbb{R}\}$ will henceforth be referred to as the trivial solutions. The closure of the set on nontrivial solutions of (3.6) will be denoted by Σ as in Lemma 1.1. Now we are ready to give our main results.

Theorem 3.1. *Let* (H1) *hold. In addition assume that for some $k \in \mathbb{N}$, either*

$$\frac{k^2\pi^2}{f_0} < \lambda < \frac{k^2\pi^2}{f_\infty}, \quad or \quad \frac{k^2\pi^2}{f_\infty} < \lambda < \frac{k^2\pi^2}{f_0}.$$

Then problem (1.1) *has at least two solutions x_k^+ and x_k^-, x_k^+ has exactly $k-1$ zeros in (0,1) and is positive near $t=0$, and x_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near $t=0$.*

Theorem 3.2. *Let* (H1) *hold. Suppose that there exist two integer $k>0$ and $j \geq 0$ such that either*

(i) $\frac{(k+j)^2\pi^2}{f_0} < \lambda < \frac{k^2\pi^2}{f_\infty}$;
(ii) $\frac{(k+j)^2\pi^2}{f_\infty} < \lambda < \frac{k^2\pi^2}{f_0}$

Then problem (1.1) *has at least $2(j+1)$ solutions x_{k+i}^+, x_{k+i}^-, $i=0,1,\dots,j$, x_{k+i}^+ has exactly $k+i-1$ zeros in $(0,1)$ and is positive near $t=0$, x_{k+i}^- has exactly $k+i-1$ zeros in $(0,1)$ and are negative near $t=0$.*

To use the terminology of Rabinowitz [20, 21], we first give the following lemmas.

Lemma 3.3. *Suppose that* (H1) *is satisfied. Then the operator A given in* (3.5) *is Fréchet differentiable at $x=\theta$, and $A'(\theta)=L_0$.*

Proof. From (3.1), we obtain

$$\begin{aligned}
&\|Ax-L_0x\|_1\\
&=\max\Big\{\sup_{t\in[0,1]}\Big|\int_0^1 G(t,s)f(s,x(s))ds-\int_0^1 G(t,s)f_0x(s)ds\Big|,\\
&\quad \sup_{t\in[0,1]}\Big|\int_0^1 G_t(t,s)f(s,x(s))ds-\int_0^1 G_t(t,s)f_0x(s)ds\Big|\Big\}\\
&=\max\Big\{\sup_{t\in[0,1]}\Big|\int_0^1 G(t,s)\zeta(s,x(s))ds\Big|,\sup_{t\in[0,1]}\Big|\int_0^1 G_t(t,s)\zeta(s,x(s))ds\Big|\Big\}\\
&\leq C\int_0^1|\zeta(s,x(s))|ds,
\end{aligned}$$

where C depends on bounds for G and G_t. Consequently, from (3.2),

$$\lim_{\|x\|_1\to 0}\frac{\|Ax-L_0x\|_1}{\|x\|_1}\leq \lim_{\|x\|_1\to 0}\frac{\int_0^1|\zeta(s,x(s))|ds}{\|x\|_1}=0,\quad \forall x\in E.$$

This means that the operator A given in (3.5) is Fréchet differentiable at $x=\theta$, and $A'(\theta)=L_0$. The proof is complete. □

Lemma 3.4. *Suppose that* (H1) *is satisfied. Then the operator A given in* (3.5) *is Fréchet differentiable at $x=\infty$, and $A'(\infty)=L_\infty$.*

Proof. For each $\varepsilon>0$, by (3.2), there exists $R>0$ such that

$$|\xi(t,x)|\leq\varepsilon|x|,\quad \text{for } |x|>R, t\in[0,1].$$

Let $M=\max_{|x|\leq R}|\xi(t,x)|$. Then we have

$$|\xi(t,x)|\leq\varepsilon|x|+M,\quad \forall x\in\mathbb{R}, t\in[0,1].$$

Therefore, for any $x \in E$, $t \in [0,1]$,

$$\begin{aligned}
&\|Ax - L_\infty x\|_1 \\
&= \max\Big\{ \sup_{t\in[0,1]} |\int_0^1 G(t,s)\xi(s,x(s))ds|, \sup_{t\in[0,1]} |\int_0^1 G_t(t,s)\xi(s,x(s))ds| \Big\} \\
&\le C_1 \int_0^1 |\xi(s,x(s))|ds \\
&\le C_1(M + \varepsilon \int_0^1 |x(s)|ds) \\
&\le C_1(M + \varepsilon\|x\|_1),
\end{aligned}$$

which implies

$$\lim_{\|x\|_1\to\infty} \frac{\|Ax - L_\infty x\|_1}{\|x\|_1} = 0,$$

where C_1 depends on bounds for G and G_t. This means that the operator A given in (3.5) is Fréchet differentiable at $x = \infty$, and $A'(\infty) = L_\infty$. The proof is complete. □

Under the condition (H1), (3.6) can be rewritten as

$$x = \lambda L_0 x + H(\lambda, x), \tag{3.7}$$

here $H(\lambda,x) = \lambda Ax - \lambda L_0 x$, L_0 is defined as (3.3). Clearly by Lemma 2.1, $x(t)$ is the solution of (2.1) if and only if $x(t)$ is the solution of the equation

$$x = \lambda L_0 x. \tag{3.8}$$

Therefore, the results of Lemma 2.2 and Lemma 2.3 satisfy (3.8).

From Lemma 3.3, it can be seen that $H(\lambda,x)$ is $o(\|x\|_1)$ for x near 0 uniformly on bounded λ.

Lemma 3.5. *For each integer $k > 0$ and each $\upsilon = +$, or $-$, there exists a continua C_k^υ of solutions of (3.6) in $\Phi_k^\upsilon \cup \{(\frac{k^2\pi^2}{f_0}, 0)\}$, which meets $\{(\frac{k^2\pi^2}{f_0}, 0)\}$ and ∞ in Σ.*

Proof. By Lemma 2.2 and Lemma 2.3, we know that for each integer $k > 0$, $\frac{k^2\pi^2}{f_0}$ is a simple characteristic value of operator L_0. So with Lemma 3.3, (3.7) can be considered as a bifurcation problem from the trivial solution. From Lemma 1.1 and condition (H1) it follows that Σ contains a component C_k which can be decomposed into two subcontinua C_k^+, C_k^- such that for some neighborhood B of $(\frac{k^2\pi^2}{f_0}, 0)$,

$$(\lambda, x) \in C_k^+(C_k^-) \cap B, \quad \text{and} \quad (\lambda, x) \ne (\frac{k^2\pi^2}{f_0}, 0),$$

implies $(\lambda, x) = (\lambda, \alpha u_k + w)$, where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \frac{k^2\pi^2}{f_0}| = o(1)$, $\|w\|_1 = o(|\alpha|)$ at $\alpha = 0$.

Since S_k is open and $u_k \in S_k$, we know

$$\frac{x}{\alpha} = u_k + \frac{w}{\alpha} \in S_k,$$

for $\alpha \ne 0$ sufficiently small. Then there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$, we have

$$(C_k \setminus \{(\frac{k^2\pi^2}{f_0}, 0)\}) \cap B_\delta \subset \Phi_k,$$

where B_δ is an open ball in Γ of radius δ centered at $(\frac{k^2\pi^2}{f_0}, 0)$. From the assumption in section 2 and (3.8) we know $C_k \setminus \{(\frac{k^2\pi^2}{f_0}, 0)\} \cap \partial\Phi_k = \emptyset$. Consequently, C_k lies in $\Phi_k \cup \{(\frac{k^2\pi^2}{f_0}, 0)\}$.

From the the same reasoning it can be seen that C_k^ϑ lies in $\Phi_k^\vartheta \cup \{(\frac{k^2\pi^2}{f_0}, 0)\}(\vartheta = +,$ or $-)$.

Next we show that the alternative (2) of Lemma 1.1 is impossible. Suppose, on the contrary and without loss of generality that C_k^+ meets $(\frac{j^2\pi^2}{f_0}, 0)$ with $k \neq j$ and $\frac{j^2\pi^2}{f_0} \in \sigma(L_0)$. Then there exists a sequence $(\gamma_m, z_m) \in C_k^+$ with $\gamma_m \to \frac{j^2\pi^2}{f_0}$ and $z_m \to 0$ as $m \to +\infty$. Notice that

$$z_m = \gamma_m L_0 z_m + H(\gamma_m, z_m). \tag{3.9}$$

Dividing this equation by $\|z_m\|_1$ and noticing that L_0 is compact on E and that $H(\gamma_m, z_m) = o(\|z_m\|_1)$ as $m \to \infty$, we may assume without loss of generality that $\frac{z_m}{\|z_m\|_1} \to z$ as $m \to \infty$. Thus by (3.9) it follows that

$$z = \frac{j^2\pi^2}{f_0} L_0 z.$$

Since $z \neq 0$, by Lemma 2.2, z belongs to S_j^+ or S_j^-. Notice that $\|\frac{z_m}{\|z_m\|_1} - z\|_1 \to 0$, S_j^+ and S_j^- are open, so $z_m \in S_j^+$ or S_j^- for m sufficiently large. This is a contradiction with $z_m \in S_k^+ (m \geq 1)$ and $j \neq k$. Hence alternative (2) is impossible.

Finally since S_k^+ and S_k^- are disjoint, thus C_k^υ does not contain a pair of points $(\lambda, x), (\lambda, -x), x \neq 0$, which means alternative (3) of Lemma 1.1 is impossible.

Therefore alternative (1) of Lemma 1.1 holds. This implies that for each integer $k > 0$, and each $\upsilon = +$ or $-$, there exists a continua C_k^υ of solutions of (3.6) in $\Phi_k^\upsilon \cup \{(\frac{k^2\pi^2}{f_0}, 0)\}$, which meets $\{(\frac{k^2\pi^2}{f_0}, 0)\}$ and ∞ in Σ. The proof is complete. $\square$

Lemma 3.6. *For each integer $k > 0$ and each $\upsilon = +$, or $-$, there exists a continua $\hat{C}_k^\upsilon$ of Σ in $\Phi_k^\upsilon \cup \{(\frac{k^2\pi^2}{f_\infty}, \infty)\}$ coming from $\{(\frac{k^2\pi^2}{f_\infty}, \infty)\}$, which meets $\{(\frac{k^2\pi^2}{f_0}, 0)\}$ or has an unbounded projection on $\mathbb{R}$.*

Proof. Let $T(\lambda, x) = \lambda A x - \lambda L_\infty x$, where L_∞ is defined as in (3.4). Then (3.6) can be rewritten as

$$x = \lambda L_\infty x + T(\lambda, x). \tag{3.10}$$

By Lemma 3.4, we know that $T(\lambda, x)$ is $o(\|x\|_1)$ for $x \in E$ near ∞ uniformly on bounded λ intervals. Notice that L_∞ is a compact linear map on E. By Lemma 2.2 and Lemma 2.3, we know that for each $k > 0$, $\frac{k^2\pi^2}{f_\infty}$ is a simple characteristic value of operator L_∞. So with Lemma 3.4, (3.10) can be considered as a bifurcation problem from infinity. A similar reasoning as in the proof of [22, Theorem 2.4] shows that $\|x\|_1^2 T(\lambda, \frac{x}{\|x\|_1^2})$ is compact. By Lemma 1.2 and Lemma 1.3, Σ contains a component D_k which can be decomposed into two subcontinua D_k^+, D_k^- which meets $(\frac{k^2\pi^2}{f_\infty}, \infty)$.

Next we show that for a smaller neighborhood $\mathcal{O} \subset \mathcal{M}$ of $(\frac{k^2\pi^2}{f_\infty}, \infty)$, $(\lambda, x) \in D_k \cap \mathcal{O}$ and $(\lambda, x) \neq (\frac{k^2\pi^2}{f_\infty}, \infty)$ implies that $x \in S_k$. In fact, by Lemma 1.3 we already know that there exists a neighborhood $\mathcal{O} \subset \mathcal{M}$ of $(\frac{k^2\pi^2}{f_\infty}, \infty)$ satisfying $(\lambda, x) \in D_k \cap \mathcal{O}$ and $(\lambda, x) \neq (\frac{k^2\pi^2}{f_\infty}, \infty)$ implies $(\lambda, x) = (\lambda, \alpha u_k + w)$ where $\alpha >$

$0(\alpha<0)$ and $|\lambda-\frac{k^2\pi^2}{f_\infty}|=o(1), \|w\|_1=o(|\alpha|)$ at $|\alpha|=\infty$. Since S_k is open and $\frac{w}{\alpha}$ is small compared to $u_k\in S_k$ near $\alpha=+\infty$, and therefore $x=\alpha u_k+w\in S_k$ for α near $+\infty$. Therefore, $D_k\cap\mathcal{O}\subset(\mathbb{R}\times S_k)\cup(\frac{k^2\pi^2}{f_\infty},\infty)$.

From the same reasoning it can be seen that $D_k^\upsilon\cap\mathcal{O}\subset(\mathbb{R}\times S_k^\upsilon)\cup(\frac{k^2\pi^2}{f_\infty},\infty)$, where $\upsilon=+$, or $-$.

Let $\hat{C}_k^+$ denote the maximal subcontinuum of D_k^+ lying in $\mathbb{R}\times S_k^+$. First suppose $\hat{C}_k^+\setminus\mathcal{O}$ is bounded. Then there exists $(\lambda,x)\in\partial\hat{C}_k^+$ with $x\in\partial S_k^+$. Hence x has a double zero. By the assumption in section 2 we know $x\equiv0$. Thus there exists a sequence $(\gamma_m,z_m)\in\hat{C}_k^+$ satisfying (3.10) with $z_m\to x\equiv0$ as $m\to+\infty$. By Lemma 1.2, like in the proof of Lemma 3.5, one can see that $\hat{C}_k^+$ meets $(\frac{k^2\pi^2}{f_0},0)$.

Finally suppose $\hat{C}_k^+\setminus\mathcal{O}$ is unbounded. In this case we show $\hat{C}_k^+\setminus\mathcal{O}$ has an unbounded projection on $\mathbb{R}$. Suppose, on the contrary, that there exists a sequence $(\mu_m,x_m)\in\hat{C}_k^+\setminus\mathcal{O}$ with $\mu_m\to\mu$ and $\|x_m\|_1\to+\infty$ as $m\to+\infty$. Let $y_m:=\frac{x_m}{\|x_m\|_1}, m\geq1$. From the fact that

$$x_m=\mu_mL_\infty x_m+T(\mu_m,x_m),$$

it follows that

$$y_m=\mu_mL_\infty y_m+\frac{T(\mu_m,x_m)}{\|x_m\|_1}. \tag{3.11}$$

Notice that L_∞ is compact. We may assume that there exists $y\in E$ with $\|y\|_1=1$ such that $\|y_m-y\|_1\to0$ as $m\to+\infty$.

Letting $m\to+\infty$ in (3.11) and noticing $\frac{T(\mu_m,x_m)}{\|x_m\|_1}\to0$ as $m\to+\infty$ one obtains

$$y=\mu L_\infty y. \tag{3.12}$$

Since $\mu\neq0$ is an eigenvalue of operator L_∞ and $y\neq0$; that is, $\mu=\frac{i^2\pi^2}{f_\infty}$ for some $i\neq k$. Then by Lemma 2.2 y belongs to S_i^+ or S_i^-. Notice the fact that $\|y_m-y\|_1\to0$ as $m\to+\infty$. Thus $x_m\in S_i^+$ or S_i^- for m sufficiently large since S_i^+ or S_i^- are open. This is a contradiction with $x_m\in S_k^+(m\geq1)$. Thus $\hat{C}_k^+\setminus\mathcal{O}$ has an unbounded projection on $\mathbb{R}$.

Similarly, the same argument works if $+$ is replaced by $-$ in the above cases. The proof is complete. □

Proof of Theorem 3.1. Case 1. $\frac{k^2\pi^2}{f_0}<\lambda<\frac{k^2\pi^2}{f_\infty}$. Consider (3.7) as a bifurcation problem from the trivial solution. To obtain the desired results we need only to show that

$$C_k^\vartheta\cap(\{\lambda\}\times E)\neq\emptyset,\quad \vartheta=+,-.$$

By Lemma 3.5 we know that C_k^ϑ joins $(\frac{k^2\pi^2}{f_0},0)$ to infinity Φ_k^ϑ. Therefore, there exists a sequence $(\mu_n,x_n)\in C_k^\vartheta$ such that

$$\lim_{n\to\infty}(\mu_n+\|x_n\|_1)=\infty.$$

We note that $\mu_n>0$ for all $n\in\mathbb{N}$ since $(\lambda,x)=(0,0)$ is the unique solution of (3.6) with $\lambda=0$ in E and $C_k^\vartheta\cap(\{0\}\times E)=\emptyset$. If

$$\lim_{n\to\infty}\mu_n=\infty,$$

then

$$C_k^\vartheta\cap(\{\lambda\}\times E)\neq\emptyset.$$

Assume that there exists $M > 0$, such that for all $n \in \mathbb{N}$,

$$\mu_n \in (0, M].$$

In this case it follows that $\|x_n\|_1 \to \infty$. We divide the equation

$$x_n = \mu_n L_\infty x_n + T(\mu_n, x_n)$$

by $\|x_n\|_1$ and set $y_n = \frac{x_n}{\|x_n\|_1}$. Since y_n is bounded in E, choosing a subsequence and relabelling, if necessary, we see that $y_n \to y$ for some $y \in E$ with $\|y\|_1 = 1$, and

$$y = \mu L_\infty y$$

where $\mu = \lim_{n\to\infty} \mu_n$. From the proof of Lemma 3.6 one can see

$$\mu = \frac{k^2\pi^2}{f_\infty}.$$

Thus C_k^ϑ joins $(\frac{k^2\pi^2}{f_0}, 0)$ to $(\frac{k^2\pi^2}{f_\infty}, \infty)$ which implies

$$C_k^\vartheta \cap (\{\lambda\} \times E) \neq \emptyset.$$

Case 2. $\frac{k^2\pi^2}{f_\infty} < \lambda < \frac{k^2\pi^2}{f_0}$. Consider (3.10) as a bifurcation problem from infinity. As above we need only to prove that

$$\hat{C}_k^\vartheta \cap (\{\lambda\} \times E) \neq \emptyset, \quad \vartheta = +, -.$$

From Lemma 3.6, we know that $\hat{C}_k^\vartheta$ comes from $(\frac{k^2\pi^2}{f_\infty}, \infty)$ meets $(\frac{k^2\pi^2}{f_0}, 0)$ or has an unbounded projection on $\mathbb{R}$.

If it meets $(\frac{k^2\pi^2}{f_0}, 0)$, then the connectedness of $\hat{C}_k^\vartheta$ and $\frac{k^2\pi^2}{f_0} > \lambda$ guarantee that

$$\hat{C}_k^\vartheta \cap (\{\lambda\} \times E) \neq \emptyset, \quad \vartheta = +, -.$$

If $\hat{C}_k^\vartheta$ has an unbounded projection on $\mathbb{R}$, notice that $(\lambda, x) = (0, 0)$ is the unique solution of (3.6), so

$$\hat{C}_k^\vartheta \cap (\{\lambda\} \times E) \neq \emptyset, \quad \vartheta = +, -.$$

The proof is complete. □

Proof of Theorem 3.2. Repeating the arguments used in the proof in Theorem 3.1, we see that for each $\vartheta \in \{+, -\}$ and each $i \in \{1, 2, \dots, j\}$

$$(C_{k+i}^\vartheta \cup \hat{C}_{k+i}^\vartheta) \cap (\{\lambda\} \times E) \neq \emptyset.$$

□

Remark 3.7. From [5], we know if $\beta = 0$, we have also the same results as Theorem 3.1 and Theorem 3.2.

4. Applications

In this section, we give an example to illustrate the applications of Theorem 3.2.

Example 4.1. Consider the second-order impulsive differential equation

$$\begin{gathered} x''(t) + f(x(t)) = 0, \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta x|_{x=\frac{1}{2}} = x(\frac{1}{2}), \\ \Delta x'|_{x=\frac{1}{2}} = -x'(\frac{1}{2} - 0), \\ x(0) = x(1) = 0, \end{gathered} \tag{4.1}$$

where $f(x) = 100 \sin x + 5x$.

It is not difficult to see that f here satisfies the assumption in section 2 and (H1) with $f_0 = 105, f_\infty = 5$ and the eigenvalues of the boundary-value problem

$$\begin{gathered} x''(t) + \lambda x(t) = 0, \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta x|_{x=\frac{1}{2}} = \lambda x(\frac{1}{2}), \\ \Delta x'|_{x=\frac{1}{2}} = -\lambda x'(\frac{1}{2} - 0), \\ x(0) = x(1) = 0, \end{gathered}$$

can be written as $\lambda_k = k^2\pi^2$, $k = 1, 2, \dots$. By calculation, we know that there exist $k = 1$ and $j = 2$ such that

$$\frac{(k+j)^2\pi^2}{f_0} < 1 < \frac{k^2\pi^2}{f_\infty}.$$

Therefore, Theorem 3.2 guarantees that (4.1) has at least six nontrivial solutions.

Acknowledgements. We thank the anonymous referees for their suggestions.

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