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GENERAL BOUNDARY CONDITIONS FOR THE KAWAHARA EQUATION ON BOUNDED INTERVALS

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Abstract. This article is concerned with initial boundary value problems for the Kawahara equation on bounded intervals. For general linear boundary conditions and small initial data, we prove the existence and uniqueness of a global regular solution and exponential decay as $t \to \infty$.

1. Introduction

This work concerns the existence and uniqueness of global solutions for the Kawahara equation posed on a bounded interval with general linear boundary conditions. Initial value problems for the Kawahara equation have been considered in [7, 11, 23] due to various applications of those results in mechanics and physics such as dynamics of long small-amplitude waves in various media. On the other hand, last years appeared publications on solvability of initial boundary value problems for dispersive equations (which included KdV and Kawahara equations) in bounded domains [1, 2, 3, 4, 6, 9, 10, 12, 13, 15, 16, 17, 18, 19, 20, 24, 25, 26]. In spite of the fact that there is not any clear physical interpretation for the problems in bounded intervals, their study is motivated by numerics.

Dispersive equations such as KdV and Kawahara equations have been developed for unbounded regions of wave propagations. However, if one is interested in implementing numerical schemes to calculate solutions in these regions, there arises the issue of cutting off a spatial domain approximating unbounded domains by bounded ones. In this occasion, some boundary conditions are needed to specify the solution. Therefore, precise mathematical analysis of mixed problems in bounded domains for dispersive equations is welcome and attracts attention of specialists in this area [2, 3, 4, 5, 6, 9, 10, 12, 13, 17, 18, 19, 20, 24, 26].

As a rule, simple boundary conditions at $x = 0$ and $x = 1$ such as $u = u_x = 0|_{x=0}$, $u = u_x = u_{xx} = 0|_{x=1}$ for the Kawahara equation were imposed. Different kind of boundary conditions was considered in [6, 24, 25]. On the other hand, general initial boundary value problems for odd-order evolution equations attracted little attention. We must mention [14] where general mixed problems for linear $(2b+1)$-hyperbolic equations were studied by means of functional analysis methods.

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It is difficult to apply their method directly to nonlinear dispersive equations due to complexity of this theory. General mixed problems for the KdV equation posed on bounded intervals, [3, 4, 15, 18, 24], and on unbounded one, [19], were considered.

The main difficulty in studying of boundary value problems with general linear boundary conditions is that for nonlinear equations such as the KdV and Kawahara equations there is no the first global in t estimate which is crucial in proving global solvability [2]. Because of that, only local in t solvability of corresponding initial boundary value problems was proved in [3, 15]. In order to prove global solvability, nonlinear boundary conditions were considered in [4, 19, 25] which allowed to prove the first global estimate without smallness of initial data. Global solvability and exponential decay of small solutions to an initial boundary value problem with general linear boundary conditions for the KdV equation have been proved in [18].

Here we study mixed problems for the Kawahara equation on bounded intervals with general linear homogeneous boundary conditions and prove the existence and uniqueness of global regular solutions as well as exponential decay while $t \to \infty$ for small initial data.

It has been shown in [13, 17] that for simple boundary conditions the KdV and Kawahara equations are implicitly dissipative. This means that for small initial data and simple boundary conditions, the energy decays exponentially as $t \to +\infty$ without any additional damping terms in equations. In the present paper, we prove that for the Kawahara equation this phenomenon also takes place for general linear dissipative boundary conditions as well as the effect of smoothing of initial data.

The paper has the following structure. Section 1 is Introduction. Section 2 contains formulation of the problem, notations and definitions. The main results on well-posedness of the considered problem are also formulated in this section. In Section 3, we study a corresponding boundary value problem for a stationary part of equation. Section 4 is devoted to a mixed problem for a complete linear evolution equation. In Section 5, local well-posedness of the original problem is established. Section 6 contains a global existence result and decay of small solutions while $t \to +\infty$. To prove our results, we use the semigroup theory in order to solve the linear problem, the Banach fixed point theorem for local in t existence and uniqueness results and, finally, a priori estimates, independent of t, for the nonlinear problem.

2. Formulation of the problem and main results

Let T and L be finite positive numbers and Q_T be a bounded domain: $Q_T = \{(x,t) \in \mathbb{R}^2 :\ x \in (0,L), t \in (0,T)\}$. Consider in Q_T the Kawahara equation

$$u_t + uDu + D^3u - D^5u = 0 \tag{2.1}$$

subject to the initial and boundary conditions:

$$u(x,0) = u_0(x), \quad x \in (0,L), \tag{2.2}$$

$$\begin{gathered}
D^3u(0,t) = a_{32}D^2u(0,t) + a_{31}Du(0,t) + a_{30}u(0,t),\\
D^4u(0,t) = a_{42}D^2u(0,t) + a_{41}Du(0,t) + a_{40}u(0,t),\\
D^2u(L,t) = b_{21}Du(L,t) + b_{20}u(L,t),\\
D^3u(L,t) = b_{31}Du(L,t) + b_{30}u(L,t),\\
D^4u(L,t) = b_{41}Du(L,t) + b_{40}u(L,t), \quad t > 0,
\end{gathered} \tag{2.3}$$

where the coefficients a_{ij}, $i=3,4$, $j=0,1,2$, and b_{ij}, $i=2,3,4$, $j=0,1$ are such that

$$\begin{gathered}
B_1 = b_{20} - b_{40} - b_{20}^2 - \frac{1}{2}|b_{21}| - \frac{1}{2}b_{41} - \frac{1}{2}|b_{30}| > 0,\\
B_2 = b_{31} - \frac{1}{2} - b_{21}^2 - \frac{1}{2}|b_{21}| - \frac{1}{2}b_{41} - \frac{1}{2}|b_{30}| > 0,\\
A_1 = a_{40} - 1 - \frac{1}{2}|a_{41}| - \frac{1}{2}|a_{42}| - \frac{1}{2}|a_{30}| > 0,\\
A_2 = \frac{1}{2} - a_{31} - \frac{1}{2}|a_{41}| - \frac{1}{2}|a_{30}| - \frac{1}{2}|a_{32}| > 0,\\
A_3 = \frac{1}{4} - \frac{1}{2}|a_{42}| - \frac{1}{2}|a_{32}| > 0;\\
D^i = \frac{\partial^i}{\partial x^i}, \quad D = D^1, \quad i \in \mathbb{N}.
\end{gathered} \tag{2.4}$$

Remark 2.1. We call (2.3) general boundary conditions because they follow naturally from a more general form. At $x=0$:

$$\begin{gathered}
k_{41}D^4u(0,t) + k_{31}D^3u(0,t) + k_{21}D^2u(0,t) + k_{11}Du(0,t) + k_{01}u(0,t) = 0,\\
k_{42}D^4u(0,t) + k_{32}D^3u(0,t) + k_{22}D^2u(0,t) + k_{12}Du(0,t) + k_{02}u(0,t) = 0.
\end{gathered} \tag{2.5}$$

Whenever the determinant $\Delta_0 = \det\begin{pmatrix} k_{41} & k_{31}\\ k_{42} & k_{32}\end{pmatrix} \neq 0$, we arrive to the system

$$\begin{gathered}
D^3u(0,t) = a_{32}D^2u(0,t) + a_{31}Du(0,t) + a_{30}u(0,t),\\
D^4u(0,t) = a_{42}D^2u(0,t) + a_{41}Du(0,t) + a_{40}u(0,t).
\end{gathered}$$

Similarly, at $x = L$:

$$\begin{gathered}
p_{41}D^4u(L,t) + p_{31}D^3u(L,t) + p_{21}D^2u(L,t) + p_{11}Du(L,t) + p_{01}u(L,t) = 0,\\
p_{42}D^4u(L,t) + p_{32}D^3u(L,t) + p_{22}D^2u(L,t) + p_{12}Du(L,t) + p_{02}u(L,t) = 0,\\
p_{43}D^4u(L,t) + p_{33}D^3u(L,t) + p_{23}D^2u(L,t) + p_{13}Du(L,t) + p_{03}u(L,t) = 0.
\end{gathered} \tag{2.6}$$

If $\Delta_L = \det\begin{pmatrix} p_{41} & p_{31} & p_{21}\\ p_{42} & p_{32} & p_{22}\\ p_{43} & p_{33} & p_{23}\end{pmatrix} \neq 0$, then

$$\begin{gathered}
D^2u(L,t) = b_{21}Du(L,t) + b_{20}u(L,t),\\
D^3u(L,t) = b_{31}Du(L,t) + b_{30}u(L,t),\\
D^4u(L,t) = b_{41}Du(L,t) + b_{40}u(L,t).
\end{gathered}$$

Note that, according to (2.4), must be $b_{40} < 0$, $b_{31} > 1/2$, $a_{40} > 1$ and $a_{31} < 1/2$. The remaining coefficients should be sufficiently small or zero. For simplicity, we consider these coefficients equal to zero and get the following boundary conditions:

$$\begin{gathered}
D^3u(0,t) = a_{31}Du(0,t),\\
D^4u(0,t) = a_{40}u(0,t),\\
D^2u(L,t) = 0,\\
D^3u(L,t) = b_{31}Du(L,t),\\
D^4u(L,t) = b_{40}u(L,t), \quad t > 0.
\end{gathered} \tag{2.7}$$

Assumptions (2.4) become

$$
\begin{gathered}
B_1 = -b_{40} > 0, \quad B_2 = b_{31} - \frac{1}{2} > 0,\\
A_1 = a_{40} - 1 > 0, \quad A_2 = \frac{1}{2} - a_{31} > 0,\\
A_3 = \frac{1}{4}.
\end{gathered} \tag{2.8}
$$

Throughout this article, we adopt the usual notation $\|\cdot\|$ and $(\cdot,\cdot)$ for the norm and the inner product in $L^2(0,1)$ respectively. Our main result is the following theorem.

Theorem 2.2. *Let $u_0 \in H^5(0,L)$ satisfy (2.7). Then for all finite real $L>0$ and $T>0$ there exists a positive real number γ ($L\gamma<1$) such that if $(1+\gamma x, u_0^2) < \frac{\gamma^2}{2L^3}$, then (2.1)–(2.3) has a unique regular solution $u = u(x,t)$:*

$$
\begin{gathered}
u \in L^\infty\big(0,T;H^5(0,L)\big) \cap L^2\big(0,T;H^7(0,L)\big),\\
u_t \in L^\infty\big(0,T;L^2(0,L)\big) \cap L^2\big(0,T;H^2(0,L)\big)
\end{gathered}
$$

and the inequality holds

$$
\|u\|^2(t) \le 2\|u_0\|^2 e^{-\chi t},
$$

where $\chi = \frac{\gamma(4L^2+1)}{4L^4(1+\gamma L)}$.

3. Stationary Problem

In this section, we solve the stationary boundary problem

$$
A_\lambda v \equiv \lambda v + D^3 v - D^5 v = f \quad \text{in } (0,L); \tag{3.1}
$$

$$
D^i v(0) = \sum_{j=0}^{2} a_{ij} D^j v(0), \quad i=3,4; \quad D^i v(L) = \sum_{j=0}^{1} b_{ij} D^j v(L), \quad i = 2,3,4, \tag{3.2}
$$

where $\lambda > 0$, $f \in H^s(0,L)$, $s \in \mathbb{N}$, a_{ij} and b_{ij} satisfy (2.7), (2.8). Denote

$$
V(v) \equiv \begin{pmatrix}
0 & 1 & 0 & -a_{31} & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & -a_{40} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -b_{31} & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -b_{40}
\end{pmatrix}
\begin{pmatrix}
D^4 v(0)\\ D^3 v(0)\\ D^2 v(0)\\ Dv(0)\\ v(0)\\ D^4 v(L)\\ D^3 v(L)\\ D^2 v(L)\\ Dv(L)\\ v(L)
\end{pmatrix}.
$$

Suppose initially that $f \in C^s\big([0,L]\big)$. Consider the problem

$$
A_\lambda v = f, \tag{3.3}
$$

$$
V(v) = 0 \tag{3.4}
$$

and the associated homogeneous problem

$$
A_\lambda v = 0, \tag{3.5}
$$

$$
V(v) = 0. \tag{3.6}
$$

It is known [8, 21], problem (3.3)-(3.4) has a unique classical solution if and only if problem (3.5)-(3.6) has only the trivial solution.

Let v_1, v_2 be nontrivial solutions of (3.5)-(3.6) and $w = v_1 - v_2$. Then

$$A_\lambda w = 0, \tag{3.7}$$

$$V(w) = 0. \tag{3.8}$$

Multiplying (3.7) by w and integrating over $(0, L)$, we obtain

$$\lambda \|w\|^2 + (D^3 w - D^5 w, w) = 0. \tag{3.9}$$

Integrating by parts and using (2.8), we find

$$(D^3 w, w) = -w(0)D^2 w(0) - \frac{1}{2}\big[Dw(L)\big]^2 + \frac{1}{2}\big[Dw(0)\big]^2 \tag{3.10}$$

and

$$\begin{aligned} -(D^5 w, w) = -b_{40} w^2(L) + a_{40} w^2(0) + b_{31}\big[Dw(L)\big]^2 + a_{31}\big[Dw(0)\big]^2 \\ + \frac{1}{2}\big[D^2 w(0)\big]^2. \end{aligned} \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\begin{aligned} (D^3 w - D^5 w, w) \geq -b_{40} w^2(L) + \big[b_{31} - \frac{1}{2}\big]\big[Dw(L)\big]^2 + \big[a_{40} - 1\big] w^2(0) \\ + \big[\frac{1}{2} - a_{31}\big]\big[Dw(0)\big]^2 + \frac{1}{4}\big[D^2 w(0)\big]^2. \end{aligned} \tag{3.12}$$

According to (2.8),

$$\begin{aligned} (D^3 w - D^5 w, w) \geq K_1 \Big(w^2(L) + [Dw(L)]^2 + w^2(0) \\ + [Dw(0)]^2 + [D^2 w(0)]^2 \Big) \geq 0, \end{aligned} \tag{3.13}$$

where

$$K_1 = \min\{A_1, A_2, A_3, B_1, B_2\} > 0. \tag{3.14}$$

From (3.7) and (3.8),

$$\lambda \|w\|^2 + (D^3 w - D^5 w, w) = 0$$

and (3.13) implies $\lambda \|w\|^2 \leq 0$. Since $\lambda > 0$, then $w \equiv 0$ and $v_1 \equiv v_2$. Hence, (3.3)-(3.4) has a unique classical solution.

Theorem 3.1. *Let $f \in H^s(0, L)$, $s \in \mathbb{N}$. Then for all $\lambda > 0$, problem* (3.1)-(3.2) *admits a unique solution $u(x)$ such that*

$$\|u\|_{H^{s+5}(0,L)} \leq C \|f\|_{H^s(0,L)}, \tag{3.15}$$

where C is a positive constant independent of u and f.

Proof. To prove this theorem, we need some estimates. First, multiplying (3.1) by u and integrating over $(0, L)$, we obtain

$$\lambda \|u\|^2 + (D^3 u - D^5 u, u) = (f, u). \tag{3.16}$$

Since

$$(D^3 u - D^5 u, u) \geq 0,$$

it follows that

$$\|u\| \leq \frac{1}{\lambda} \|f\|. \tag{3.17}$$

Using (3.13), (3.17), from (3.16), we obtain

$$\begin{aligned}&\lambda\|u\|^2+2K_1\Big(u^2(L)+\big[Du(L)\big]^2+u^2(0)+\big[Du(0)\big]^2+\big[D^2u(0)\big]^2\Big)\\&\le\frac{1}{\lambda}\|f\|^2.\end{aligned}\tag{3.18}$$

Next, multiply (3.1) by $(-Du)$ and integrate over $(0,L)$ to obtain

$$-\lambda\Big(u,Du\Big)-\Big(D^3u,Du\Big)+\Big(D^5u,Du\Big)=-\Big(f,Du\Big).$$

By (3.18),

$$\begin{aligned}I_1&=-\lambda\big(u,Du\big)\ge -C_1\|f\|^2,\\I_2&=-\big(D^3u,Du\big)=-D^2(L)Du(L)+D^2u(0)Du(0)+\|D^2u\|^2\\&\ge\|D^2u\|^2-C_2\|f\|^2\\I_3&=\big(D^5u,Du\big)=D^4u(x)Du(x)\big|_{x=0}^{x=L}-D^3u(x)D^2u(x)\big|_{x=0}^{x=L}+\|D^3u\|^2\\&\ge\|D^3u\|^2-C_3\|f\|^2.\end{aligned}$$

Summing $I_1+I_2+I_3$, we have

$$\|D^2u\|^2+\|D^3u\|^2\le C_4\|f\|^2+\frac{1}{2}\|Du\|^2.\tag{3.19}$$

On the other hand, using (3.18), we calculate

$$\begin{aligned}\|Du\|^2&=-(u,D^2u)+u(L)Du(L)-u(0)Du(0)\\&\le\frac{1}{2}\|D^2u\|^2+\|u\|^2+|u(L)Du(L)|+|u(0)Du(0)|\\&\le\frac{1}{2}\|D^2u\|^2+C_5\|f\|^2.\end{aligned}$$

This and (3.19) give

$$\|u\|_{H^3(0,L)}\le K_2\|f\|.\tag{3.20}$$

Now, directly from (3.1)

$$\|D^5u\|\le\|u\|_{H^3(0,L)}+\|f\|\le K_3\|f\|.\tag{3.21}$$

Multiplying (3.1) by D^3u, we obtain

$$\lambda\Big(u,D^3u\Big)+\Big(D^3u,D^3u\Big)-\Big(D^5u,D^3u\Big)=\Big(f,D^3u\Big).\tag{3.22}$$

Integrating by parts, we calculate

$$\begin{gathered}I_4=\lambda\Big(u,D^3u\Big)\le\lambda\|u\|\|D^3u\|,\quad I_5=\Big(D^3u,D^3u\Big)=\|D^3u\|^2,\\I_6=-\Big(D^5u,D^3u\Big)=-D^3u(L)D^4u(L)+D^3u(0)D^4u(0)+\|D^4u\|^2.\end{gathered}$$

Hence

$$\|D^4u\|^2\le\|D^5u\|\|D^3u\|+C_7\Big(u^2(L)+|Du(L)|^2+u^2(0)+|Du(0)|^2+|D^2u(0)|^2\Big).$$

Taking into account (3.18), (3.20) and (3.21), we find

$$\|u\|_{H^5(0,L)}\le C(\lambda)\|f\|,\tag{3.23}$$

where the constant $C(\lambda)$ depends only on $\lambda>0$. This means that $u\in H^5(0,L)$. Moreover, differentiating sequentially s times equation (3.1), we obtain $D^{s+5}u=$

$\lambda D^s u + D^{s+3}u - D^s f$ which implies $u \in H^{s+5}(0,L)$ provided that $f \in H^s(0,L)$. The proof is complete. $\square$

4. Linear evolution problem

Consider the linear problem

$$u_t + D^3u - D^5u = f \quad \text{in } Q_T, \tag{4.1}$$

$$u(x,0) = u_0(x), \quad x \in (0,L), \tag{4.2}$$

$$V(v) = 0 \tag{4.3}$$

and define in $L^2(0,L)$ the linear operator A by

$$Au := D^3u - D^5u, \quad D(A) := \big\{u \in H^5(0,L);\, V(u) = 0\big\}. \tag{4.4}$$

Theorem 4.1. *Let $u_0 \in D(A)$ and $f \in H^1(0,T,L^2(0,L))$. Then for every $T > 0$, problem (4.1)–(4.3) has a unique solution $u = u(x,t)$;*

$$u \in C\Big([0,T], D(A)\Big) \cap C^1\Big([0,T], L^2(0,L)\Big).$$

Proof. To solve (4.1)–(4.3), we use the semigroup theory. According to Theorem 3.1, for all $\lambda > 0$ and $f \in L^2(0,L)$ there exists $u(x)$ such that $A_\lambda u = f$, hence, $R(A+\lambda I) = L^2(0,L)$. Moreover, by (3.13), $\Big(Au, u\Big) \geq 0\, \forall u \in D(A)$. Its means that A is a m-acretive operator. By the Lumer-Phillips theorem, [22, 27], A is a infinitesimal generator of a semigroup of contractions of class C_0. Therefore the following abstract Cauchy problem:

$$u_t + Au = f, \tag{4.5}$$

$$u(0) = u_0 \tag{4.6}$$

has a unique solution

$$u \in C\Big([0,T]; D(A)\Big) \cap C^1\Big([0,T]; L^2(0,L)\Big)$$

for all $f \in L^2\Big([0,T]; L^2(0,L)\Big)$ such that $f_t \in L^2\Big([0,T]; L^2(0,L)\Big)$ and $u_0 \in D(A)$. $\square$

Remark 4.2. If $u_0 \in D(A^2)$, $f \in H^2(0,T; L^2(0,L))$, then $u \in C([0,T]; D(A^2))$, $u_t \in C([0,T]; D(A)) \cap C^1([0,T]; L^2(0,L))$.

5. Nonlinear evolution problem. Local solutions

In this section we prove the existence and uniqueness of local regular solutions of (2.1)–(2.3).

Theorem 5.1. *Let $u_0 \in H^5(0,L)$ satisfy (2.7). Then there exists a real $T_0 > 0$ such that (2.1)–(2.3) has a unique regular solution $u(x,t)$ in Q_{T_0};*

$$u \in L^\infty(0,T; H^5(0,L)) \cap L^2(0,T; H^7(0,L)),$$
$$u_t \in L^\infty(0,T; L^2(0,L)) \cap L^2(0,T; H^2(0,L)).$$

Proof. We prove this theorem using the Banach Fixed Point Theorem. Define the spaces:

$$X = L^\infty(0,T;H^5(0,L));$$
$$Y = L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L));$$
$$V = \Big\{ v : [0,L]\times[0,T] \to \mathbb{R}; v \in X,\ v_t \in Y,\ v(x,0) = u_0(x) \Big\}$$

with the norm

$$\|v\|_V^2 = \sup_{t\in(0,T)} \Big\{ \|v\|^2(t) + \|v_t\|^2(t) \Big\} + \int_0^T \sum_{i=1}^2 \Big(\|D^i v\|^2(t) + \|D^i v_t\|^2(t) \Big) dt. \quad (5.1)$$

The space V equipped with the norm (5.1) is a Banach space. Define the ball

$$B_R = \{ v \in V; \|v\|_V \le \sqrt{10} R \},$$

where $R > 1$ is such that

$$(1+L)\Big[\sum_{i=0}^5 \|D^i u_0\|^2 + \|u_0 D u_0\|^2 \Big] < R^2. \quad (5.2)$$

For any $v \in B_R$ consider the linear problem

$$u_t + D^3 u - D^5 u = -vDv, \quad \text{in } Q_T; \quad (5.3)$$
$$u(x,0) = u_0(x), \quad x \in (0,L); \quad (5.4)$$

$$\begin{aligned} D^i u(0,t) &= \sum_{j=0}^2 a_{ij} D^j u(0,t), \quad i = 3,4, \quad t > 0; \\ D^i u(L,t) &= \sum_{j=0}^1 b_{ij} D^j u(L,t), \quad i = 2,3,4, \quad t > 0 \end{aligned} \quad (5.5)$$

with a_{ij}, b_{ij} defined by (2.7), (2.8).

It will be shown that $f(x,t) = -vDv$ satisfies

$$f, f_t \in L^2(0,T;L^2(0,L)).$$

We will need the following lemma.

Lemma 5.2. *For all $u \in H^1(0,L)$ we have:*

(1) *If $u(\alpha) = 0$ for some $\alpha \in [0,L]$, then*

$$\sup_{x\in(0,L)} |u(x)| \le \sqrt{2}\|u\|^{1/2}\|Du\|^{1/2}.$$

(2) *If $u(x) \neq 0, \forall x \in [0,L]$ then*

$$\sup_{x\in(0,L)} |u(x)| \le 2\|u\|_{H^1(0,L)}.$$

Proof. (1) Let $\alpha \in [0,L]$ be such that $u(\alpha) = 0$. Then for any $x \in (0,L)$

$$u^2(x) = \int_\alpha^x D_s u^2(s) ds \le 2\int_\alpha^x |u(s) D_s u(s)| ds \le 2\|u\|_{L^2(0,L)}(t)\|Du\|_{L^2(0,L)}.$$

Therefore,

$$\sup_{x\in(0,L)} |u(x)| \le \sqrt{2}\|u\|^{1/2}(t)\|Du\|^{1/2}.$$

(2) If $u(x) \neq 0 \, \forall x \in [0, L], \ L \geq 1$, consider the extension

$$\widetilde{u}(x) = \begin{cases} (1+x)u(-x), & \text{for } x \in [-1, 0] \\ u(x), & \text{for } x \in [0, L]. \end{cases}$$

Obviously, $\widetilde{u} \in H^1(-1, L)$ and $\widetilde{u}(-1) = 0$. By part 1 of this Lemma,

$$\begin{aligned} \sup_{x \in (-1,L)} |\widetilde{u}(x,t)|^2 &\leq 2\|\widetilde{u}\|_{L^2(-1,L)}(t)\|D\widetilde{u}\|_{L^2(-1,L)}(t) \\ &\leq \|\widetilde{u}\|^2_{L^2(-1,L)}(t) + \|D\widetilde{u}\|^2_{L^2(-1,L)}(t) \\ &= \|\widetilde{u}\|^2_{H^1(-1,L)}(t). \end{aligned} \tag{5.6}$$

We have

$$\begin{aligned} \|\widetilde{u}\|^2_{L^2(-1,L)}(t) &= \int_{-1}^0 (1+x)^2 u^2(-x)dx + \int_0^L u^2(x)dx \\ &\leq \int_0^1 u^2(x)dx + \int_0^L u^2(x)dx \leq 2\|u\|^2_{L^2(0,L)}(t). \end{aligned}$$

Similarly,

$$\|D\widetilde{u}\|^2_{L^2(-1,L)}(t) \leq 2\|u\|^2_{L^2(0,L)}(t) + 3\|Du\|^2_{L^2(0,L)}(t).$$

Returning to (5.6), we obtain

$$\sup_{x \in (-1,L)} |\widetilde{u}(x)|^2 \leq 4\|u\|^2_{H^1(0,L)}(t)$$

or

$$\sup_{x \in (0,L)} \|u(x)\| = \sup_{x \in (0,L)} |\widetilde{u}(x)| \leq \sup_{x \in (-1,L)} |\widetilde{u}(x)| \leq 2\|u\|_{H^1(0,L)}(t).$$

In the case $L < 1$, we use the extension

$$\widetilde{u}(x) = \begin{cases} (L+x)u(-x), & \text{for } x \in [-L, 0] \\ u(x), & \text{for } x \in [0, L], \end{cases}$$

and repeating calculations of the case $L \geq 1$, come to the same result. □

Proposition 5.3. *If $v \in B_R$, then for all $t \in (0, T)$*

$$\|Dv\|^2(t) \leq 11R^2.$$

Proof.

$$\begin{aligned} \|Dv\|^2(t) &= \|Dv\|^2(0) + \int_0^t \frac{\partial}{\partial s}\Big(\int_0^L \|Dv\|^2 dx\Big)ds \\ &\leq \|Du_0\|^2 + \int_0^T \big[\|Dv\|^2 + \|Dv_t\|^2\big]dt \\ &\leq \|Du_0\|^2 + \|v\|^2_V \leq 11R^2. \end{aligned}$$

□

Using Lemma 5.2, we obtain

$$\sup_{(x,t) \in Q_T} |v(x,t)|^2 \leq 84R^2,$$

$$\sup_{(x,t) \in Q_T} |v_t(x,t)|^2 \leq 4\|v_t\|^2_{H^1(0,L)}(t) \leq 4(10R^2 + \|Dv_t\|^2(t)).$$

For $f = -vDv$ it follows that
$$f, f_t \in L^2(0,T; L^2(0,L)).$$
Indeed,
$$\begin{aligned}
\int_0^T \int_0^L |f|^2 dxdt &= \int_0^T \int_0^L |vDv|^2 dxdt \\
&\le \int_0^T \sup_{x\in(0,L)} |v(t)|^2 \Big[\int_0^L |Dv|^2 dx \Big] dt \\
&\le \int_0^T 4\|v\|^2_{H^1(0,L)}(t)\|Dv\|^2(t)dt \\
&= 4\int_0^T \|v\|^2(t)\|Dv\|^2(t)dt + 4\int_0^T \|Dv\|^4(t)dt \\
&\le 4 \sup_{t\in(0,T)} \{\|v\|^2(t)\} \int_0^T \|Dv\|^2(t)dt + 4(11)^2R^4T \\
&\le 528R^4T < +\infty.
\end{aligned}$$
On the other hand, $f_t = -(vDv)_t$. Hence
$$\begin{aligned}
\int_0^T \int_0^L |(vDv)_t|^2 dxdt &= \int_0^T \int_0^L |v_tDv + vDv_t|^2 dxdt \\
&\le 2\Big[\int_0^T \int_0^L |v_tDv|^2 dxdt + \int_0^T \int_0^L |vDv_t|^2 dxdt \Big].
\end{aligned}$$
By Lemma 5.2 and Proposition 5.3,
$$\begin{aligned}
I_1 &= \int_0^T \int_0^L |v_tDv|^2 dxdt \\
&\le \int_0^T \sup_{x\in(0,L)} |v_t(x,t)|^2 \Big[\int_0^L |Dv|^2 dx \Big] dt \\
&\le \int_0^T \big[4\|v_t\|^2_{H^1(0,L)}(t)\|Dv\|^2(t)\big] dt \\
&= \int_0^T \Big(4\big[\|v_t\|^2_{L^2(0,L)}(t) + \|Dv_t\|^2(t)\big]\|Dv\|^2(t)\Big) dt < +\infty
\end{aligned}$$
and
$$\begin{aligned}
I_2 &= \int_0^T \int_0^L |vDv_t|^2 dxdt \\
&\le \int_0^T \sup_{x\in(0,L)} |v(x,t)|^2 \Big[\int_0^L |Dv_t|^2 dx \Big] dt \\
&\le \int_0^T 4\|v\|^2_{H^1(0,L)}(t)\|Dv_t\|^2(t)dt \\
&= 4\int_0^T \|v\|^2(t)\|Dv_t\|^2(t)dt + 4\int_0^T \|Dv\|^2(t)\|Dv_t\|^2(t)dt \\
&\le 4 \sup_{t\in(0,T)} \{\|v\|^2(t)\} \int_0^T \|Dv_t\|^2(t)dt + 4\int_0^T (11)^2R^2\|Dv_t\|^2(t)dt
\end{aligned}$$

$$\le 4\|v\|_V^4 + 4(11)^2R^2\|v\|_V^2 \le 488R^4 < +\infty.$$

Hence

$$\int_0^T\int_0^L |f_t|^2dxdt = \int_0^T\int_0^L |-(vDv)_t|^2dxdt < +\infty$$

and $f, f_t \in L^2(0,T;L^2(0,L))$.

By Theorem 4.1, we may define an operator P, related to (5.3)–(5.5), such that $u = Pv$.

Lemma 5.4. *There are a real $T_0 : 0 < T_0 \le T \le 1$ and $\gamma > 0$ such that the operator P maps B_R into B_R.*

Proof. To prove this lemma, we will need the following estimates:

Estimate 1. Multiplying (5.3) by $2u$ and integrating over $(0,L)$, we have

$$\Big(u_t,u\Big)(t) + \Big(D^3u,u\Big)(t) - \Big(D^5u,u\Big)(t) = \Big(-vDv,u\Big)(t)$$

or

$$\begin{aligned}&\frac{d}{dt}\|u\|^2(t) + K_1\Big(u^2(L,t)+[Du(L,t)]^2+u^2(0,t)+[Du(0,t)]^2+[D^2u(0,t)]^2\Big)\\ &\le \|u\|^2(t) + 484R^4.\end{aligned} \tag{5.7}$$

By the Gronwall lemma,

$$\|u\|^2(t) \le e^TR^2\Big(1+484R^2T\Big). \tag{5.8}$$

Taking $0 < T_1 \le T$ such that $e^{T_1} \le 2$ and $484R^2T_1 \le 1$, we obtain

$$\|u\|^2(t) \le 4R^2, \quad t \in [0,T_1].$$

Returning to (5.7), we obtain

$$\begin{aligned}&\|u\|^2(t) + K_1\int_0^t\Big(u^2(L,s)+[Du(L,s)]^2+u^2(0,s)+[Du(0,s)]^2+[D^2u(0,s)]^2\Big)ds\\ &\le \big[4+484R^2\big]R^2T + \|u_0\|^2.\end{aligned}$$

Taking $0 < T_2 \le T \le 1$ such that $[4+484R^2]R^2T_2 < R^2$, we obtain

$$\begin{aligned}&\|u\|^2(t) + K_1\int_0^t\Big(u^2(L,s)+[Du(L,s)]^2+u^2(0,s)+[Du(0,s)]^2+[D^2u(0,s)]^2\Big)ds\\ &\le 2R^2.\end{aligned}$$

Estimate 2. Multiply (5.3) by $(1+\gamma x)u$ to obtain

$$\begin{aligned}&\Big(u_t,(1+\gamma x)u\Big)(t) + \Big(D^3u,(1+\gamma x)u\Big)(t) - \Big(D^5u,(1+\gamma x)u\Big)(t)\\ &= -\Big(vDv,(1+\gamma x)u\Big)(t).\end{aligned} \tag{5.9}$$

We estimate:

$$I_1 = \Big(-vDv,(1+\gamma x)u\Big)(t) \le 882(1+\gamma L)R^4 + \frac12\Big(1+\gamma x,u^2\Big)(t).$$

Substituting I_1 into (5.9) gives

$$(K_1-\gamma C_L)\big[u^2(L,t)+[Du(L,t)]^2+u^2(0,t)+[Du(0,t)]^2+[D^2u(0,t)]^2\big]$$

$$
\begin{aligned}
&+ \frac{d}{dt}\Big(1+\gamma x, u^2\Big)(t) + 3\|Du\|^2(t) + 5\|D^2u\|^2(t) \\
&\le (1+\gamma L)\Big(2+1764R^2\Big)R^2\,,
\end{aligned}
$$

where C_L is a positive constant which depends on the coefficients a_{ij}, b_{ij} and L. Choosing $\gamma > 0$ such that $\gamma C_L = \frac{K_1}{2}$, we obtain

$$
\begin{aligned}
&\frac{K_1}{2}\big[u^2(L,t) + [Du(L,t)]^2 + u^2(0,t) + [Du(0,t)]^2 + [D^2u(0,t)]^2\big] \\
&+ \frac{d}{dt}\Big(1+\gamma x, u^2\Big)(t) + 3\|Du\|^2(t) + 5\|D^2u\|^2(t) \\
&\le (1+\gamma L)(2+1764R^2)R^2
\end{aligned}
$$

and for $0 < T_3 \le T \le 1$ such that $(1+\gamma L)(2+1764^2R^2)R^2T_3 \le R^2$, we obtain

$$
\int_0^t \big[\|Du\|^2(s) + \|D^2u\|^2(s)\big]ds \le \frac{2}{3}R^2. \tag{5.10}
$$

Estimate 3. Differentiating (5.3) with respect to t, multiplying the result by u_t, we have

$$
\begin{aligned}
&\Big(u_{tt}, u_t\Big)(t) + \Big(D^3u_t, u_t\Big)(t) - \Big(D^5u_t, u_t\Big)(t) \\
&= -\Big(v_tDv, u_t\Big)(t) - \Big(vDv_t, u_t\Big)(t).
\end{aligned} \tag{5.11}
$$

Using Proposition 5.3, we calculate

$$
\begin{aligned}
I_1 = \Big(-v_tDv, u_t\Big)(t) &\le \frac{1}{2\epsilon^2}\|u_t\|^2(t) + \frac{\epsilon^2}{2}\|v_tDv\|^2(t) \\
&\le \frac{1}{2\epsilon^2}\|u_t\|^2(t) + 220\epsilon^2R^4 + 22\epsilon^2R^2\|Dv_t\|^2(t)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 = \Big(-vDv_t, u_t\Big)(t) &\le \frac{1}{2\epsilon^2}\|u_t\|^2(t) + \frac{\epsilon^2}{2}\|vDv_t\|^2(t) \\
&\le \frac{1}{2\epsilon^2}\|u_t\|^2(t) + 42\epsilon^2R^2\|Dv_t\|^2(t),
\end{aligned}
$$

where ϵ is an arbitrary positive number. Substituting $I_1 - I_2$ into (5.11), we find

$$
\frac{d}{dt}\|u_t\|^2(t) \le \frac{2}{\epsilon^2}\|u_t\|^2(t) + 128R^2\epsilon^2\|Dv_t\|^2(t) + 440\epsilon^2R^4. \tag{5.12}
$$

By the Gronwall lemma,

$$
\|u_t\|^2(t) \le e^{\int_0^t \frac{2}{\epsilon^2}ds}\Big(\|u_t\|^2(0) + 128R^2\epsilon^2\int_0^t \|Dv_s\|^2(s)ds + 440\epsilon^2R^4t\Big).
$$

Taking $\epsilon > 0$ such that $1280R^2\epsilon^2 = 1$, we obtain

$$
440\epsilon^2R^4 = \frac{44}{128}R^2,
$$

$$
\|u_t\|^2(t) \le e^{\frac{2}{\epsilon^2}t}\Big(\|u_t\|^2(0) + \frac{1}{10}\int_0^t \|Dv_s\|^2(s)ds + \frac{44}{128}R^2t\Big).
$$

Since

$$
\|u_t\|^2(0) \le 3[\|u_0Du_0\|^2 + \|D^3u_0\|^2 + \|D^5u_0\|^2] \le 3R^2,
$$

and using Proposition 5.3, we obtain

$$\begin{aligned}\|u_t\|^2(t) &\le e^{\frac{2}{\epsilon^2}t}\Big(3R^2 + \int_0^t \|Dv_s\|^2(s)ds + \frac{44}{128}R^2 t\Big)\\ &\le e^{\frac{3}{\epsilon^2}t}\Big(4R^2 + \frac{44}{128}R^2 T\Big).\end{aligned}$$

Taking $0 < T_4 \le T \le 1$ such that $e^{\frac{2}{\epsilon^2}T_4} \le 2$ and $\frac{44}{128}R^2T_4 \le R^2$, we obtain

$$\|u_t\|^2(t) \le 10R^2.$$

Returning to (5.11), we obtain

$$\begin{aligned}&K_1\int_0^t \Big(u_s^2(L,s) + [Du_s(L,s)]^2 + u_s^2(0,s) + [Du_s(0,s)]^2\\ &+ [D^2u_s(0,s)]^2\Big)ds + \|u_t\|^2(t)\\ &\le \frac{20}{\epsilon^2}R^2T + \frac{44}{128}R^2T + 4R^2.\end{aligned}$$

For $0 < T_5 \le T \le 1$ sufficiently small, we obtain

$$\begin{aligned}&K_1\int_0^t \Big(u_s^2(L,s) + [Du_s(L,s)]^2 + u_s^2(0,s)\\ &+ [Du_s(0,s)]^2 + [D^2u_s(0,s)]^2\Big)ds + \|u_t\|^2(t)\\ &\le 5R^2.\end{aligned}$$

Estimate 4: Differentiating (5.3) with respect to t, multiplying the result by $(1+\gamma x)u_t$ and integrating over $(0,t)$, we have

$$\begin{aligned}&\Big(u_{tt}, (1+\gamma x)u_t\Big)(t) + \Big(D^3u_t, (1+\gamma x)u_t\Big)(t) - \Big(D^5u_t, (1+\gamma x)u_t\Big)(t)\\ &= \Big(-v_tDv, (1+\gamma x)u_t\Big)(t) + \Big(-vDv_t, (1+\gamma x)u_t\Big)(t).\end{aligned} \tag{5.13}$$

We estimate

$$\begin{aligned}I_1 &= \Big(-v_tDv, (1+\gamma x)u_t\Big)(t)\\ &\le (1+\gamma L)\big[220\epsilon^2R^4 + 22\epsilon^2R^2\|Dv_t\|^2(t) + \frac{1}{2\epsilon^2}\|u_t\|^2(t)\big],\end{aligned}$$

$$I_2 = \Big(-vDv_t, (1+\gamma x)u_t\Big)(t) \le (1+\gamma L)\big[42\epsilon^2R^2\|Dv_t\|^2(t) + \frac{1}{2\epsilon^2}\|u_t\|^2(t)\big],$$

where ϵ is an arbitrary positive number. Substituting $I_1 - I_2$ into (5.13) and using previous estimates, we find

$$\begin{aligned}&(K_1 - \gamma C_L)\Big(u_t^2(L,t) + [Du_t(L,t)]^2 + u_t^2(0,t) + [Du_t(0,t)]^2\\ &+ [D^2u_t(0,t)]^2\Big) + \frac{d}{dt}(1+\gamma x, u_t^2)(t) + 3\|Du_t\|^2(t) + 5\|D^2u_t\|^2(t)\\ &\le (1+\gamma L)\big[\frac{2}{\epsilon^2}\|u_t\|^2(t) + 128R^2\epsilon^2\|Dv_t\|^2(t) + 440\epsilon^2R^4\big]\\ &\le (1+\gamma L)\big[\frac{10}{\epsilon^2}R^2 + 128R^2\epsilon^2\|Dv_t\|^2(t) + 440\epsilon^2R^4\big].\end{aligned} \tag{5.14}$$

Integrating over $(0,t)$, we find

$$\begin{aligned}
&3\int_0^t \big[\|Du_s\|^2(s)+\|D^2u_s\|^2(s)\big]ds\\
&\leq (1+\gamma L)\big[\frac{10}{\epsilon^2}R^2T+128R^2\epsilon^2\int_0^t \|Dv_s\|^2(s)ds+440\epsilon^2R^4T\big]+3R^2.
\end{aligned}$$

Taking $\epsilon>0$ such that $1280(1+\gamma L)R^2\epsilon^2=1$ for a fixed $\gamma>0$, we obtain

$$3\int_0^t \big[\|Du_s\|^2(s)+\|D^2u_s\|^2(s)\big]ds \leq \frac{10}{\epsilon^2}(1+\gamma L)R^2T+4R^2+\frac{44}{128}R^2T$$

and choosing $0<T_6\leq T\leq 1$ such that $(1+\gamma L)\frac{10}{\epsilon^2}R^2T_6\leq \frac{R^2}{2}$ for fixed γ,ϵ^2 and $\frac{44}{128}R^2T_6\leq \frac{R^2}{2}$, we obtain

$$\int_0^t \big[\|Du_s\|^2(s)+\|D^2u_s\|^2(s)\big]ds\leq \frac{5}{3}R^2.$$

Putting $T_0=\min_{1\leq i\leq 6}\{T_i\}$, we find

$$\|u\|_V^2\leq \frac{28}{3}R^2;$$

therefore $\|u\|_V\leq \sqrt{10}R$. The proof is complete. $\square$

Lemma 5.5. *For $T_0>0$ sufficiently small, the operator P is a contraction mapping in B_R.*

Proof. For $v_1,v_2\in B_R$ denote

$$u_i=Pv_i,\quad i=1,2,\quad w=v_1-v_2\quad\text{and}\quad z=u_1-u_2$$

which satisfies the initial boundary problem

$$z_t+D^3z-D^5z=-\frac{1}{2}(v_1+v_2)Dw-\frac{1}{2}wD\big(v_1+v_2\big)\quad\text{in } Q_{T_0}, \tag{5.15}$$

$$z(x,0)=0,\quad x\in(0,L), \tag{5.16}$$

$$\begin{gathered}
D^iz(0,t)=\sum_{j=0}^{2}a_{ij}D^jz(0,t),\quad i=3,4,\quad t\in[0,T_0],\\
D^iz(L,t)=\sum_{j=0}^{1}b_{ij}D^jz(L,t),\quad i=2,3,4,\quad t\in[0,T_0].
\end{gathered}\tag{5.17}$$

Define the metric

$$\begin{aligned}
\rho^2(v_1,v_2)&=\rho^2(w)\\
&=\sup_{t\in[0,T_0]}\Big\{\|w\|^2(t)+\|w_t\|^2(t)\Big\}\\
&\quad+\int_0^{T_0}\sum_{i=1}^{2}\big[\|D^iw\|^2(t)+\|D^iw_t\|^2(t)\big]dt.
\end{aligned}$$

Multiplying (5.15) by z, we obtain

$$\begin{aligned} &\frac{d}{dt}\|z\|^2(t) + K_1\Big(z^2(L,t) + [Dz(L,t)]^2 + z^2(0,t) \\ &+ [Dz(0,t)]^2 + [D^2z(0,t)]^2\Big) \\ &\le -\Big((v_1+v_2)Dw, z\Big)(t) - \Big(wD(v_1+v_2), z\Big)(t). \end{aligned} \tag{5.18}$$

We estimate

$$I_1 = -\Big((v_1+v_2)Dw, z\Big)(t) \le 84\epsilon^2R^2\|Dw\|^2(t) + \frac{1}{\epsilon^2}\|z\|^2(t),$$

$$I_2 = -\Big(wD(v_1+v_2), z\Big)(t) \le 44R^2\epsilon^2\Big(\|w\|^2(t) + \|Dw\|^2(t)\Big) + \frac{1}{\epsilon^2}\|z\|^2(t),$$

where ϵ is an arbitrary positive number. Substituting $I_1 - I_2$ in (5.18), we obtain

$$\frac{d}{dt}\|z\|^2(t) \le \frac{2}{\epsilon^2}\|z\|^2(t) + 128R^2\epsilon^2[\|w\|^2(t) + \|Dw\|^2(t)].$$

Choosing $\epsilon^2 = 128R^2/8$ and using the Gronwall Lemma,

$$\|z\|^2(t) \le \frac{1}{8}e^{\frac{2}{\epsilon^2}T_0}\Big(T_0 \sup_{t\in(0,T_0)}\{\|w\|^2(t)\} + \int_0^{T_0}\|Dw\|^2(t)dt\Big).$$

Taking $0 < T_0 \le 1$ such that $e^{\frac{2}{\epsilon^2}T_0} < 2$, we have

$$\|z\|^2(t) \le \frac{1}{4}\rho^2(w), \quad t \in (0,T_0).$$

Returning to (5.18) and integrating over $(0,t)$, we obtain

$$\begin{aligned} &K_1\int_0^t \Big(z^2(L,s) + [Dz(L,s)]^2 + z^2(0,s) + [Dz(0,s)]^2 \\ &+ [D^2z(0,s)]^2\Big)ds + \|z\|^2(t) \\ &\le \big[\frac{1}{2\epsilon^2}T_0 + 128R^2\epsilon^2\big]\rho^2(w), \quad \forall t \in [0,T_0]. \end{aligned} \tag{5.19}$$

Multiplying (5.15) by $(1+\gamma x)z$ and integrating over $(0,L)$, we obtain

$$\begin{aligned} &\Big(z_t, (1+\gamma x)z\Big)(t) + \Big(D^3z, (1+\gamma x)z\Big)(t) - \Big(D^5z, (1+\gamma x)z\Big)(t) \\ &= -\frac{1}{2}\Big((v_1+v_2)Dw, (1+\gamma x)z\Big)(t) - \frac{1}{2}\Big(wD(v_1+v_2), (1+\gamma x)z\Big)(t). \end{aligned} \tag{5.20}$$

We estimate

$$\begin{aligned} I_3 &= -\frac{1}{2}\Big((v_1+v_2)Dw, (1+\gamma x)z\Big)(t) \\ &\le (1+\gamma L)\Big(42\epsilon^2R^2\|Dw\|^2(t) + \frac{1}{2\epsilon^2}\|z\|^2(t)\Big) \end{aligned}$$

and

$$\begin{aligned} I_4 &= -\frac{1}{2}\Big(wD(v_1+v_2), (1+\gamma x)z\Big)(t) \\ &\le (1+\gamma L)\big[22R^2\epsilon^2\Big(\|w\|^2(t) + \|Dw\|^2(t)\Big) + \frac{1}{2\epsilon^2}\|z\|^2(t)\big]. \end{aligned}$$

Substituting $I_3 - I_4$ in (5.20) and integrating over $(0,t)$, we obtain

$$
\begin{aligned}
&(K_1-\gamma C_L)\int_0^t \Big(z^2(L,s)+[Dz(L,s)]^2+z^2(0,s)+[Dz(0,s)]^2+[D^2z(0,s)]^2\Big)ds\\
&+\|z\|^2(t)+3\int_0^t\Big(\|Dz\|^2(s)+\|D^2z\|^2(s)\Big)ds\\
&\le 128(1+\gamma L)R^2\epsilon^2\int_0^t\|Dw\|^2(s)ds+44(1+\gamma L)R^2\epsilon^2\int_0^t\|w\|^2(s)ds\\
&\quad+\frac{2}{\epsilon^2}(1+\gamma L)\Big(\frac{1}{2\epsilon^2}T_0+128R^2\epsilon^2\Big)\rho^2(w)t.
\end{aligned}
$$

Taking $\epsilon > 0$ such that for a fixed $\gamma > 0$

$$
128(1+\gamma L)R^2\epsilon^2=\frac{1}{4},
$$

we have

$$
\begin{aligned}
&\|z\|^2(t)+3\int_0^t\Big(\|Dz\|^2(s)+\|D^2z\|^2(s)\Big)ds\\
&\le\frac{1}{4}\rho^2(w)+\frac{2}{\epsilon^2}(1+\gamma L)\Big(\frac{1}{2\epsilon^2}T_0+128R^2\epsilon^2\Big)T_0\rho^2(w).
\end{aligned}
$$

Taking $0 < T_0 \le 1$ such that $\frac{2}{\epsilon^2}(1+\gamma L)\Big(\frac{1}{2\epsilon^2}T_0+128R^2\epsilon^2\Big)T_0\le\frac{1}{4}$, we obtain

$$
\|z\|^2(t)+\int_0^t\Big(\|Dz\|^2(s)+\|D^2z\|^2(s)\Big)ds\le\frac{1}{2}\rho^2(w). \tag{5.21}
$$

Then

$$
\rho^2(z)\le\frac{1}{2}\rho^2(w).
$$

This completes the proof. □

Remark 5.6. The estimate (5.21) partially implies that the data-solution map is continuous. More precisely, let u_0, $\overline{u_0}$ satisfy the conditions of Theorem 2.2 and let $u, \overline{u}$ be corresponding solutions of (2.1)-(2.3). Then $\forall\varepsilon\ \exists\delta=\delta(\varepsilon,T,max\{u_0,\overline{u_0}\})$ such that

$$
\|u_0-\overline{u_0}\|<\delta \implies \|u-\overline{u}\|(t)<\varepsilon \ \ \textit{for all}\ \ 0<t<T.
$$

Lemmas 5.4 and 5.5 imply that P is a contraction mapping in B_R. By the Banach fixed-point theorem, there exists a unique generalized solution $u = u(x,t)$ of the problem (2.1)–(2.3) such that

$$
u,u_t\in L^\infty(0,T_0;L^2(0,L))\cap L^2(0,T_0;H^2(0,L)).
$$

Consequently, $Du \in L^\infty(0,T_0;L^2(0,L))$.

Rewriting (2.1) in the form

$$
D^3u-D^5u+u=u-u_t-uDu=G(x,t),
$$

it is easy to see that $G(x,t) \in L^\infty(0,T_0;L^2(0,L))$. By Theorem 3.1, we have that $u \in L^\infty(0,T_0;H^5(0,L))$. Hence, $G \in L^2(0,T_0;H^2(0,L))$ which implies $u \in L^\infty(0,T_0;H^5(0,L))\cap L^2(0,T_0;H^7(0,L))$. Theorem 5.1 is proved. □

6. Global solutions. Exponential decay

In this section we prove global solvability and decay of small solutions for the nonlinear problem

$$u_t + uDu + D^3u - D^5u = 0, \quad x \in (0, L), \quad t > 0; \tag{6.1}$$

$$u(x, 0) = u_0(x), \quad x \in (0, L); \tag{6.2}$$

$$\begin{gathered} D^iu(0, t) = \sum_{j=0}^{2} a_{ij}D^ju(0, t), \quad i = 3, 4, \quad t > 0, \\ D^iu(L, t) = \sum_{j=0}^{1} b_{ij}D^ju(L, t), \quad i = 2, 3, 4, \quad t > 0, \end{gathered} \tag{6.3}$$

where the coefficients a_{ij} and b_{ij} are real constants satisfying (2.8).

Proof of Theorem 2.2. The existence of local regular solutions follows from Theorem 5.1. Hence, we need global in t a priori estimates of these solutions in order to prolong them for all $t > 0$.

Estimate 1. Multiplying (6.1) by $2(1 + \gamma x)u$, integrating the result by parts and taking into account (6.3), one gets

$$\begin{aligned} &\frac{d}{dt}\Big(1 + \gamma x, u^2\Big)(t) + 2\Big((1 + \gamma x)u^2, Du\Big)(t) + (K_1 - \gamma C_L)\Big(u^2(L, t) \\ &+ \big[Du(L, t)\big]^2 + u^2(0, t) + \big[Du(0, t)\big]^2 + \big[D^2u(0, t)\big]^2\Big) + 3\gamma\|Du\|^2(t) \\ &+ 5\gamma\|D^2u\|^2(t) \le 0. \end{aligned} \tag{6.4}$$

Taking γ such that $0 < L\gamma \le 1$, we estimate,

$$2\Big(1 + \gamma x, u^2Du\Big)(t) \le 2\delta|u(0, t)|^2 + \Big(2\delta L + \frac{4}{\delta}\|u\|^2(t)\Big)\|Du\|^2(t),$$

where δ is an arbitrary positive number. Then (6.4) reads

$$\begin{aligned} &\frac{d}{dt}\Big(1 + \gamma x, u^2\Big)(t) - \big[2\delta L + \frac{4}{\delta}\|u\|^2(t)\big]\|Du\|^2(t) + (K_1 - \gamma C_L - 2\delta)\Big(u^2(L, t) \\ &+ \big[Du(L, t)\big]^2 + u^2(0, t) + \big[Du(0, t)\big]^2 + \big[D^2u(0, t)\big]^2\Big) + 3\gamma\|Du\|^2(t) \\ &+ 5\gamma\|D^2u\|^2(t) \le 0. \end{aligned}$$

Since

$$\begin{gathered} \|Du\|^2(t) \ge \frac{1}{2L^2}\|u\|^2(t) - \frac{1}{L}|u(0, t)|^2, \\ \|D^2u\|^2(t) \ge \frac{1}{2L^2}\|Du\|^2(t) - \frac{1}{L}|Du(0, t)|^2, \\ \|D^2u\|^2(t) \ge \frac{1}{4L^4}\|u\|^2(t) - \frac{1}{2L^3}|u(0, t)|^2 - \frac{1}{L}|Du(0, t)|^2, \end{gathered}$$

it follows that

$$\begin{aligned} &\frac{d}{dt}\Big(1 + \gamma x, u^2\Big)(t) + 2\big[\gamma\Big(1 + \frac{2}{L^2}\Big) - \delta L - \frac{2}{\delta}\|u\|^2(t)\big]\|Du\|^2(t) + \gamma\|Du\|^2(t) \\ &+ \gamma\|D^2u\|^2(t) + (K_1 - \gamma C_L - 2\delta - \frac{4\gamma}{L})\big[u^2(L, t) + \big[Du(L, t)\big]^2 + u^2(0, t) \\ &+ \big[Du(0, t)\big]^2 + \big[D^2u(0, t)\big]^2\big] \le 0. \end{aligned}$$

Taking $\delta = 2\gamma/L^3$, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(1+\gamma x, u^2\Big)(t) + 2\Big(\gamma - \frac{L^3}{\gamma}\Big(1+\gamma x, u^2\Big)(t)\Big)\|Du\|^2(t) + \gamma\|Du\|^2(t)\\&+ \gamma\|D^2u\|^2(t) + \Big(K_1 - \gamma C_L - \frac{4\gamma}{L^3} - \frac{4\gamma}{L}\Big)\big[u^2(L,t) + \big[Du(L,t)\big]^2 + u^2(0,t)\\&+ \big[Du(0,t)\big]^2 + \big[D^2u(0,t)\big]^2\big] \le 0.\end{aligned}$$

Choosing $\gamma > 0$ sufficiently small, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(1+\gamma x, u^2\Big)(t) + 2\big[\gamma - \frac{L^3}{\gamma}\Big(1+\gamma x, u^2\Big)(t)\big]\|Du\|^2(t) + \gamma\|Du\|^2(t)\\&+ \gamma\|D^2u\|^2(t) + \frac{K_1}{2}\Big[u^2(L,t) + \big[Du(L,t)\big]^2 + u^2(0,t)\\&+ \big[Du(0,t)\big]^2 + \big[D^2u(0,t)\big]^2\Big] \le 0.\end{aligned}$$

Since $\big(1+\gamma x, u_0^2\big) < \frac{\gamma^2}{2L^3}$, then $(1+\gamma x, u^2)(t) < \frac{\gamma^2}{2L^3}$ for all $t > 0$ [13]. Hence, for $\gamma > 0$ sufficiently small

$$\frac{d}{dt}\Big(1+\gamma x, u^2\Big)(t) + \Big(\frac{4L^2+1}{4L^4}\Big)\Big(\frac{\gamma}{1+\gamma L}\Big)\Big(1+\gamma x, u^2\Big)(t) \le 0.$$

By the Gronwall lemma,

$$\big(1+\gamma x, u^2\big)(t) \le e^{-\chi t}\big(1+\gamma x, u_0^2\big),$$

where $\chi = \frac{(4L^2+1)\gamma}{4L^4(1+\gamma L)}$.

Returning to (6.4), using assumption (2.8) and choosing $\gamma > 0$ sufficiently small, we obtain

$$\begin{aligned}&\int_0^t \big[|u(L,s)|^2 + |Du(L,s)|^2 + |u(0,s)|^2 + |Du(0,s)|^2 + |D^2u(0,s)|^2\big]\,ds\\&+ \Big(1+\gamma x, u^2\Big)(t) + \|u\|^2(t) + \int_0^t \big[\|Du\|^2(s) + \|D^2u\|^2(s)\big]\,ds \le C\|u_0\|^2,\end{aligned} \tag{6.5}$$

where C is a positive number.

Estimate 2. Differentiate (6.1)–(6.2) with respect to t, multiply the result by $2(1+\gamma x)u_t$ to obtain

$$\begin{aligned}&\frac{d}{dt}\Big(1+\gamma x, u_t^2\Big)(t) + 2\Big((1+\gamma x)uu_t, Du_t\Big)(t) + 2\Big((1+\gamma x)u_t^2, Du\Big)(t)\\&+ (K_1 - \gamma C_L)\big[u_t^2(L,t) + \big[Du_t(L,t)\big]^2 + u_t^2(0,t) + \big[Du_t(0,t)\big]^2\\&+ \big[D^2u_t(0,t)\big]^2\big] + 3\gamma\|Du_t\|^2(t) + 5\gamma\|D^2u_t\|^2(t) \le 0.\end{aligned}$$

For $\delta \in (0,1)$ and $0 < L\gamma \le 1$, we estimate

$$\begin{aligned}&2\Big((1+\gamma x)uu_t, Du_t\Big)(t)\\&\le 2\gamma\|Du_t\|^2(t) + \frac{4}{\gamma}\Big(|u(0,t)|^2 + L\|Du\|^2(t)\Big)\Big(1+\gamma x, u_t^2\Big)(t)\end{aligned}$$

and

$$2\Big((1+\gamma x)u_t^2, Du\Big)(t) \le \Big(1 + 2[Du(0,t)]^2 + 2L\|D^2u\|^2(t)\Big)\Big(1+\gamma x, u_t^2\Big)(t).$$

This implies

$$
\begin{aligned}
&(K_1-\gamma C_L)(u_t^2(L,t)+[Du_t(L,t)]^2+u_t^2(0,t)+[Du_t(0,t)]^2+[D^2u_t(0,t)]^2)\\
&+\frac{d}{dt}\Big(1+\gamma x,u_t^2\Big)(t)\le\Big(\frac{4}{\gamma}\big[u(0,t)^2+L\|Du\|^2(t)\big]\\
&+\big[1+2[Du(0,t)]^2+2L\|D^2u\|^2(t)\big]\Big)\Big(1+\gamma x,u_t^2\Big)(t).
\end{aligned}
\tag{6.6}
$$

Taking $2C_L\gamma\in(0,K_1)$ and remembering that due to (6.5) $u^2(0,t)+\|Du\|^2(t)\in L^1(0,t)$, by the Gronwall lemma,

$$
\Big(1+\gamma x,u_t^2\Big)(t)\le C\|u_0\|^2_{H^5(0,L)}. \tag{6.7}
$$

Returning to (6.6), we obtain

$$
\begin{aligned}
&\int_0^t\Big(u_s^2(L,s)+\big[Du_s(L,s)\big]^2+u_s^2(0,s)+\big[Du_s(0,s)\big]^2+\big[D^2u_s(0,s)\big]^2\Big)ds\\
&+\Big(1+\gamma x,u_t^2\Big)(t)+\int_0^t\big[\|Du_s\|^2(s)+\|D^2u_s\|^2(s)\big]ds\\
&\le C\|u_0\|^2_{H^5(0,L)}.
\end{aligned}
$$

It remains to prove that

$$
u\in L^\infty(0,T;H^5(0,L))\cap L^2(0,T;H^7(0,L)).
$$

We estimate

$$
\begin{aligned}
\|uDu\|(t)&\le\sup_{x\in(0,L)}\Big\{|u(x,t)|\Big\}\|Du\|(t)\\
&\le\Big(|u(0,t)|+\sqrt{L}\|Du\|(t)\Big)\|Du\|(t)\\
&\le\Big(|u(0,0)|+\int_0^t|u_s(0,s)|ds+\sqrt{L}\|Du\|(t)\Big)\|Du\|(t)\\
&\le2\big[|u(0,0)|^2+L\int_0^T|u_t(0,t)|^2dt\big]\\
&\quad+(2L+1)\big[\|Du_0\|^2+\int_0^T\Big\{\|Du\|^2(t)+\|Du_t\|^2(t)\Big\}\,dt\big]\\
&\le C\big[\|u_0\|^2_{H^1(0,L)}+\int_0^T\Big(u_t^2(0,t)+\|Du\|^2(t)+\|Du_t\|^2(t)\Big)\,dt\big]<+\infty.
\end{aligned}
$$

Hence $\|uDu\|(t)\in L^\infty(0,T)$ and $uDu\in L^\infty(0,T;L^2(0,L))$. Rewriting (6.1) as

$$
u+D^3u-D^5u=u-u_t-uDu,
$$

we have $u-u_t-uDu\in L^\infty(0,T;L^2(0,L))$. By Theorem 3.1, $u\in L^\infty(0,T;H^5(0,L))$. In turn, this implies $u-u_t-uDu\in L^\infty(0,T;H^2(0,L))$. And again by Theorem 3.1, $u\in L^\infty(0,T;H^5(0,L))\cap L^2(0,T;H^7(0,L))$.

Finally, a unique solution of (6.1)-(6.3) is from the class

$$
\begin{gathered}
u\in L^\infty(0,T;H^5(0,L))\cap L^2(0,T;H^7(0,L))\\
u_t\in L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2(0,L)).
\end{gathered}
$$

The proof of Theorem 2.2 is complete. $\square$

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