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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SEMILINEAR DIFFERENTIAL EQUATIONS WITH SUBQUADRATIC POTENTIALS

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Abstract. Existence and multiplicity of nontrivial solutions of a class of semilinear differential equations with subquadratic potentials are studied using Clark's theorem. As an application of the above results, periodic solutions for a second Hamiltonian systems are studied.

1. Introduction

In physical, chemical and biological sciences, many models can be set as equation of the form

$$\mathcal{L}u = V_u(t,u), \quad t \in \mathbb{R}, \tag{1.1}$$

where $\mathcal{L}$ is a linear differential operator which is self-adjoint and positive in some suitable space, and the potential $V(t,q) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a C^1-function, with $V_u(t,q) = \frac{\partial V}{\partial u}$; see for example the references in this article. If V satisfies

$$\lim_{u\to\infty} V(t,u)/|u|^2 \leqslant c < \infty,$$

then we say that (1.1) is subquadratic. If $\lim_{|u|\to\infty} V(u)/|u|^2 = \infty$, then (1.1) is superquadratic. Our main goal is to find periodic solutions of (1.1) with subquadratic potentials by variational methods; that is, periodic solutions of (1.1) are critical points of the corresponding functional in an appropriate Hilbert space $(X, \|\cdot\|)$, given by

$$\varphi_T(u) = \frac{1}{2}\|u\|^2 - J(u), \quad u \in \mathbb{X}, \tag{1.2}$$

where $\langle \mathcal{L}u, u\rangle = \frac{1}{2}\|u\|^2$, $\langle , \rangle$ denotes the inner product of X. Under some assumptions, we have an abstract result as follows.

Theorem 1.1. Let $(X, \|\cdot\|) \subset L^2 \equiv L^2([0,T], \mathbb{R}^n)$ be a Hilbert space such that

$$\|u\|_{L^2(0,T)} \leqslant \big(\frac{T}{\pi}\big)^k \|u\|, \quad \forall u \in X \tag{1.3}$$

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for some $k \in \mathbb{N}$, and $\{e_j(t)\}_{j=1}^{\infty}$ be an orthogonal sequence in X and L^2 such that $|e_j(t)| \leqslant 1$, for all $j \geqslant 1$, $t \in [0,T]$ and

$$\|e_j(t)\|^2_{L^2(0,T)} \geqslant \mu, \quad \|e_j(t))\|^2 \leqslant \mu(\frac{j\pi}{T})^{2k}, \quad \forall j \geqslant 1 \tag{1.4}$$

for some $\mu > 0$, the functional $\varphi_T(u)$ in (1) is defined in X such that $J(u) \in \mathbb{C}^1(X,\mathbb{R})$, $J(0) = 0$ and $J'(u)$ is completely continuous. Furthermore, we assume that $J(u)$ satisfies:

- (J1) $J(u) = J(-u)$ for all $u \in X$,
- (J2) there exist $m > 0$ and $b > 0$ such that for all $u \in X$, $T < \pi/\sqrt[2k]{m}$, and $J(u) \leqslant b + \frac{1}{2}m\|u\|^2_{L^2(0,T)}$,
- (J3) there exist $p \in \mathbb{N}$ and $M > 0$, $\rho > 0$such that $T > p\pi/\sqrt[2k]{M}$, $M > mp^{2k}$, and
$$J(u) \geqslant \frac{1}{2}M\|u\|^2_{L^2(0,T)}, \quad \forall u \in X \text{ with } \|u\|_{L^\infty(0,T)} \leqslant \rho\sqrt{p}.$$

Then, for each $T \in (p\pi/\sqrt[2k]{M}, \pi/\sqrt[2k]{m})$, there exist at least p distinct critical point pairs $(u_i, -u_i)$ of $\varphi_T(u)$ such that $\varphi_T(u_i) < 0 (1 \leqslant i \leqslant p)$.

In section 2, we will give the proof of Theorem 1.1. For this, we recall a compactness condition introduced by Palais and Smale.

Palais-Smale condition. Let X be a Banach space and $\varphi \in \mathbb{C}^1(X,\mathbb{R})$ is said to satisfy Palais-Smale condition if any sequence $\{u_j\} \subset X$ such that $\varphi(u_j)$ is bounded and $\varphi'(u_j) \to 0$ possesses a convergent subsequence.

We also need the following theorem.

Theorem 1.2 (Clark Theorem [2])**.** *Let Xbe a Banach space and $\varphi \in \mathbb{C}^1(X,\mathbb{R})$ be even satisfying the Palais-Smale condition. Suppose that (i) φ is bounded from below; (ii) there exist a closed, symmetric set $K \subset X$ and $p \in \mathbb{N}$ such that K is homeomorphism to S^{p-1} by an odd map, and* $\sup\{\varphi(x) : x \in K\} < \varphi(0)$*. Then φ possesses at least $p$distinct pairs $(u, -u)$ of critical point with corresponding critical values less than $\varphi(0)$.*

In section 3, we will apply Theorem 1.1 to Hamiltonian systems and fourth-order differential equations with bi-even subquadratic potentials. First, we consider second order Hamiltonian systems

$$\ddot{u}(t) + V_u(t, u(t)) = 0, \quad t \in \mathbb{R} \tag{1.5}$$

where $V(t,u) \in \mathbb{C}^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ is a T-periodic function in the variable t,$V(t,0) \equiv 0$. The existence of periodic solutions is one of the most important problems in the theory of Hamiltonian systems. In the past thirty years, many authors studied periodic solutions for Hamiltonian systems via the critical point theory. Here we only mention some results for subquadratic Hamiltonian systems. Clarke and Ekeland [3] studied a family of convex subquadratic Hamiltonian systems where $V(t,u) = V(u)$ satisfies $\lim_{|u|\to\infty} V(u)/|u|^2 = 0$, $\lim_{|u|\to 0} V(u)/|u|^2 = \infty$, and they used the dual variational method to obtain the first variational result on periodic solutions having a prescribed minimal period. Later, Mawhin and Willem [6] made an improvement, they supposed that $V(u)$ is convex, also satisfies $\lim_{|u|\to\infty} V(u)/|u|^2 = 0$, $\lim_{|u|\to 0} V(u)/|u|^2 = \infty$, and proved that there is a $T_0 > 0$ such that for all $T > T_0$, (1.5) has a periodic solution with minimal period T. Rabinowitz [7, 8], Tang [10] and others proved the existence, where authors used the subquadratic condition:

$uV_u(t,u) \leqslant \alpha V(t,u)(0<\alpha<2)$, which plays an important role. Schechter [9] assumed that $V(t,u)$ is subquadratic, and $2V(t,u)-uV_u(t,u)\to-\infty(|u|\to\infty)$ or $2V(t,u)-uV_u(t,u)\leqslant W(t)$, then he proved that (1.5) has one non-constant periodic solution. Long [5] studied this problem for bi-even subquadratic potentials, and get the existence of one odd T-periodic solution. Inspired by the above papers, using Theorem 1.1, we give a multiplicity result for (1.5) as follows.

Theorem 1.3. *Let $V(t,u)\in \mathbb{C}^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ be τ-periodic in t and satisfy*

- (V1) *$V(t,u)=V(t,-u)=V(-t,-u)$, for all $t\in\mathbb{R}$, $u\in\mathbb{R}^n$;*
- (V2) *there exist $m>0, b>0$ such that $\tau<2\pi/\sqrt{m}$ and $V(t,u)\leqslant b+\frac{1}{2}m|u|^2$, for all $t\in\mathbb{R}$, $u\in\mathbb{R}^n$;*
- (V3) *there exist $p\in\mathbb{N}$ and constants $M>0,\rho>0$ such that $\tau>2p\pi/\sqrt{M}$, $M>mp^2$ and*

$$V(t,u)\geqslant \frac{1}{2}M|u|^2,\quad \forall t\in\mathbb{R},\ |u|\leqslant\rho\sqrt{p}.$$

Then, for $\tau\in(2p\pi/\sqrt{M},2\pi/\sqrt{m})$,(1.5) has p distinct pairs $(u(t),-u(t))$ of non-trivial odd τ-periodic solutions.

Remark 1.4. If $V(t,u)=a(t)W(u)$, $a(t)$ and $W(u)$ are even, then (V1) holds.

Note that our method and results in this article are different from the earlier ones in [5, 6, 7, 8, 9, 10] and references therein.

As the second application, in the study of formation of spatial patterns in bistable systems, we consider a fourth-order differential equation [1, 4, 11],

$$u^{(4)}(t)-V_u(t,u(t))=0,\quad 0\leqslant t\leqslant T \tag{1.6}$$

with the boundary condition $u(0)=u(T)=u''(0)=u''(T)=0$. For (1.6), we also have a result similar to Theorem 1.3.

Theorem 1.5. *Let $V(t,u)\in\mathbb{C}^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ satisfy*

- (V4) *$V(t,u)=V(t,-u)$, for all $t\in\mathbb{R},u\in\mathbb{R}$;*
- (V5) *there exist $m>0,b>0$ such that $T<\pi/\sqrt[4]{m}$ and $V(t,u)\leqslant b+\frac{1}{2}m|u|^2$, for all $t\in\mathbb{R},u\in\mathbb{R}$;*
- (V6) *there exist $p\in\mathbb{N}$ and constants $M>0,\rho>0$ such that $T>p\pi/\sqrt[4]{M}$,$M>mp^4$ and*

$$V(t,u)\geqslant\frac{1}{2}M|u|^2,\quad\forall t\in\mathbb{R},\ |u|\leqslant\rho\sqrt{p}.$$

Then, for each $T\in(p\pi/\sqrt[4]{M},\pi/\sqrt[4]{m})$, (1.6) has at least p distinct pairs $(u(t),-u(t))$ of solutions.

To the best of our knowledge, only a few multiplicity results for fourth-differential equations, similar to Theorem 1.5, have been reported in the literature.

2. Proof of Theorem 1.1

Lemma 2.1. *Under the hypotheses of Theorem 1.1, the functional $\varphi_T(u)$ is coercive in X, and bounded from below.*

Proof. The condition (J2) implies the estimate

$$\varphi_T(u)\geqslant\frac{1}{2}\|u\|^2-\frac{1}{2}m\|u\|^2_{L^2(0,T)}-b, \tag{2.1}$$

which combined with the inequality (1.3) yields

$$\varphi_T(u) \geqslant \frac{1}{2}\big[1 - m(\frac{T}{\pi})^{2k}\big]\|u\|^2 - b = \frac{1}{2}B\|u\|^2 - b \geqslant -b, \tag{2.2}$$

with $B = 1 - m(\frac{T}{\pi})^{2k} > 0$. Thus, we conclude that $\varphi_T(u)$ is coercive in X, and bounded from below. □

Lemma 2.2. *Under the hypotheses of Theorem 1.1, the functional $\varphi_T(u)$ satisfies Palais-Smale condition in X.*

Proof. Let $\{u_j\} \subset X$ be such that $\varphi_T(u_j)$ is bounded and ${\varphi_T}'(u_j) \to 0 (j \to \infty)$. Then by Lemma 2.1,

$$\|u_j\|^2 \leqslant \frac{2}{B}(\varphi_T(u_j) + b) \tag{2.3}$$

which implies that $\{u_j\}$ is bounded, so we may assume, by passing to a subsequence if necessary, that

$$u_j \rightharpoonup u \quad \text{weakly in } X. \tag{2.4}$$

As we know

$${\varphi_T}'(u_j)u = \langle u_j, u\rangle - J'(u_j)u \tag{2.5}$$

letting $j \to \infty$, by the completely continuousness of J', we have

$$0 = \|u\|^2 - J'(u)u\,. \tag{2.6}$$

Since $|{\varphi_T}'(u_j)u_j| \leqslant \|{\varphi_T}'(u_j)\|\|u_j\| \to 0$, using (2.6), we obtain

$$\|u_j\|^2 = {\varphi_T}'(u_j)u_j + J'(u_j)u_j \to J'(u)u = \|u\|^2 \tag{2.7}$$

Thus, we conclude that $u_j \to u$ in X. The proof is complete. □

Proof of Theorem 1.1. For $p \in \mathbb{N}$ and $\rho > 0$ defined in (J3), let the subset K of X as follows

$$K = \{\lambda_1 e_1(t) + \lambda_2 e_2(t) + \dots + \lambda_p e_p(t) : \lambda_1, \lambda_2, \dots, \lambda_p \in \mathbb{R}, \sum_{k=1}^{p} \lambda_k^2 = \rho^2\} \tag{2.8}$$

We know the map

$$\lambda_1 e_1(t) + \lambda_2 e_2(t) + \dots + \lambda_p e_p(t) \to (-\frac{\lambda_1}{\rho}, -\frac{\lambda_2}{\rho}, \dots, -\frac{\lambda_p}{\rho}) \tag{2.9}$$

is an odd homeomorphism from K to S^{p-1}. For all $u(t) = \lambda_1 e_1(t) + \lambda_2 e_2(t) + \dots + \lambda_p e_p(t) \in K$, we have the estimate

$$|u(t)|^2 \leqslant (\lambda_1^2 + \lambda_2^2 + \dots + \lambda_p^2)(|e_1(t)|^2 + |e_2(t)|^2 + \dots + |e_p(t)|^2) \leqslant p\rho^2. \tag{2.10}$$

Combining (2.10) and (J3) with (1.4) shows that

$$\begin{aligned}
\varphi_T(u) &\leqslant \frac{1}{2}\|u\|^2 - \frac{1}{2}M\|u\|^2_{L^2(0,T)} \\
&= \frac{1}{2}\sum_{j=1}^{p} \lambda_j^2 \|e_j(t)\|^2 - \frac{1}{2}M\sum_{j=1}^{p} \lambda_j^2 \|e_j(t)\|^2_{L^2(0,T)} \\
&\leqslant \frac{\mu}{2}\sum_{j=1}^{p} \lambda_j^2 [(\frac{j\pi}{T})^{2k} - M].
\end{aligned} \tag{2.11}$$

Since $T > p\pi/\sqrt[2k]{M}$, we have $0 < \frac{j\pi}{T} < \sqrt[2k]{M}$, for $1 \leqslant j \leqslant p$. Therefore, we obtain

$$\varphi_T(u) \leqslant \frac{\mu}{2}\rho^2((\frac{p\pi}{T})^{2k} - M) < 0, \quad \forall u \in K \tag{2.12}$$

which implies $sup\{\varphi_T(u) : u \in K\} < 0 = \varphi_T(0)$. Thus, Lemma 2.1, Lemma 2.2 and Clark Theorem imply that $\varphi_T(u)$ possesses at least p distinct pairs $(u_i, -u_i)$ of critical points such that $\varphi_T(u_i) < 0$ clearly $u_i \neq 0 (1 \leqslant i \leqslant p)$. □

Corollary 2.3. *In Theorem 1.1, for any $T > 0$, if conditions* (J2)-(J3) *are replaced by*

- (J2') $\lim_{|u|\to\infty} V(t,u)/|u|^2 = 0$ *uniformly in* $t \in \mathbb{R}$,
- (J3') $\lim_{|u|\to 0} V(t,u)/|u|^2 = \infty$ *uniformly in* $t \in \mathbb{R}$,

then the functional $\varphi_T(u)$ has infinitely many distinct pairs $(-u, u)$ of critical points.

Proof. For any fixed $p \in \mathbb{N}$, by (J2') and (J3'), we may take m sufficiently small and M large enough such that

$$0 < m < (\frac{\pi}{T})^{2k}, \quad M > (\frac{p\pi}{T})^{2k}, \quad M > mp^{2k}, \tag{2.13}$$

thus (J2)-(J3) are all satisfied, and $T \in (p\pi/\sqrt[2k]{M}, \pi/\sqrt[2k]{m})$. By Theorem 1.1, the functional $\varphi_T(u)$ has at least p distinct pairs $(u, -u)$ of critical points. Since p is arbitrary, there exist infinitely many distinct pairs $(u, -u)$ of critical points of $\varphi_T(u)$. □

3. Applications of Theorem 1.1

The first application is for to Hamiltonian systems. To prove Theorem 1.3, we study the related boundary value problem

$$\begin{gathered} \ddot{u}(t) + V_u(t, u(t)) = 0, \quad 0 < x < T, \\ u(0) = u(T) = 0, \end{gathered} \tag{3.1}$$

with $T = \tau/2$. For a solution $u(t)$ of (3.1), we define

$$\bar{u}(t) = \begin{cases} u(t) & 0 \leqslant t \leqslant T, \\ -u(-t) & -T \leqslant t \leqslant 0. \end{cases} \tag{3.2}$$

For any $t \in [-T, 0]$, by (V1), we see that

$$\begin{aligned} (\bar{u})''(t) + V_u(t, \bar{u}(t)) &= -\ddot{u}(-t) + V_u(t, -u(-t)) \\ &= -[\ddot{u}(-t) - V_u(t, -u(-t))] \\ &= -[\ddot{u}(-t) + V_u(t, -u(-t))] = 0. \end{aligned}$$

Hence, $\bar{u} = \bar{u}(t)$ is a solution of (1.5) over $[-T, T]$,and its $2T$-periodic extension over $\mathbb{R}$, still denoted by $\bar{u} = \bar{u}(t)$, is an odd τ-periodic solution of (1.5) with $\tau = 2T$. Let $X = H_0^1([0, T]; \mathbb{R}^n)$ be a Hilbert space with the inner product $(u, \omega) = \int_0^T [\dot{u}(t)\dot{\omega}(t) + u(t)\omega(t)]dt$ and the corresponding norm

$$\|u\|_\Delta = (u, u)^{1/2} = (\int_0^T [|\dot{u}(t)|^2 + |u(t)|^2]dt)^{1/2}.$$

The Pointcare inequality $\int_0^T |u(t)|^2 dt \leqslant (\frac{T}{\pi})^2 \int_0^T |\dot{u}(t)|^2 dt$ implies that

$$\|u\| = (\int_0^T |\dot{u}(t)|^2 dt)^{1/2}$$

is also a norm in X, and is equivalent to $\|u\|_\Delta$. Now we define the functional $\varphi_T(u)$ on X:

$$\varphi_T(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dx - \int_0^T V(t, u(t))dt \equiv \frac{1}{2}\|u\|^2 - J(u) \tag{3.3}$$

From $V(t,u) \in \mathbb{C}^1(\mathbb{R} \times \mathbb{R}^n)$, we know that $\varphi_T \in \mathbb{C}^1(X, \mathbb{R})$, and $J'(u)$ is completely continuous.

Proof of Theorem 1.3. Under the assumptions of Theorem 1.3, $\{\sin \frac{j\pi t}{T} e\}_{j=1}^\infty$ is an orthogonal sequence with $e = (1, 0, 0, \dots, 0) \in \mathbb{R}^n$ in both X and L^2 such that $\|\sin \frac{j\pi t}{T} e\|^2_{L^2(0,T)} = \frac{T}{2}$, $\|sin \frac{j\pi t}{T} e\|^2 = \frac{T}{2}(\frac{j\pi}{T})^2$, for all $j \geqslant 1$, the functional (3.3) satisfies $(J_1) - (J_3)$ of Theorem 1.1. Thus, $\varphi_T(u)$ possesses at least p distinct pairs $(u_i, -u_i)$ of critical points such that $\varphi_T(u_i) < 0$ with $u_i \neq 0 (1 \leqslant i \leqslant p)$. Since $X \cap \mathbb{R}^n = 0$,we conclude that $u_i \neq$ any constant $(1 \leqslant i \leqslant p)$. Thus, in the way of (3.2), the extensions of $\pm u_i(t)$ $(1 \leqslant i \leqslant p)$ are p distinct pairs of nontrivial odd $\tau-$ periodic solutions of (1.5). □

Remark 3.1 ([5])**.** For $\alpha \in (0, 1/2)$, we can choose a function $h \in \mathbb{C}^1([0, \infty), \mathbb{R})$ such that

$$r^{1+2\alpha} \leqslant h(r) \leqslant r^{1+\alpha} \quad \text{for } 0 \leqslant r \leqslant 1,$$
$$-r^4 \leqslant h(r) \leqslant \frac{1}{8}r^2 \quad \text{for } r \geqslant 2.$$

Define $V(t, u) = (1 + \frac{1}{2}\cos t)h(|u|)$ then for $\tau = 2\pi$, for all $p \geqslant 1$, (V1)-(V3) are satisfied. Thus, by Theorem 1.3, (1.5) has infinitely many nontrivial pairs $(u(t), -u(t))$ of odd 2π-periodic solutions.

Finally, since the proof of Theorem 1.5, is similar to that of Theorem 2, so we briefly sketch it.

Proof of Theorem 1.5. Set

$$X = H^2(0, T) \cap H_0^1(0, T) \tag{3.4}$$

and the functional

$$\varphi_T(u) = \frac{1}{2}\int_0^T |\ddot{u}(t)|^2 dx - \int_0^T V(t, u(t))dt, u \in X. \tag{3.5}$$

Then the critical points of φ_T in (3.5) are the classical solutions of the problem (1.6). By [4, Lemma 2.1],

$$\|u\| = (\int_0^T |\ddot{u}(t)|^2 dt)^{1/2}$$

is a norm in X, $\|u\|_{L^2(0,T)} \leqslant (\frac{T}{\pi})^2 \|u\|$, and the set of functions $\{\sin \frac{j\pi t}{T}\}_{j=1}^\infty$ is an orthogonal sequence in both X and L^2 such that

$$\|\sin \frac{j\pi t}{T}\|^2_{L^2(0,T)} = \frac{T}{2}, \quad \|\sin \frac{j\pi t}{T}\|^2 = \frac{T}{2}(\frac{j\pi}{T})^4, \quad \forall j \geqslant 1.$$

Therefore, by Theorem 1.1, the proof is complete. □

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