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# THE FORM OF THE SPECTRAL FUNCTION ASSOCIATED WITH STURM-LIOUVILLE PROBLEMS FOR SMALL VALUES OF THE SPECTRAL PARAMETER

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Abstract. We study the linear second-order differential equation

$$-y'' + q(x)y = \lambda y$$

where, amongst other conditions, $q \in L^1[0, \infty)$. We obtain a convergent series expansion for the spectral function which is valid for small values of $\lambda$. We also derive an asymptotic representation.

## 1. Introduction

We consider the linear, second-order differential equation

$$-y'' + q(x)y = \lambda y \text{ for } x \in [0, \infty), \tag{1.1}$$

$$y(0) = 0 \tag{1.2}$$

in the case where $q$ is a real-valued member of $L^1[0, \infty)$. It is well known, see for example [5] that under these circumstances the spectral function $\rho_0(\lambda)$ associated with (1.1), (1.2) is such that $\rho_0'(\lambda)$ exists and is continuous on $(0, \infty)$. In recent years many papers have investigated the form of $\rho_0(\lambda)$ for large values of $\lambda$. In particular we mention the asymptotic results in [1, 2] and the explicit representations derived in [3, 4, 6] which are valid for all $\lambda \geq \Lambda_0$ where $\Lambda_0$ is computable. In [4] the condition $q \in L^1[0, \infty)$ was relaxed to the requirement that $q$ be of Wigner-von Neumann type or be slowly decreasing. The situation for small values of $\lambda$ is somewhat more complicated as the form of the derived series will show. In particular the conditions on $q$ and the form of the series representation are in terms of the solution of a particular Riccati equation. A necessary condition for the existence of such a solution on $(0, \infty)$ is the finiteness of $\int_0^\infty (1+t)^2 q(t)\, dt$. It follows that the results require $q$ to be small at infinity. A consequence of our main result is a representation of $\lim_{\lambda \to 0^+} \rho_0'(\lambda)$. We also, in §4, show that the convergent series may be truncated and an asymptotic representation obtained.

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## 2. Results

We assume the existence of a solution, $v_0(x)$, of the Riccati equation

$$v_0' = q(x) - v_0^2 \tag{2.1}$$

which is defined on $[0, \infty)$ and satisfies

$$\lim_{x\to\infty} x v_0(x) = 0. \tag{2.2}$$

We further assume that

$$(1+t)|v_0(t)| \in L^1[0,\infty). \tag{2.3}$$

Under these conditions it will be shown that there exists a sequence of functions $\{v_n(x,\lambda)\}$ defined recursively as follows:

$$v_1(x,\lambda) := 2i\lambda^{1/2} \int_x^\infty e^{2i\lambda^{1/2}(t-x) - 2\int_x^t v_0(s)\,ds} v_0(t)\,dt \tag{2.4}$$

and

$$v_n(x,\lambda) := \int_x^\infty e^{2i\lambda^{1/2}(t-x) - 2\int_x^t v_0(s)\,ds} \Big( v_{n-1}^2 + 2v_{n-1} \sum_{m=1}^{n-2} v_m \Big)\,dt. \tag{2.5}$$

**Theorem 2.1.** *Under conditions* (2.1)–(2.3) *there exists* $\Lambda > 0$ *so that for* $\lambda \in (0, \Lambda)$

$$\rho_0'(\lambda) = \frac{1}{\pi}\big\{\lambda^{1/2} + \operatorname{Im} \sum_{n=1}^\infty v_n(0,\lambda)\big\}. \tag{2.6}$$

*In particular*

$$\lim_{\lambda\to 0^+} \rho_0'(\lambda) = 0. \tag{2.7}$$

**Example 2.2.** If $q(x) := -e^{-x}(1-e^{-x})$ then it is easy to see that $v_0(x) = e^{-x}$ satisfies (2.1), (2.2), and (2.3) and $\lim_{\lambda\to 0^+} \rho_0'(\lambda) = 0$.

**Remark 2.3.** If $v_0$ satisfies (2.1) then

$$(1+t)^2 v_0'(t) = (1+t)^2 q(t) - (1+t)^2 v_0(t)^2$$

and an integration by parts and (2.2) gives

$$-v_0(0) - 2\int_0^\infty (1+t) v_0(t)\,dt = \int_0^\infty (1+t)^2 q(t)\,dt - \int_0^\infty (1+t)^2 v_0(t)^2.$$

The boundedness of $\int_0^\infty (1+t)^2 q(t)\,dt$ now follows from 2.1–(2.3).

**Remark 2.4.** It is shown below that the requirements (2.1)–(2.3) ensure that $v_0(x)$ is real-valued.

## 3. Proof of Theorem 2.1

Following the analysis employed in [5], we seek a solution of the Riccati equation

$$v' = -\lambda + q - v^2 \tag{3.1}$$

which satisfies

$$\lim_{x\to\infty} v(x,\lambda) = i\lambda^{1/2}. \tag{3.2}$$

Then, from [5, (4.4)],

$$\rho_0'(\lambda) = \frac{1}{\pi} \operatorname{Im}\{v(\lambda)\}. \tag{3.3}$$

We try for a solution of (3.1) a series of the form

$$v(x,\lambda) = i\lambda^{1/2} + v_0(x) + \sum_{n=1}^{\infty} v_n(x,\lambda). \tag{3.4}$$

If term by term differentiation of the terms of the series of (3.4) is justified, substitution of (3.4) into (3.1) leads to a choice of the $\{v_n\}$ such that

$$v_1' + (2i\lambda^{1/2} + v_0)v_1 = -2i\lambda^{1/2} v_0 \tag{3.5}$$

and for $n = 2, 3, \dots$,

$$v_n' + 2(i\lambda^{1/2} + v_0)v_n = -v_{n-1}^2 - 2v_{n-1}\sum_{m=1}^{n-2} v_m. \tag{3.6}$$

It is straightforward to check that the functions defined in (2.4) and (2.5) satisfy (3.5) and (3.6). We now bound the $\{v_n\}$ and show that the series $\sum v_n'$ is absolutely uniformly convergent on compact subsets of $[0,\infty)$ which is sufficient to justify the term by term differentiation.

**Lemma 3.1.** *Let*

$$K := \sup_{0\le x\le t<\infty} \left| e^{-2\int_x^t v_0(s)\,ds} \right| \tag{3.7}$$

*and suppose there exists $a(x)$ which is a decreasing member of $L^1[0,\infty)$ such that*

$$|v_1(x,\lambda)| \le \lambda^{1/2} a(x) \tag{3.8}$$

*for $x \in [0,\infty)$ and $\lambda \in [0,\Lambda]$ where $\Lambda$ is so small that $10K\lambda^{1/2}\int_0^\infty a(t)\,dt \le 1$ for $\lambda \in [0,\Lambda]$. Then $|v_n(x,\lambda)| \le \frac{\lambda^{1/2}a(x)}{2^{n-1}}$ for $x \in [0,\lambda)$ and $\lambda \in [0,\Lambda]$.*

*Proof.* We use induction on $n$. When $n = 1$, the result follows from the hypothesis (3.8). Suppose now the result is true for all subscripts up to the $(n-1)$st. Then from (2.4), (3.7), and the induction hypothesis:

$$\begin{aligned}
|v_n(x,\lambda)| &\le K\int_x^\infty |v_{n-1}|^2 + 2|v_{n-1}|\sum_{m=1}^{n-2}|v_m|\,dt \\
&\le K\int_x^\infty \frac{\lambda a(t)^2}{2^{2n-4}} + \frac{2\lambda a(t)^2}{2^{n-2}}\sum_{m=1}^{n-2}\frac{1}{2^{m-1}}\,dt \\
&\le \frac{\lambda^{1/2}a(x)}{2^{n-1}}\lambda^{1/2}\big\{\frac{1}{2^{n-3}} + 8\big\}\int_0^\infty a(t)\,dt
\end{aligned}$$

since $a(\cdot)$ is a decreasing function. The result now follows from the choice of $\Lambda$.

It may now be seen from the Lemma and (3.6) that the series $\sum v_n'$ is absolutely uniformly convergent which justifies the term by term differentiation. To complete the proof of the theorem we observe that, since $v_0(\cdot) \in L^1[0,\infty)$, there exists a $K$ which satisfies (3.7) and also, from (2.4), that

$$|v_1(x,\lambda)| \le 2\lambda^{1/2}K\int_x^\infty |v_0(t)|\,dt.$$

We now choose $a(x) := 2K\int_x^\infty |v_0(t)|\,dt$ and note that

$$\int_0^\infty a(x)\,dx = \int_0^\infty 2K\int_x^\infty |v_0(t)|\,dt\,dx = 2K\int_0^\infty t|v_0(t)|\,dt.$$

The first part of the theorem now follows.

It remains to show that, under the assumptions (2.1)–(2.3), $v_0$ is real-valued. Suppose not; if $v_0(t) = u(t) + iw(t)$ then upon substitution into (2.1) and the separation of real and imaginary parts we see that

$$w' = -2uw$$

whence

$$w(t) = Ce^{-2\int_0^t u(s)\,ds}$$

The requirement $\lim_{t\to\infty} v_0(t) = 0$ then requires either $C = 0$ or $\lim_{t\to\infty} \int_0^t u(s)\,ds = \infty$. But the latter case contradicts (2.3) which requires that $(1+t)v_0(t)$ and hence $(1+t)u(t) \in L^1[0,\infty)$, so the only possibility is that $v_0$ is real-valued. □

## 4. An asymptotic expansion

The bounds derived in Lemma 3.1 lead to estimates for the $\{v_n\}$ which show that $\sum_{n=1}^{\infty} v_n(x,\lambda)$ is uniformly, absolutely convergent for $x \in [0,\infty)$ and $0 \le \lambda < \Lambda$ for some $\Lambda$ which is, in principle at least, computable. In terms of $\lambda$ however the bounds are all of order $\lambda^{1/2}$. We now show that the terms of the series are decreasing asymptotically with increasing powers of $\lambda$.

**Lemma 4.1.** *With $K$ as in* (3.7) *and with $v_1$ satisfying* (3.8) *there exist sequences of constants $\{C_n\}$ and $\{\Lambda_n\}$ so that for $x \in [0,\infty)$ and $0 \le \lambda \le \Lambda_n \le \Lambda_{n-1}$*

$$|v_n(x,\lambda)| \le C_n \lambda^{n/2} a(x). \tag{4.1}$$

*Proof.* We proceed by induction. From (2.5),

$$\begin{aligned}|v_2(x,\lambda)| &\le \int_x^\infty e^{-2\int_x^t v_0(s)\,ds} |v_1(t,\lambda)|^2\,dt \\ &\le \lambda K \int_x^\infty a(t)^2\,dt \le \lambda a(x) K \int_0^\infty a(t)\,dt\end{aligned}$$

Suppose the result is true up to $n \ge 2$, then from (2.5):

$$\begin{aligned}|v_{n+1}(x,\lambda)| &\le K \int_x^\infty |v_n|^2 + 2|v_n| \sum_{m=1}^{n-1} |v_m|\,dt \\ &\le K \int_x^\infty C_n^2 \lambda^n a(t)^2 + 2C_n \lambda^{n/2} a(t) \sum_{m=1}^{n-1} C_m \lambda^{m/2} a(t)\,dt \\ &\le K \lambda^{\frac{n+1}{2}} a(x) \big\{ C_n^{\frac{n-1}{2}} \lambda - 2C_n \sum_{m=1}^{n-1} C_m \lambda^{\frac{m-1}{2}} \big\} \int_0^\infty a(t)\,dt\end{aligned}$$

since the $\Lambda_n$ form a decreasing sequence. The result now follows. □

In consequence of Lemma 4.1 we have that for every $N$,

$$\rho_0'(\lambda) = \frac{1}{\pi} \big\{ \lambda^{1/2} + \operatorname{Im} \sum_{n=1}^{N} v_n(0,\lambda) \big\} + O\left(\lambda^{\frac{N+1}{2}}\right)$$

as $\lambda \to 0^+$ and, in particular,

$$\rho_0'(\lambda) = \frac{\lambda^{1/2}}{\pi} \big\{ 1 + 2\int_0^\infty \cos(2\lambda^{1/2} t) e^{-2\int_0^t v_0(s)\,ds} v_0(t)\,dt \big\} + O(\lambda)$$

as $\lambda \to 0^+$.

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