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OSCILLATION OF SOLUTIONS TO NONLINEAR FORCED FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the oscillation of solutions to a nonlinear forced fractional differential equation. The fractional derivative is defined in the sense of the modified Riemann-Liouville derivative. Based on a transformation of variables and properties of the modified Riemann-liouville derivative, the fractional differential equation is transformed into a second-order ordinary differential equation. Then by a generalized Riccati transformation, inequalities, and an integration average technique, we establish oscillation criteria for the fractional differential equation.

1. Introduction

Recently, research on oscillation of various equations including differential equations, difference equations and dynamic equations on time scales has been a hot topic in the literature. Much effort has been done to establish oscillation criteria for these equations; see for example the references in this article. We notice that in these publications very little attention is paid to oscillation of fractional differential equations.

In this article, we are concerned with the oscillation of solutons to the nonlinear forced fractional differential equation

$$D_t^\alpha[r(t)\psi(x(t))D_t^\alpha x(t)] + q(t)f(x(t)) = e(t), t \geq t_0 > 0,\ 0 < \alpha < 1, \tag{1.1}$$

where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative [15] with respect to the variable t, the functions $r \in C^\alpha([t_0,\infty), R_+)$, which is the set of functions with continuous derivative of order α, the functions q, e belong to $C([t_0,\infty), R)$, and the functions f, ψ belong to $C(R, R)$, $0 < \psi(x) \leq m$ for some positive constant m, and $xf(x) > 0$ for all $x \neq 0$.

The definition and some important properties for the Jumarie's modified Riemann-Liouville derivative of order α are listed next (see also in [10, 29, 30]):

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, & 1 \leq n \leq \alpha < n+1. \end{cases}$$

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$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{1.2}$$

$$D_t^\alpha (f(t)g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \tag{1.3}$$

$$D_t^\alpha f[g(t)] = f_g'[g(t)] D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha. \tag{1.4}$$

A solution of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

For the sake of convenience, in this article, we denote:

$$\xi_0 = \frac{t_0^\alpha}{\Gamma(1+\alpha)}, \quad \xi = \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad \widetilde{\rho}(\xi) = \rho(t), \quad \widetilde{r}(\xi) = r(t),$$
$$\widetilde{q}(\xi) = q(t), \quad \xi_{a_i} = \frac{a_i^\alpha}{\Gamma(1+\alpha)}, \quad \xi_{b_i} = \frac{b_i^\alpha}{\Gamma(1+\alpha)}, \quad \mathbb{R}_+ = (0,\infty).$$

Let $h_1,\ h_2,\ H \in C([\xi_0,\infty), R)$ satisfy

$$H(\xi,\xi) = 0, \quad H(\xi,s) > 0, \quad \xi > s \geq \xi_0.$$

Let H have continuous partial derivatives $\frac{\partial H(\xi,s)}{\partial \xi}$ and $\frac{\partial H(\xi,s)}{\partial s}$ on $[\xi_0,\infty)$ such that

$$\frac{\partial H(\xi,s)}{\partial \xi} = -h_1(\xi,s)\sqrt{H(\xi,s)}, \quad \frac{\partial H(\xi,s)}{\partial s} = -h_2(\xi,s)\sqrt{H(\xi,s)}.$$

For $\mathbf{s},\ \xi \in [\xi_{\mathbf{0}},\infty)$, denote

$$Q_1(s,\xi) = h_1(s,\xi) - \frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\sqrt{H(s,\xi)}, \quad Q_2(\xi,s) = h_2(\xi,s) - \frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\sqrt{H(\xi,s)},$$

We organize this article as follows. In Section 2, we establish some new oscillation criteria for (1.1) under the condition that $f(x)$ is increasing. In Section 3, we establish oscillation criteria for (1.1) without the condition $f(x)$ being increasing. In the proof for the main results in Sections 2 and 3, we use a generalized Riccati transformation method. This Riccati transformation and the function H defined above are widely used for proving oscillation of ordinary differential equations of integer order; see for example [13, 18, 26, 27, 28]. Yet this approach has scarcely been used to prove oscillation of fractional differential equations. In Section 4, we present some examples that apply the results established. Finally, some conclusions are presented at the end of this article.

2. Oscillation criteria when $f(x)$ is increasing

Theorem 2.1. *Assume $f'(x)$ exists and $f'(x) \geq \mu$ for some $\mu > 0$ and for all $x \neq 0$. Also assume that for any $T \geq t_0$, there exist a_1, b_1, a_2, b_2 such that $T \leq a_1 < b_1 \leq a_2 < b_2$ satisfying*

$$e(t) \begin{cases} \leq 0, & t \in [a_1,b_1], \\ \geq 0, & t \in [a_2,b_2]. \end{cases} \tag{2.1}$$

If there exist $y_i \in (\xi_{a_i},\xi_{b_i})$ and $\rho \in C^\alpha([t_0,\infty),\mathbb{R}_+)$ such that

$$\begin{aligned} &\frac{1}{H(y_i,\xi_{a_i})}\int_{\xi_{a_i}}^{y_i} \widetilde{\rho}(s)[H(s,\xi_{a_i})\widetilde{q}(s) - \frac{\widetilde{r}(s)}{4k_1}Q_1^2(s,\xi_{a_i})]ds \\ &+ \frac{1}{H(\xi_{b_i},y_i)}\int_{y_i}^{\xi_{b_i}} \widetilde{\rho}(s)[H(\xi_{b_i},s)\widetilde{q}(s) - \frac{\widetilde{r}(s)}{4k_1}Q_2^2(\xi_{b_i},s)]ds > 0 \end{aligned} \tag{2.2}$$

for $i = 1, 2$, where $k_1 = \mu/m$, then every solution of (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ be a non-oscillatory solution of (1.1), say $x(t) \neq 0$ on $[T_0, \infty)$ for some sufficient large $T_0 \geq t_0$. Define the following Riccati transformation function:

$$\omega(t) = \rho(t)\frac{r(t)\psi(x(t))D_t^\alpha x(t)}{f(x(t))}, \quad t \geq T_0. \tag{2.3}$$

Then for $t \geq T_0$, from (1.2)-(1.4) we deduce that

$$\begin{aligned} D_t^\alpha \omega(t) &= -\rho(t)q(t) + \frac{D_t^\alpha \rho(t)}{\rho(t)}\omega(t) - \frac{f'(x(t))}{\rho(t)r(t)\psi(x(t))}\omega^2(t) + \frac{e(t)\rho(t)}{f(x(t))} \\ &\leq -\rho(t)q(t) + \frac{D_t^\alpha \rho(t)}{\rho(t)}\omega(t) - k_1\frac{\omega^2(t)}{\rho(t)r(t)} + \frac{e(t)\rho(t)}{f(x(t))}. \end{aligned} \tag{2.4}$$

By assumption, if $x(t) > 0$, then we can choose $a_1, b_1 \geq T_0$ with $a_1 < b_1$ such that $e(t) \leq 0$ on the interval $[a_1, b_1]$. If $x(t) < 0$, then we can choose $a_2, b_2 \geq T_0$ with $a_2 < b_2$ such that $e(t) \geq 0$ on the interval $[a_2, b_2]$. So $\frac{e(t)\rho(t)}{f(x(t))} \leq 0$, $t \in [a_i, b_i]$, $i = 1, 2$, and from (2.4) one can deduce that

$$D_t^\alpha \omega(t) \leq -\rho(t)q(t) + \frac{D_t^\alpha \rho(t)}{\rho(t)}\omega(t) - k_1\frac{\omega^2(t)}{\rho(t)r(t)}, \quad t \in [a_i, b_i],\ i = 1, 2. \tag{2.5}$$

Let $w(t) = \widetilde{w}(\xi)$. Then $D_t^\alpha w(t) = \widetilde{w}'(\xi)$ and $D_t^\alpha \rho(t) = \widetilde{\rho}'(\xi)$. So (2.5) is transformed into

$$\widetilde{\omega}'(\xi) \leq -\widetilde{\rho}(\xi)\widetilde{q}(\xi) + \frac{\widetilde{\rho}'(\xi)}{\widetilde{\rho}(\xi)}\widetilde{\omega}(\xi) - k_1\frac{\widetilde{\omega}^2(\xi)}{\widetilde{\rho}(\xi)\widetilde{r}(\xi)}, \quad \xi \in [\xi_{a_i}, \xi_{b_i}],\ i = 1, 2. \tag{2.6}$$

Let c_i be an arbitrary point in (ξ_{a_i}, ξ_{b_i}). Substituting ξ with s, multiplying both sides of (2.6) by $H(\xi, s)$ and integrating it over $[c_i, \xi)$ for $\xi \in [c_i, \xi_{b_i})$, $i = 1, 2$, we obtain

$$\begin{aligned} &\int_{c_i}^{\xi} H(\xi, s)\widetilde{\rho}(s)\widetilde{q}(s)ds \\ &\leq -\int_{c_i}^{\xi} H(\xi, s)\widetilde{\omega}'(s)ds + \int_{c_i}^{\xi} H(\xi, s)\big[\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{\omega}(s) - k_1\frac{\widetilde{\omega}^2(s)}{\widetilde{\rho}(s)\widetilde{r}(s)}\big]ds \\ &= H(\xi, c_i)\widetilde{\omega}(c_i) - \int_{c_i}^{\xi} \widetilde{\omega}(s)h_2(\xi, s)\sqrt{H(\xi, s)}ds \\ &\quad + \int_{c_i}^{\xi} H(\xi, s)\big[\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{\omega}(s) - k_1\frac{\widetilde{\omega}^2(s)}{\widetilde{\rho}(s)\widetilde{r}(s)}\big]ds \\ &= H(\xi, c_i)\widetilde{\omega}(c_i) - \int_{c_i}^{\xi} \Big[\Big(\frac{H(\xi, s)k_1}{\widetilde{\rho}(s)\widetilde{r}(s)}\Big)^{1/2}\widetilde{\omega}(s) - \frac{1}{2}\Big(\frac{\widetilde{\rho}(s)\widetilde{r}(s)}{k_1}\Big)^{1/2} Q_2(\xi, s)\Big]^2 ds \\ &\quad + \int_{c_i}^{\xi} \frac{\widetilde{\rho}(s)\widetilde{r}(s)}{4k_1}Q_2^2(\xi, s)ds \\ &\leq H(\xi, c_i)\widetilde{\omega}(c_i) + \int_{c_i}^{\xi} \frac{\widetilde{\rho}(s)\widetilde{r}(s)}{4k_1}Q_2^2(\xi, s)ds. \end{aligned} \tag{2.7}$$

Letting $\xi \to \xi_{b_i}^-$ and dividing it by $H(\xi_{b_i}, c_i)$, we obtain

$$\begin{aligned} &\frac{1}{H(\xi_{b_i}, c_i)} \int_{c_i}^{\xi_{b_i}} H(\xi_{b_i}, s)\widetilde{\rho}(s)\widetilde{q}(s)ds \\ &\leq \widetilde{\omega}(c_i) + \frac{1}{H(\xi_{b_i}, c_i)} \int_{c_i}^{\xi_{b_i}} \frac{\widetilde{\rho}(s)\widetilde{r}(s)}{4k_1} Q_2^2(\xi_{b_i}, s)ds. \end{aligned} \tag{2.8}$$

On the other hand, substituting ξ by s, multiplying both sides of (2.6) by $H(s, \xi)$ and integrating it over (ξ, c_i) for $\xi \in [\xi_{a_i}, c_i)$, we obtain

$$\begin{aligned} &\int_{\xi}^{c_i} H(s,\xi)\widetilde{\rho}(s)\widetilde{q}(s)ds \\ &\leq -\int_{\xi}^{c_i} H(s,\xi)\widetilde{\omega}'(s)ds + \int_{\xi}^{c_i} H(s,\xi)\Big[\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{\omega}(s) - k_1\frac{\widetilde{\omega}^2(s)}{\widetilde{\rho}(s)\widetilde{r}(s)}\Big]ds \\ &= -H(\xi, c_i)\omega(c_i) - \int_{\xi}^{c_i} \widetilde{\omega}(s)h_1(s,\xi)\sqrt{H(s,\xi)}ds \\ &\quad + \int_{\xi}^{c_i} H(s,\xi)\Big[\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{\omega}(s) - k_1\frac{\widetilde{\omega}^2(s)}{\widetilde{\rho}(s)\widetilde{r}(s)}\Big]ds \\ &\leq -H(c_i,\xi)\widetilde{\omega}(c_i) + \int_{\xi}^{c_i} \frac{\widetilde{\rho}(s)\widetilde{r}(s)}{4k_1} Q_1^2(s,\xi)ds. \end{aligned} \tag{2.9}$$

Letting $\xi \to \xi_{a_i}^+$ and dividing by $H(c_i, \xi_{a_i})$, we obtain

$$\frac{1}{H(c_i,\xi_{a_i})} \int_{\xi_{a_i}}^{c_i} H(s,\xi_{a_i})\widetilde{\rho}(s)\widetilde{q}(s)ds \leq -\widetilde{\omega}(c_i) + \frac{1}{H(c_i,\xi_{a_i})} \int_{\xi_{a_i}}^{c_i} \frac{\widetilde{\rho}(s)\widetilde{r}(s)}{4k_1} Q_1^2(s,\xi_{a_i})ds.$$

A combination of (2.8) and the above inequality yields

$$\begin{aligned} &\frac{1}{H(c_i,\xi_{a_i})} \int_{\xi_{a_i}}^{c_i} H(s,\xi_{a_i})\widetilde{\rho}(s)\widetilde{q}(s)ds + \frac{1}{H(\xi_{b_i},c_i)} \int_{c_i}^{\xi_{b_i}} H(\xi_{b_i},s)\widetilde{\rho}(s)\widetilde{q}(s)ds \\ &\leq \frac{1}{4H(c_i,\xi_{a_i})} \int_{\xi_{a_i}}^{c_i} \frac{\rho(s)\widetilde{r}(s)}{k_1} Q_1^2(s,\xi_{a_i})ds + \frac{1}{4H(\xi_{b_i},c_i)} \int_{c_i}^{\xi_{b_i}} \frac{\rho(s)\widetilde{r}(s)}{k_1} Q_2^2(\xi_{b_i},s)ds. \end{aligned}$$

which contradicts to (2.2) since c_i is arbitrary in (ξ_{a_i}, ξ_{b_i}). The proof is complete. $\square$

Theorem 2.2. *Under the conditions of Theorem 2.1, suppose* (2.2) *does not hold, and* $\widetilde{q}(\xi) > 0$ *for any* $\xi \geq \xi_0$. *If for some* $u \in C[\xi_{a_i}, \xi_{b_i}]$ *satisfying* $u' \in L^2[\xi_{a_i}, \xi_{b_i}]$, $u(\xi_{a_i}) = u(\xi_{b_i}) = 0$, $i = 1, 2$, *and* u *is not identically zero, there exists* $\rho \in C^1([\xi_0, \infty), R_+)$ *such that*

$$\int_{\xi_{a_i}}^{\xi_{b_i}} \Big\{u^2(s)\widetilde{\rho}(s)\widetilde{q}(s) - \Big[\frac{\widetilde{\rho}(s)\widetilde{r}(s)}{k_1}\Big(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2\Big\}ds > 0 \tag{2.10}$$

for $i = 1, 2$, *where* $\widetilde{\rho}, \widetilde{r}, \widetilde{q}, \xi_{a_i}, \xi_{b_i}, k_1$ *are defined as in Theorem 2.1, then* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ be a non-oscillatory solution of (1.1), say $x(t) \neq 0$ on $[T_0, \infty)$ for some sufficient large $T_0 \geq t_0$. Let $\widetilde{\omega}$ be defined as in Theorem 2.1. Then we obtain (2.6). Substituting ξ by s, multiplying both sides of (2.6) by

$u^2(s)$, integrating it with respect to s from ξ_{a_i} to ξ_{b_i} and using $u(\xi_{a_i}) = u(\xi_{b_i}) = 0$, we obtain

$$
\begin{aligned}
&\int_{\xi_{a_i}}^{\xi_{b_i}} u^2(s)\widetilde{\rho}(s)\widetilde{q}(s)ds \\
&\leq -\int_{\xi_{a_i}}^{\xi_{b_i}} u^2(s)\widetilde{\omega}'(s)ds + \int_{\xi_{a_i}}^{\xi_{b_i}} u^2(s)\Big[\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{\omega}(s) - k_1\frac{\widetilde{\omega}^2(s)}{\widetilde{\rho}(s)\widetilde{r}(s)}\Big]ds \\
&= 2\int_{\xi_{a_i}}^{\xi_{b_i}} \widetilde{\omega}(s)u(s)u'(s)ds + \int_{\xi_{a_i}}^{\xi_{b_i}} u^2(s)\Big[\frac{\widetilde{\rho}'(s)}{\rho(s)}\widetilde{\omega}(s) - k_1\frac{\widetilde{\omega}^2(s)}{\widetilde{\rho}(s)\widetilde{r}(s)}\Big]ds \\
&= -\int_{\xi_{a_i}}^{\xi_{b_i}} \Big[\sqrt{\frac{k_1}{\widetilde{\rho}(s)\widetilde{r}(s)}}u(s)\widetilde{\omega}(s) - \sqrt{\frac{\widetilde{\rho}(s)\widetilde{r}(s)}{k_1}}\Big(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2 ds \\
&\quad + \int_{\xi_{a_i}}^{\xi_{b_i}} \Big[\frac{\widetilde{\rho}(s)\widetilde{r}(s)}{k_1}\Big(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2 ds.
\end{aligned}
$$

$$
\int_{\xi_{a_i}}^{\xi_{b_i}} \Big\{u^2(s)\widetilde{\rho}(s)\widetilde{q}(s) - \Big[\frac{\widetilde{\rho}(s)\widetilde{r}(s)}{k_1}\Big(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2\Big\}ds \leq 0
$$

which contradicts to (2.10). So every solution of (1.1) is oscillatory. The proof is complete. □

Corollary 2.3. *Under the conditions of Theorem 2.1, suppose that* (2.2) *does not hold, and* $\widetilde{q}(\xi) > 0$ *for any* $\xi \geq \xi_0$. *If for each* $r \geq \xi_0$,

$$
\begin{aligned}
&\limsup_{\xi\to\infty} \int_r^{\xi} \{\widetilde{\rho}(s)(\xi - s)^2(s - r)^2\widetilde{q}(s) \\
&- \Big[\frac{\widetilde{\rho}(s)\widetilde{r}(s)}{k_1}\Big((\xi + r - 2s) + \frac{1}{2}(\xi - s)(s - r)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2\}ds > 0,
\end{aligned}
\tag{2.11}
$$

then (1.1) *is oscillatory.*

The proof of the above corollary is done by setting $u(s) = (\xi_{b_i} - s)(s - \xi_{a_i})$ in the proof of Theorem 2.2.

Remark 2.4. The results established above provide sufficient conditions for oscillation of (1.1) with $f(x)$ increasing. These results are similar to those for ordinary differential equations of integer order. The Riccati transformation methods are similar, However, they are essentially different. The most significant difference lies in the fact that the functions $\widetilde{\rho}, \widetilde{r}, \widetilde{q}$ are compound functions, the variable ξ has a special form $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$. The main difficulty to overcome in using the Riccati transformation for (1.1) can be summarized in two aspects. One is the computation of the α-order derivative for the Riccati transformation function $\omega(t)$, in which two important properties (1.3) and (1.4) for the modified Riemann-Liouville derivative are used The other is how to transform (2.5) into (2.6), in which the property (1.4) for the modified Riemann-Liouville derivative and a suitable variable transformation from the original variable t to a new variable ξ denoted by $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$ are used. In summary, the oscillation criteria presented above are established under the combination of the Riccati transformation method and the properties of the modified Riemann-Liouville derivative.

3. Oscillation criteria with $f(x)$ not necessarily increasing

Theorem 3.1. *Suppose $f(x)/x \ge k_2 > 0$ for all $x \neq 0$, and for any $T \ge \xi_0$, there exist $T \le a_1 < b_1 \le a_2 < b_2$ such that (2.1) holds. If there exist $y_i \in (\xi_{a_i}, \xi_{b_i})$ and $\rho \in C^1([\xi_0,\infty), R_+)$ such that*

$$
\begin{aligned}
&\frac{1}{H(y_i,\xi_{a_i})}\int_{\xi_{a_i}}^{y_i} H(s,\xi_{a_i})k_2\widetilde{\rho}(s)\widetilde{q}(s)ds + \frac{1}{H(\xi_{b_i},y_i)}\int_{y_i}^{\xi_{b_i}} H(\xi_{b_i},s)k_2\widetilde{\rho}(s)\widetilde{q}(s)ds \\
&> \frac{1}{4H(y_i,\xi_{a_i})}\int_{\xi_{a_i}}^{y_i} m\rho(s)\widetilde{r}(s)Q_1^2(s,\xi_{a_i})ds \\
&\quad + \frac{1}{4H(\xi_{b_i},y_i)}\int_{y_i}^{\xi_{b_i}} m\rho(s)\widetilde{r}(s)Q_2^2(\xi_{b_i},s)ds.
\end{aligned} \tag{3.1}
$$

for $i = 1, 2$, where $Q_1, Q_2, \widetilde{q}, \widetilde{r}, \widetilde{\rho}, \xi_{a_i}, \xi_{b_i}$ are defined as in Theorem 2.1. Then every solution of (1.1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ be a non-oscillatory solution of (1.1), say $x(t) \neq 0$ on $[T_0,\infty)$ for some sufficient large $T_0 \ge t_0$. Define the Riccati transformation function

$$
\omega(t) = \rho(t)\frac{r(t)\psi(x(t))D_t^\alpha x(t)}{x(t)}, \quad t \ge T_0. \tag{3.2}
$$

Then for $t \ge T_0$, from (1.2)-(1.4) we deduce that

$$
\begin{aligned}
&D_t^\alpha \omega(t) \\
&= -\frac{f(x(t))\rho(t)q(t)}{x(t)} + \frac{D_t^\alpha \rho(t)}{\rho(t)}\omega(t) - \frac{1}{\rho(t)r(t)\psi(x(t))}\omega^2(t) + \frac{e(t)\rho(t)}{x(t)} \\
&\le -k_2\rho(t)q(t) + \frac{D_t^\alpha\rho(t)}{\rho(t)}\omega(t) - \frac{\omega^2(t)}{m\rho(t)r(t)} + \frac{e(t)\rho(t)}{x(t)}.
\end{aligned} \tag{3.3}
$$

By assumption, if $x(t) > 0$, then we can choose $a_1, b_1 \ge T_0$ with $a_1 < b_1$ such that $e(t) \le 0$ on the interval $[a_1, b_1]$. If $x(t) < 0$, then we can choose $a_2, b_2 \ge T_0$ with $a_2 < b_2$ such that $e(t) \ge 0$ on the interval $[a_2, b_2]$. So $\frac{e(t)\rho(t)}{x(t)} \le 0$, $t \in [a_i, b_i]$, $i = 1, 2$, and from (3.3) one can deduce that

$$
D_t^\alpha\omega(t) \le -k_2\rho(t)q(t) + \frac{D_t^\alpha\rho(t)}{\rho(t)}\omega(t) - \frac{\omega^2(t)}{m\rho(t)r(t)}, \quad t \in [a_i, b_i],\ i = 1, 2. \tag{3.4}
$$

Let $w(t) = \widetilde{w}(\xi)$. Then we have $D_t^\alpha w(t) = \widetilde{w}'(\xi)$ and $D_t^\alpha\rho(t) = \widetilde{\rho}'(\xi)$. So (3.4) is transformed into

$$
\widetilde{\omega}'(\xi) \le \frac{\widetilde{\rho}'(\xi)}{\widetilde{\rho}(\xi)}\widetilde{\omega}(\xi) - k_2\widetilde{\rho}(\xi)\widetilde{q}(\xi) - \frac{1}{m\widetilde{\rho}(\xi)\widetilde{r}(\xi)}\widetilde{\omega}^2(\xi), \quad \xi \in [\xi_{a_i}, \xi_{b_i}],\ i = 1, 2. \tag{3.5}
$$

Let c_i be selected from (ξ_{a_i}, ξ_{b_i}) arbitrarily. Substituting ξ with s, multiplying both sides of (3.5) by $H(\xi, s)$ and integrating it over $[c_i, \xi)$ for $\xi \in [c_i, \xi_{b_i})$, after similar computation to (2.7), we obtain

$$\begin{aligned}
&\int_{c_i}^{\xi} H(\xi,s)k_2\widetilde{\rho}(s)\widetilde{q}(s)ds \\
&\le -\int_{c_i}^{\xi} H(\xi,s)\widetilde{\omega}'(s)ds + \int_{c_i}^{\xi} H(\xi,s)\Big[\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{\omega}(s) - \frac{\widetilde{\omega}^2(s)}{m\widetilde{\rho}(s)\widetilde{r}(s)}\Big]ds \qquad (3.6)\\
&\le H(\xi,c_i)\widetilde{\omega}(c_i) + \int_{c_i}^{\xi} \frac{m\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_2^2(\xi,s)ds.
\end{aligned}$$

Letting $\xi \to \xi_{b_i}^-$ in (3.6) and dividing it by $H(\xi_{b_i},c_i)$, we obtain

$$\begin{aligned}
&\frac{1}{H(\xi_{b_i},c_i)}\int_{c_i}^{\xi_{b_i}} H(\xi_{b_i},s)k_2\widetilde{\rho}(s)\widetilde{q}(s)ds \\
&\le \widetilde{\omega}(c_i) + \frac{1}{H(\xi_{b_i},c_i)}\int_{c_i}^{\xi_{b_i}} \frac{m\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_2^2(\xi_{b_i},s)ds.
\end{aligned} \tag{3.7}$$

On the other hand, substituting ξ with s, multiplying both sides of (3.5) by $H(s,\xi)$, and integrating it over (ξ,c_i) for $\xi \in [\xi_{a_i},c_i)$, we deduce that

$$\int_{\xi}^{c_i} H(s,\xi)k_2\widetilde{\rho}(s)\widetilde{q}(s)ds \le -H(c_i,\xi)\widetilde{\omega}(c_i) + \int_{\xi}^{c_i} \frac{m\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_1^2(s,\xi)ds$$

Letting $\xi \to \xi_{a_i}^+$ and dividing by $H(c_i,\xi_{a_i})$, we obtain

$$\begin{aligned}
&\frac{1}{H(c_i,\xi_{a_i})}\int_{\xi_{a_i}}^{c_i} H(s,\xi_{a_i})k_2\widetilde{\rho}(s)\widetilde{q}(s)ds \\
&\le -\widetilde{\omega}(c_i) + \frac{1}{H(c_i,\xi_{a_i})}\int_{\xi_{a_i}}^{c_i} \frac{m\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_1^2(s,\xi_{a_i})ds.
\end{aligned} \tag{3.8}$$

A combination of (3.7) and (3.8) yields the inequality

$$\begin{aligned}
&\frac{1}{H(c_i,\xi_{a_i})}\int_{\xi_{a_i}}^{c_i} H(s,\xi_{a_i})k_2\widetilde{\rho}(s)\widetilde{q}(s)ds + \frac{1}{H(\xi_{b_i},c_i)}\int_{c_i}^{\xi_{b_i}} H(\xi_{b_i},s)k_2\widetilde{\rho}(s)\widetilde{q}(s)ds \\
&\le \frac{1}{4H(c_i,\xi_{a_i})}\int_{\xi_{a_i}}^{c_i} m\rho(s)\widetilde{r}(s)Q_1^2(s,\xi_{a_i})ds + \frac{1}{4H(\xi_{b_i},c_i)}\int_{c_i}^{\xi_{b_i}} m\widetilde{\rho}(s)\widetilde{r}(s)Q_2^2(\xi_{b_i},s)ds,
\end{aligned}$$

which contradicts to (3.1) since c_i is selected from (ξ_{a_i},ξ_{b_i}) arbitrarily. Therefore, every solution of (1.1) is oscillatory, and the proof is complete. □

Theorem 3.2. *Under the conditions of Theorem 3.1, furthermore, suppose* (3.1) *does not hold, and $\widetilde{q}(\xi) > 0$ for any $\xi \ge \xi_0$. If for some $u \in C[\xi_{a_i},\xi_{b_i}]$ satisfying $u' \in L^2[\xi_{a_i},\xi_{b_i}]$, $u(\xi_{a_i}) = u(\xi_{b_i}) = 0$, $i = 1,2$, and u is not identically zero, there exists $\rho \in C^1([\xi_0,\infty),R_+)$ such that*

$$\int_{\xi_{a_i}}^{\xi_{b_i}} \Big\{u^2(s)k_2\widetilde{\rho}(s)\widetilde{q}(s) - \Big[m\widetilde{\rho}(s)\widetilde{r}(s)\Big(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2\Big\}ds > 0 \tag{3.9}$$

for $i = 1,2$, then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ be a non-oscillatory solution of (1.1), say $x(t) \ne 0$ on $[T_0,\infty)$ for some sufficient large $T_0 \ge t_0$. Let ω be defined as in Theorem 3.1. Then we obtain (3.5). Substituting ξ by s, multiplying both sides of (3.5) by

$u^2(s)$, integrating it with respect to s from ξ_{a_i} to ξ_{b_i} and using $u(\xi_{a_i}) = u(\xi_{b_i}) = 0$, we deduce that

$$\begin{aligned}
&\int_{\xi_{a_i}}^{\xi_{b_i}} u^2(s)k_2\widetilde{\rho}(s)\widetilde{q}(s)ds \\
&\leq -\int_{\xi_{a_i}}^{\xi_{b_i}} u^2(s)\widetilde{\omega}'(s)ds + \int_{\xi_{a_i}}^{\xi_{b_i}} u^2(s)\Big[\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{\omega}(s) - \frac{\widetilde{\omega}^2(s)}{m\widetilde{\rho}(s)\widetilde{r}(s)}\Big]ds \\
&= -\int_{\xi_{a_i}}^{\xi_{b_i}} \Big[\sqrt{\frac{1}{m\widetilde{\rho}(s)\widetilde{r}(s)}}u(s)\widetilde{\omega}(s) - \sqrt{m\widetilde{\rho}(s)\widetilde{r}(s)}\Big(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2 ds \\
&\quad + \int_{\xi_{a_i}}^{\xi_{b_i}} \Big[m\widetilde{\rho}(s)\widetilde{r}(s)\Big(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2 ds.
\end{aligned}$$

Then

$$\int_{\xi_{a_i}}^{\xi_{b_i}} \Big\{u^2(s)k_2\widetilde{\rho}(s)\widetilde{q}(s) - \Big[m\widetilde{\rho}(s)\widetilde{r}(s)\Big(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2\Big\}ds \leq 0,$$

which contradicts (3.9). The proof is complete. □

The following corollary has a proof similar to the one of Corollary 2.3.

Corollary 3.3. *Under the conditions of Theorem 3.2, if for each* $r \geq \xi_0$,

$$\begin{aligned}
&\limsup_{\xi\to\infty} \int_r^{\xi} \Big\{k_2\widetilde{\rho}(s)(\xi - s)^2(s - r)^2\widetilde{q}(s) \\
&- \Big[m\widetilde{\rho}(s)\widetilde{r}(s)\Big((\xi + r - 2s) + \frac{1}{2}(\xi - s)(s - r)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\Big)\Big]^2\Big\}ds > 0,
\end{aligned}$$

then (1.1) *is oscillatory.*

4. Applications

Example 4.1. Consider the nonlinear fractional differential equation with forced term

$$D_t^{\alpha}\Big(\sin^2(\frac{t^{\alpha}}{\Gamma(1+\alpha)})e^{-x^2(t)}D_t^{\alpha}x(t)\Big) + (x(t) + x^3(t)) = \sin(\frac{t^{\alpha}}{\Gamma(1+\alpha)}), \tag{4.1}$$

$t \geq 2$, $0 < \alpha < 1$. This corresponds to (1.1) with $t_0 = 2$, $r(t) = \sin^2(\frac{t^{\alpha}}{\Gamma(1+\alpha)})$, $\psi(x) = e^{-x^2}$, $q(t) \equiv 1$, $f(x) = x + x^3$, $e(t) = \sin(\frac{t^{\alpha}}{\Gamma(1+\alpha)})$. Therefore, $\psi(x) \leq 1$, $f'(x) = 1 + 3x^2 \geq 1$, which implies $\mu = m = 1$. Since $\xi = \frac{t^{\alpha}}{\Gamma(1+\alpha)}$, it follows that $\widetilde{r}(\xi) = r(t) = \sin^2(\frac{t^{\alpha}}{\Gamma(1+\alpha)}) = \sin^2\xi$.

In (2.10), we have $k_1 = \mu/m = 1$. Furthermore, letting $u(s) = \sin s$, $\xi_{a_i} = (2k+i)\pi$, $\xi_{b_i} = (2k+i)\pi + \pi$ such that ξ_{a_i}, ξ_{b_i} is sufficiently large, we obtain $u(\xi_{a_i}) = u(\xi_{b_i}) = 0$, and considering $\widetilde{q}(s) \equiv 1$, $\widetilde{\rho}(s) \equiv 1$, it holds that

$$\int_{(2k+i)\pi}^{(2k+i)\pi+\pi} \big(\sin^2 s - \sin^2 s\cos^2 s\big)\, ds = \int_{(2k+i)\pi}^{(2k+i)\pi+\pi} \sin^4 s ds > 0.$$

On the other hand, by the connection between a_i, b_i and ξ_{a_i}, ξ_{b_i} we have

$$a_i = [\Gamma(1+\alpha)(2k+i)\pi]^{\frac{1}{\alpha}}, \quad b_i = [\Gamma(1+\alpha)(2k+i)\pi + \pi]^{\frac{1}{\alpha}}.$$

So for $e(t) = \sin(\frac{t^\alpha}{\Gamma(1+\alpha)})$, one can see (2.1) holds with k selected enough large. Therefore, by Theorem 2.2 Equation (4.1) is oscillatory.

Example 4.2. Consider the nonlinear fractional differential equation with forced term:

$$D_t^\alpha\Big(\sin^2(\frac{t^\alpha}{\Gamma(1+\alpha)})\frac{1}{1+x^2(t)}D_t^\alpha x(t)\Big)+\frac{x(t)(2+x^2(t))}{1+x^2(t)}=\sin(\frac{t^\alpha}{\Gamma(1+\alpha)}), \quad (4.2)$$

$t \geq 2$, $0 < \alpha < 1$. This corresponds to (1.1) with $t_0 = 2$, $r(t) = \sin^2(\frac{t^\alpha}{\Gamma(1+\alpha)})$, $\psi(x) = \frac{1}{1+x^2}$, $q(t) \equiv 1$, $f(x) = \frac{2x+x^3}{1+x^2}$, $e(t) = \sin(\frac{t^\alpha}{\Gamma(1+\alpha)})$.

Therefore, $\widetilde{r}(\xi) = r(t) = \sin^2(\frac{t^\alpha}{\Gamma(1+\alpha)}) = \sin^2\xi$, $\psi(x) \leq 1$, which implies $m = 1$. Furthermore, we notice that it is complicated in obtaining the lower bound of $f'(x)$. So Theorems 2.1 and 2.2 are not applicable, while one can easily see $f(x)/x \geq 1$, which implies $k_2 = 1$. Then by Theorem 3.2, and analysis similar to the last paragraph in Example 4.1, Equation (4.2) is oscillatory.

Conclusions. We have established some new oscillation criteria for a nonlinear forced fractional differential equation. As one can see, the variable transformation used in ξ is very important, transforms a fractional differential equation into an ordinary differential equation of integer order, whose oscillation criteria can be established using a generalized Riccati transformation, inequalities, and an integration average technique. Finally, we note that this approach can also be applied to the oscillation for other fractional differential equations involving the modified Riemann-liouville derivative.

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5. Addendum posted on November 17, 2016

In response to a message from a reader, the authors want to point out that the first equality in (1.4) is the chain rule for Jumarie's modified Riemann-Liouville derivative obtained in [15]. However. this rule is icorrect, a shown in the article

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Therefore the main result of this article is icorrect. End of addendum.

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