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NON-EXISTENCE OF SOLUTIONS FOR TWO-POINT FRACTIONAL AND THIRD-ORDER BOUNDARY-VALUE PROBLEMS

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Abstract. In this article, we provide sufficient conditions for the non-existence of solutions of the boundary-value problems with fractional derivative of order $\alpha \in (2,3)$ in the Riemann-Liouville sense

$$D_{0+}^{\alpha} x(t) + \lambda a(t) f(x(t)) = 0, \quad t \in (0,1),$$
$$x(0) = x'(0) = x'(1) = 0,$$

and in the Caputo sense

$${}^{C}D^{\alpha} x(t) + f(t, x(t)) = 0, \quad t \in (0,1),$$
$$x(0) = x'(0) = 0, \quad x(1) = \lambda \int_0^1 x(s) ds;$$

and for the third-order differential equation

$$x'''(t) + (Fx)(t) = 0, \quad \text{a.e. } t \in [0,1],$$

associated with three among the following six conditions

$$x(0) = 0, \quad x(1) = 0, \quad x'(0) = 0, \quad x'(1) = 0, \quad x''(0) = 0, \quad x''(1) = 0.$$

Thus, fourteen boundary-value problems at resonance and six boundary-value problems at non-resonanse are studied. Applications of the results are, also, given.

1. Introduction

The problem of existence of solutions of fractional differential equations and general third-order differential equations, satisfying two-point boundary conditions, has been extensively discussed in the literature, see, e.g., [1, 2, 8, 9, 11, 13, 14, 16, 20, 21, 22, 23, 27, 37, 38, 40, 41] and the references therein.

In this work we are dealing with non-existence of solutions and this, because the two problems (namely, existence and non-existence) are equally important in the theory of differential equations. To our knowledge, the problem of non-existence for such boundary-value problems has not been studied sufficiently. Indeed, only a few results have been given on it, and not in a systematic research. See, e.g., [6, 15, 28, 34, 42].

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Let I be the interval $[0,1]$. We are going to give sufficient conditions for the non-existence of solutions of the two boundary-value problems

$$D^{\alpha}_{0+}x(t)+\lambda a(t)f(x(t))=0,\quad 0<t<1,\quad 2<\alpha<3, \tag{1.1}$$
$$x(0)=x'(0)=x'(1)=0, \tag{1.2}$$

(see, e.g., [6]) with fractional Riemann-Liouville type derivative, and

$$^{C}D^{\alpha}x(t)+f(t,x(t))=0,\quad 0<t<1,\quad 2<\alpha<3, \tag{1.3}$$
$$x(0)=x''(0)=0,\quad x(1)=\lambda\int_0^1 x(s)ds \tag{1.4}$$

(see, e.g., [3]), with fractional Caputo type derivative. The same problem is, also, discussed for the third-order differential equation of the form

$$x'''(t)+(Fx)(t)=0,\quad a.e.\quad t\in I:=[0,1], \tag{1.5}$$

associated with the following boundary value conditions:

$$x(0)=x(1)=x'(0)=0, \tag{1.6}$$
$$x(0)=x(1)=x'(1)=0, \tag{1.7}$$
$$x(0)=x(1)=x''(0)=0, \tag{1.8}$$
$$x(0)=x(1)=x''(1)=0, \tag{1.9}$$
$$x(0)=x'(0)=x'(1)=0, \tag{1.10}$$
$$x(1)=x'(0)=x'(1)=0, \tag{1.11}$$
$$x(0)=x'(0)=x''(0)=0, \tag{1.12}$$
$$x(0)=x'(0)=x''(1)=0, \tag{1.13}$$
$$x(0)=x'(1)=x''(0)=0, \tag{1.14}$$
$$x(1)=x'(0)=x''(1)=0, \tag{1.15}$$
$$x(1)=x'(1)=x''(0)=0, \tag{1.16}$$
$$x(0)=x'(1)=x''(1)=0, \tag{1.17}$$
$$x(1)=x'(0)=x''(0)=0, \tag{1.18}$$
$$x(1)=x'(1)=x''(1)=0, \tag{1.19}$$
$$x'(0)=x'(1)=x''(0)=0, \tag{1.20}$$
$$x'(0)=x'(1)=x''(1)=0, \tag{1.21}$$
$$x(0)=x''(0)=x''(1)=0, \tag{1.22}$$
$$x(1)=x''(0)=x''(1)=0, \tag{1.23}$$
$$x'(0)=x''(0)=x''(1)=0, \tag{1.24}$$
$$x'(1)=x''(0)=x''(1)=0. \tag{1.25}$$

Conditions (1.6)-(1.19) lead to non-resonance boundary-value problems and they are examined in section 5. On the other hand conditions (1.20), (1.21) formulate boundary-value problems at resonance, and they are investigated in the last section 6. Existence of solutions of problems at non-resonance with this kind of boundary conditions, is investigated in a great number of works, see e.g. [4, 5], [7]-[13], [17]-[20], [24]-[26], [30]-[36], [39].

2. Preliminaries and some general results

Let $C(I,\mathbb{R})$ be the (Banach) space of all continuous functions $y : I \to \mathbb{R}$ endowed with the sup-norm $\|\cdot\|$. Let, also, U be a subspace of $C(I,\mathbb{R})$ and $F : U \to C(I,\mathbb{R})$ a continuous function. Finally, let T be an operator defined on a subset of $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ with range in $C(I,\mathbb{R})$. The question refers to the existence of solutions of the operator equation

$$y = T(y, F(y)). \tag{2.1}$$

Assume that the operator T satisfies the condition

$$\|T(y,u)\| \le K\|u\|,$$

for all (y,u) in the domain of T. Let K_T be the infimum of all such numbers. For instance, if an operator depends only on the second argument u and it is defined by an integral of the form

$$(Ty)(t) = \int_0^1 G(t,s)y(s)ds, \quad t \in I, \tag{2.2}$$

where the kernel G is a continuous function defined on the square $[0,1]\times[0,1]$, then

$$K_T = \sup_{t\in I} \int_0^1 |G(t,s)|ds.$$

From the definition of K_T we have the following result.

Theorem 2.1. *If it holds*

$$\|Fu\| < \frac{1}{K_T}\|u\|, \quad u \in C(I,\mathbb{R}), \tag{2.3}$$

then there is no solution of equation (2.1).

An equivalent result is the following.

Corollary 2.2. *Under the previous conditions, the set*

$$\{x \in C(I,\mathbb{R}) : \|Fx\| < \frac{1}{K_T}\|x\|\}$$

does not contain any solution of equation (2.1).

The inequality in Theorem 2.1 can be guaranteed if, the function F satisfies

$$L_F < \frac{1}{K_T}, \tag{2.4}$$

where

$$L_F := \sup\{\frac{\|Fx\|}{\|x\|} < +\infty : x \in U, \quad 0 < \|x\|\}. \tag{2.5}$$

Our main purpose is to give information about the number K_T for several operators generated from boundary-value problems mentioned in the introduction and then to check for the applicability of inequality (2.4).

An interesting case, which will be used in the applications given later, is when the response F is a Nemytskii-type operator. Indeed, assume that U is the space $C^{(2)}(I,\mathbb{R})$. Also, let $F : U \to C(I,\mathbb{R})$ be defined via a continuous function $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$, by the type

$$(Fx)(t) := f(t, x(t), x'(t), x''(t)), \quad t \in I.$$

Then we have the following corollary.

Corollary 2.3. *If the condition*

$$\sup_{t\in I}\sup_{u_0,u_1,u_2\in\mathbb{R}}\frac{|f(t,u_0,u_1,u_2)|}{|u_0|}<\frac{1}{K_T}$$

holds, then no solution of problem (2.1) *exists.*

Proof. Let k be such that

$$\sup_{t\in I}\sup_{u_0,u_1,u_2\in\mathbb{R}}\frac{|f(t,u_0,u_1,u_2)|}{|u_0|}<k<\frac{1}{K_T}$$

Then, the result follows from Theorem 2.1 and the fact that for any function x, with continuous derivative of second order and some points $t_1,t_2\in I$ we have

$$\begin{aligned}\frac{\|f(\cdot,x(\cdot),x'(\cdot),x''(\cdot))\|}{\|x(\cdot)\|}&=\frac{|f(t_1,x(t_1),x'(t_1),x''(t_1))|}{|x(t_2)|}\\&\leq\frac{|f(t_1,x(t_1),x'(t_1),x''(t_1))|}{|x(t_1)|}\\&\leq\sup_{t\in I}\sup_{u_0,u_1,u_2\in\mathbb{R}}\frac{|f(t,u_0,u_1,u_2)|}{|u_0|}<k.\end{aligned}$$

□

Warning: In the sequel when an operator T is generated from a boundary-value problem (a)-(b) instead of K_T we shall write $K_{(a)-(b)}$.

3. Nonexistence for the fractional BVP (1.1)-(1.2)

We start with the fractional boundary-value problem (1.1)-(1.2), studied in [6], where (the Krasnoselskii's fixed point theorem in cones is applied and sufficient conditions for) the existence of solutions is investigated.

Here the parameter λ is a positive real number, $a:(0,1)\to[0,+\infty)$ is a continuous function with $\int_0^1 a(t)dt>0$, and $f:[0,+\infty)\to[0,+\infty)$ is continuous. The symbol $D^\alpha_{0+}u$ represents the Riemann-Liouville fractional derivative of the continuous function $u:I\to\mathbb{R}$ of order $\alpha\in(2,3)$, (see, e.g., [29]), i.e. the quantity defined by

$$D^\alpha_{0+}u(t)=\frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dt}\Big)^n\int_0^t\frac{u(s)}{(t-s)^{\alpha-n+1}}ds,\quad n=\lfloor\alpha\rfloor+1.$$

Theorem 3.1. *Assume that f satisfies the condition*

$$L_f:=\sup_{|u|>0}\frac{|f(u)|}{|u|}<\frac{\Gamma(\alpha)}{\lambda\int_0^1 s(1-s)^{\alpha-2}a(s)ds}.\tag{3.1}$$

Then problem (1.1)-(1.2) *does not admit solutions.*

Proof. According to [6], the problem is equivalent to the operator equation (2.1), where the operator T has the integral form (2.2) with the Green's function G being defined by the type

$$G(t,s):=\begin{cases}\frac{(1-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)}a(s),\quad 0\leq t\leq s\leq 1,\\ \Big[\frac{(1-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big]a(s),\quad 0\leq s\leq t\leq 1.\end{cases}$$

Here $\Gamma(\alpha)$ is the gamma function at α. It is not hard to see that G takes positive values and it satisfies

$$G(t,s) \le G(1,s), \quad (t,s) \in [0,1] \times [0,1].$$

Therefore, the result will follow from Theorem 2.1 and inequality (2.4), when we know that the quantity in the right side of relation (3.1) is less than or equal to $1/K_{(1.1)-(1.2)}$. So, we have to prove this fact. Indeed, we have

$$\begin{aligned}
K_{(1.1)-(1.2)} &= \lambda \sup_{t\in I} \int_0^1 G(t,s)a(s)ds \\
&= \lambda \int_0^1 G(1,s)a(s)ds \\
&= \lambda \frac{1}{\Gamma(\alpha)} \int_0^1 \Big[(1-s)^{\alpha-2} - (1-s)^{\alpha-1}\Big] a(s)ds \\
&= \lambda \frac{1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} a(s)ds,
\end{aligned}$$

from which the result follows. □

Application 3.2. Consider the values

$$\alpha = 2.7, \quad a(t) := 2t+3, \quad f(u) := \frac{8u^2+u}{u+1}(4+\sin(u)),$$

as they appear in [6]. Then we have

$$\begin{aligned}
\int_0^1 s(1-s)^{\alpha-2} a(s)ds &= \int_0^1 (1-s)^{\alpha-2}(2s^2+3s)ds \\
&= \int_0^1 s^{\alpha-2}(2(1-s)^2 + 3(1-s))ds \\
&= \int_0^1 s^{\alpha-2}(5-7s+2s^2)ds = \frac{3\alpha+7}{(\alpha-1)\alpha(\alpha+1)}.
\end{aligned}$$

and therefore it follows that

$$\frac{\Gamma(\alpha)}{\lambda \int_0^1 s(1-s)^{\alpha-2}a(s)ds} = \frac{(\alpha-1)\Gamma(\alpha+2)}{\lambda(3\alpha+7)} \approx \frac{26.23339972}{15.1\lambda}.$$

Since it holds

$$\frac{|f(u)|}{|u|} = \frac{8u+1}{u+1}(4+\sin(u)) \le 40,$$

we have, also,

$$\frac{\|f(u(\cdot))\|}{\|u(\cdot)\|} \le 40,$$

which, due to Theorem 2.1, shows that, if the parameter λ is chosen such that

$$\lambda < \frac{26.23339972}{40 \times 15.1} = \frac{26.23339972}{604} = 0.04343278,$$

then the problem has no (any, and not necessarily positive) solution. This upper bound of the parameter λ agrees with the value of the parameter suggested in [6].

4. Nonexistence for the fractional BVP (1.3)-(1.4)

Here we discuss non-existence for the fractional boundary-value problem (1.3)-(1.4), studied in [3], where, again, (the Krasnoselskii's fixed point theorem in cones is applied and) the existence of solutions is investigated.

It is assumed that ${}^C D^\alpha u$ is the Caputo fractional derivative of the continuous function $u : I \to \mathbb{R}$ at the real number $\alpha \in (2,3)$ defined by

$$ {}^C D^\alpha u(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{(\alpha-1)} u(s) ds, \quad n = \lfloor \alpha \rfloor + 1. $$

Also, the function $f : [0,1] \times [0, \to [0,\infty)$ is continuous and $0 < \lambda < 2$.

Theorem 4.1. *Assume that f satisfies the condition*

$$ L_f := \sup_{t\in I} \sup_{|u|>0} \frac{|f(t,u)|}{|u|} < \frac{(2-\lambda)(\alpha-2)\Gamma(\alpha+2)}{2\alpha(\alpha-1)}. \tag{4.1} $$

Then problem (1.3)-(1.4) *does not admit solutions.*

Proof. According to [3], the problem is equivalent to the operator equation (2.1), where T has the integral form (2.2), with the Green's function G being given by the type

$$ G(t,s) := \begin{cases} \frac{2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (2-\lambda)\alpha(t-s)^{\alpha-1}}{(2-\lambda)\Gamma(\alpha+1)}, & 0 \le s \le t \le 1, \\ \frac{2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(2-\lambda)\Gamma(\alpha+1)}, & 0 \le t \le s \le 1. \end{cases} $$

Due to [3, Lemmas 2.3 and 2.4], the kernel G is a nonnegative function and it satisfies the inequality

$$ G(t,s) \le \frac{2\alpha}{\lambda(\alpha-2)} G(1,s), $$

for all $s, t, \in [0,1]$ and $\lambda \in (0,2)$. Hence, the result will follow from Theorem 2.1 and inequality (2.4), when we know that the quantity in the right side of relation (4.1) is less than or equal to the $1/K_{(1.3)-(1.4)}$. To prove this fact observe that

$$ \begin{aligned} K_{(1.3)-(1.4)} &= \sup_{t\in I} \int_0^1 G(t,s) ds \\ &\le \frac{2\alpha}{\lambda(\alpha-2)} \int_0^1 G(1,s) ds = \frac{2\alpha(\alpha-1)}{(2-\lambda)(\alpha-2)\Gamma(\alpha+2)}. \end{aligned} $$

This completes the proof of the theorem. □

5. Nonexistence for third-order BVPs at non-resonance

In this section we give information about non-existence for the third-order differential equation (1.5) subject to one of the boundary conditions (1.6), (1.7), $\cdots$, (1.19).

Theorem 5.1. *The boundary-value problems (at non-resonance)* (1.5)-(1.6)*,* (1.5)-(1.7)*, ...,*(1.5)-(1.19)*, do not have solutions provided that* (2.4) *is satisfied, where the number L_F is such as the following tables shows:*

	(1.5)-(1.6)	(1.5)-(1.7)	(1.5)-(1.8)	(1.5)-(1.9)
$0 < L_f <$	40.5	40.5	$\inf_{t\in I} 6/t(t^4 - 2t^3 + 1)$	$9\sqrt{3}$

	(1.5)-(1.10)	(1.5)-(1.11)	(1.5)-(1.12)	(1.5)-(1.13)	(1.5)-(1.14)
$0 < L_f <$	12	12	6	3	3

	(1.5)-(1.15)	(1.5)-(1.16)	(1.5)-(1.17)	(1.5)-(1.18)	(1.5)-(1.19)
$0 < L_f <$	3	3	6	6	6

Proof. All the boundary-value problems are equivalent to the integral equation (2.1), where the operator T is defined by (2.2). Thus the proof is implied from Theorem 2.1, provided that the numbers defined in (4.1) have values as in the tabular. Therefore what we have to (and shall) do is to obtain the Green's functions of the integral equations and then to calculate the corresponding numbers $1/K_T$.

Problem (1.5)-(1.6)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{1}{2}s(1-t)[t(1-s)+t-s], & 0 \le s \le t \le 1, \\ \frac{t^2}{2}(1-s)^2, & 0 \le t \le s \le 1 \end{cases}$$

and it is nonnegative. Hence we have

$$\begin{aligned} K_{(1.5)-(1.6)} &= \sup_{t\in I}\Big[\int_0^t G(t,s)ds + \int_t^1 G(t,s)ds\Big] \\ &= \frac{1}{2}\sup_{t\in I}\Big[\int_0^t s(1-t)[t(1-s)+t-s]ds + \int_t^1 t^2(1-s)^2 ds\Big] \\ &= \dots = \sup_{t\in I}\frac{t^2}{6}(1-t) = \frac{2}{81} = \frac{1}{40.5}. \end{aligned}$$

Problem (1.5)-(1.7)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{-1}{2}s^2(1-t)^2, & 0 \le s \le t \le 1 \\ \frac{-t}{2}(1-s)[(s-t)+s(1-t)], & 0 \le t \le s \le 1, \end{cases}$$

which is nonpositive. Thus we have

$$\begin{aligned} K_{(1.5)-(1.7)} &= \sup_{t\in I}\Big[\int_0^t |G(t,s)|ds + \int_t^1 |G(t,s)|ds\Big] \\ &= \frac{1}{2}\sup_{t\in I}\Big[\int_0^t s^2(1-t)^2 ds + \int_t^1 t(1-s)[(s-t)+s(1-t)]ds\Big] \\ &= \dots = \sup_{t\in I}\frac{t}{6}(t-1)^2 = \frac{2}{81} = \frac{1}{40.5}. \end{aligned}$$

Problem (1.5)-(1.8)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{1}{2}(1-t)(t-s^2), & 0 \le s \le t \le 1 \\ \frac{t}{2}(1-s)^2, & 0 \le t \le s \le 1, \end{cases}$$

which is nonnegative. Thus we have

$$\begin{aligned} K_{(1.5)-(1.8)} &= \sup_{t\in I}\Big[\int_0^t G(t,s)ds + \int_t^1 G(t,s)ds\Big] \\ &= \frac{1}{2}\sup_{t\in I}\Big[\int_0^t (1-t)(t-s^2)ds + \int_t^1 t(1-s)^2 ds\Big] \\ &= \dots = \sup_{t\in I}\frac{t}{6}(t^4-2t^3+1) = \frac{1}{\inf_{t\in I}\frac{6}{t(t^4-2t^3+1)}} \approx \frac{1}{14.309267}. \end{aligned}$$

Problem (1.5)-(1.9)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{-s^2}{2}(1-t) & s \le t, \\ \frac{-t}{2}[(s-t)+s(1-s)], & t \le s \end{cases}$$

and it is nonpositive. Then we have

$$K_{(1.5)-(1.9)} = \sup_{t\in I} \frac{t}{6}(1-t)(2-t) = \frac{\sqrt{3}}{27} = \frac{1}{9\sqrt{3}}.$$

Problem (1.5)-(1.10)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{1}{2}[ts(1-t)+s(t-s)], & 0 \le s \le t \le 1, \\ \frac{t^2}{2}(1-s), & 0 \le t \le s \le 1. \end{cases}$$

The latter is nonnegative. Hence we obtain

$$K_{(1.5)-(1.10)} = \sup_{t\in I} \Big[\frac{t^2}{4} - \frac{t^3}{6}\Big] = \frac{1}{12}.$$

Problem (1.5)-(1.11)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{-s}{2}(1-t)^2, & s \le t, \\ \frac{-1}{2}(1-s)(s-t^2), & t \le s \end{cases}$$

and it is nonpositive. Hence it follows that

$$K_{(1.5)-(1.11)} = \sup_{t\in I} \frac{1}{12}(1-3t^2+2t^3) = \frac{1}{12}.$$

Problem (1.5)-(1.12)*:* The corresponding Green's function is

$$G(t,s) := \frac{-(t-s)^2}{2}\chi_{[0,t]}(s),$$

which is nonpositive. Thus we have

$$K_{(1.5)-(1.12)} = \sup_{t\in I} \int_0^t \frac{(t-s)^2}{2} ds = \int_0^1 \frac{s^2}{2} ds = \frac{1}{6}.$$

Problem (1.5)-(1.13)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{1}{2}[ts+s(t-s)], & s \le t, \\ \frac{t^2}{2}, & t \le s, \end{cases}$$

which is nonnegative. Hence we obtain

$$K_{(1.5)-(1.13)} = \sup_{t\in I} \frac{t^2}{6}(3-t) = \frac{1}{3}.$$

Problem (1.5)-(1.14)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{1}{2}(t(1-t)+t-s+s(1-s)), & 0 \le s \le t \le 1 \\ t(1-s), & 0 \le t \le s \le 1, \end{cases}$$

which is nonnegative. Thus we have

$$K_{(1.5)-(1.14)} = \sup_{t\in I} \frac{t}{6}(3-t^2) = \frac{1}{3}.$$

Problem (1.5)-(1.15)*:* The corresponding Green's function G is

$$G(t,s) := \begin{cases} -s(1-t), & s \le t, \\ \frac{-1}{2}[2s - s^2 - t^2], & t \le s, \end{cases}$$

and it is nonpositive. This gives

$$K_{(1.5)-(1.15)} = \sup_{t\in I} \frac{1}{6}(2 - 3t^2 + t^3) = \frac{1}{3}.$$

Problem (1.5)-(1.16)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{-1}{2}(1-t)^2, & 0 \le s \le t \le 1 \\ \frac{-1}{2}(1-s)[(s-t) + (1-t)], & 0 \le t \le s \le 1, \end{cases}$$

which is nonpositive. Thus we obtain

$$K_{(1.5)-(1.16)} = \sup_{t\in I} \frac{1}{6}[2 - 3t + t^3] = \frac{1}{3}.$$

Problem (1.5)-(1.17)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{-1}{2}s^2, & 0 \le s \le t \le 1 \\ \frac{-1}{2}[ts + t(s-t)], & 0 \le t \le s \le 1, \end{cases}$$

which is nonpositive. Then we obtain

$$K_{(1.5)-(1.17)} = \sup_{t\in I} \frac{1}{6}[3t - 3t^2 + t^3] = \frac{1}{6}.$$

Problem (1.5)-(1.18)*:* The corresponding Green's function is

$$G(t,s) := \begin{cases} \frac{1-t}{2}(1 + t - 2s), & 0 \le s \le t \le 1 \\ \frac{1}{2}(1-s)^2, & 0 \le t \le s \le 1, \end{cases}$$

which is nonnegative. Hence, it follows that

$$K_{(1.5)-(1.18)} = \sup_{t\in I} \frac{1}{6}[1 - t^3] = \frac{1}{6}.$$

Problem (1.5)-(1.19)*:* This is a terminal value problem with Green's function given by

$$G(t,s) := \frac{(t-s)^2}{2}\chi_{[t,1]}(s),$$

which is nonnegative. Thus we have

$$K_{(1.5)-(1.19)} = \sup_{t\in I} \int_t^1 \frac{(t-s)^2}{2} ds = \frac{1}{6}.$$

The proof is complete. □

Application 5.2. Consider the differential equation

$$x''' + (\sigma + \frac{\cos(x' + x'')}{4(1+x^2)})x = 0. \tag{5.1}$$

Here we have

$$f(u_1, u_2, u_3) = (\sigma + \frac{\cos(u_2 + u_3)}{4(1+u_1^2)})u_1$$

and therefore
$$\sup_{u_1,u_2,u_3} \frac{f(u_1,u_2,u_3)}{u_1} = \sigma + \frac{1}{4}.$$
Taking into account Corollary 2.3, the previous theorem and Theorem 2.1, we can, easily, see that the problems (5.1)-(1.6), (5.1)-(1.7), $\cdots$, (5.1)-(1.19) do not have any solution, provided that the parameter σ has corresponding values, as the following tables show:

	(5.1)-(1.6)	(5.1)-(1.7)	(5.1)-(1.8)	(5.1)-(1.9)	(5.1)-(1.10)
$0 < \sigma$	< 40.25	< 40.25	< 14.0602	< 15.3385	< 11.75

	(5.1)-(1.11)	(5.1)-(1.12)	(5.1)-(1.13)	(5.1)-(1.14)	(5.1)-(1.15)
$0 < \sigma$	< 11.75	< 5.75	< 2.75	< 2.75	< 2.75

	(5.1)-(1.16)	(5.1)-(1.17)	(5.1)-(1.18)	(5.1)-(1.19)
$0 < \sigma$	< 2.75	< 5.75	< 5.75	< 5.75

The upper bounds of σ for problem (5.1)-(1.8) and (5.1)-(1.9) are approximate values.

6. Nonexistence for third-order BVPs at resonance

In this section we shall discuss the problems (6.1)-(1.20) - (6.1)-(1.25) where (6.1) is equation
$$x''' + f(x(t)) = 0, \quad \text{a. e. } t \in I. \tag{6.1}$$
Due to the boundary conditions (1.20)-(1.25), these problems are at resonance and, so, they do not have integral equivalent forms, as in the preceding cases. Hence we need to apply a suitable technique, where, we need to assume some rather simple conditions on the function f. Of course, this will restrict the family of responses. Indeed, in the sequel we shall assume that the response f satisfies the conditions:

(1) f is a continuously differentiable real valued function.
(2) There are positive real numbers $L_f > \delta$ such that
$$|f(u)| \le L_f|u| \quad \text{and} \quad |f'(u)| \ge \delta, \quad u \in \mathbb{R}. \tag{6.2}$$

Clearly, the latter implies that either $f'(u) \ge \delta$, for all u, or $f'(u) \le -\delta$, for all u. Our main theorem reads as follows.

Theorem 6.1. *The boundary-value problems* (6.1)-(1.20), …, (6.1)-(1.25) *(at resonance) do not admit solutions when the parameters L, δ satisfy the relations as in the following tables:*

	(6.1)-(1.20)	(6.1)-(1.21)	(6.1)-(1.22)
$\delta < L_f <$	$-5\delta + \sqrt{25\delta^2 + 60\delta}$	$\frac{1}{11}(-10\delta + 2\sqrt{25\delta^2 + 165\delta})$	$-\delta + \sqrt{\delta^2 + 12\delta}$

	(6.1)-(1.23)	(6.1)-(1.24)	(6.1)-(1.25)
$\delta < L_f <$	$\frac{1}{3}(-\delta + \sqrt{\delta^2 + 36\delta})$	$-2\delta + \sqrt{4\delta^2 + 24\delta}$	$\frac{1}{5}(-4\delta + 2\sqrt{4\delta^2 + 30\delta})$

Proof. These boundary-value problems can be written as operator equations of the form (2.1), where the operator T is not linear. So, for each problem, we must formulate this operator and then to calculate the quantity K_T. Also, due to (6.2), Theorem 2.1 will be applied, when we shall show that
$$L_f K_T < 1. \tag{6.3}$$

Problem (6.1)-(1.20)*:* Assume that $\delta < L_f < -5\delta + \sqrt{25\delta^2 + 60\delta}$. Then it holds

$$L_f[\frac{L_f}{60\delta} + \frac{1}{6}] < 1, \tag{6.4}$$

If there is a solution x of the problem, it will satisfy the boundary conditions $x'(0) = x'(1) = x''(0) = 0$. Integrating equation $x''' + f(x) = 0$ we get

$$x''(t) = -\int_0^t y(s)ds,$$

where $y(t) := f(x(t))$. By using the boundary conditions we obtain

$$\int_0^1 (1-s)y(s)ds = 0 \tag{6.5}$$

and $x(t) = x(0) - (g_1y)(t), \quad t \in [0,1]$, where

$$(g_1y)(t) := \int_0^t \frac{(t-s)^2}{2} y(s)ds, \quad t \in I.$$

Next, define the function

$$(A_1y)(v) := \int_0^1 (1-t)f(v - (g_1y)(t))dt, \quad v \in \mathbb{R}.$$

This is a differentiable function, which, due to (6.5), satisfies the equation

$$(A_1y)(0) + (A_1y)'(\mu x(0))x(0) = (A_1y)(x(0)) = 0,$$

for some $\mu \in [0,1]$. Namely we have

$$\int_0^1 (1-t)f(-(g_1y)(t))dt + x(0)\int_0^1 (1-t)f'(\mu x(0) - (g_1y)(t))dt = 0.$$

Thus the initial value satisfies the relation

$$x(0) = -b_1(x,y)\int_0^1 (1-t)f(-(g_1y)(t))dt,$$

where

$$b_1(x,y) := \frac{1}{\int_0^1 (1-t)f'(\mu x(0)t - (g_1y)(t))dt}.$$

Clearly, it holds $|b_1(x,y)| \le 2/\delta$. Therefore, the solution x satisfies the integral equation $x = T(x, f(x))$, where the operator T is defined on $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ by the type

$$T(x,y)(t) := -b_1(x,y)\int_0^1 (1-r)f(-(g_1y)(r))dr - (g_1y)(t), \quad t \in [0,1].$$

Then, from (6.2) we obtain

$$|T(x,y)(t)| \le K_{(6.1)-(1.20)}\|y\|,$$

where

$$K_{(6.1)-(1.20)} := \frac{2}{\delta}L_f\int_0^1 (1-r)\frac{r^3}{6}dr + \frac{1}{6},$$

which implies that

$$L_fK_{(6.1)-(1.20)} < L_f[\frac{L_f}{60\delta} + \frac{1}{6}] < 1,$$

because of (6.2) and (6.4).

Problem (6.1)-(1.21)*:* Let $\delta < L_f < \frac{1}{11}(-10\delta + 2\sqrt{25\delta^2 + 55\delta})$. Then we have

$$L_f \frac{11L_f + 20\delta}{60\delta} < 1. \tag{6.6}$$

Assume that there is a solution $x(t), \quad t \in [0,1]$ of the problem such that $x'(0) = x'(1) = x''(1) = 0$. Integrating equation $x''' + f(x) = 0$, we get

$$x''(t) = -\int_0^t y(s)ds, \tag{6.7}$$

where $y(s) := f(x(s))$. By using the boundary conditions $x''(1) = x'(1) = 0$, it follows that

$$\int_0^1 sy(s)ds = 0. \tag{6.8}$$

Integrate (6.7), once again, and get $x(t) = x(0) + (g_2 y)(t)$, where

$$(g_2 y)(t) := \frac{t^2}{2}\int_t^1 y(s)ds + \int_0^t \frac{2ts - s^2}{2} y(s)ds, \quad t \in I.$$

Define the function

$$(A_2 y)(v) := \int_0^1 tf(v + (g_2 y)(t))dt, \quad v \in \mathbb{R},$$

for which we observe that

$$(A_2 y)(0) + (A_2 y)'(\lambda x(0))x(0) = (A_2 y)(x(0)) = 0,$$

because of (6.8), for some $\lambda \in [0,1]$. This implies that

$$\int_0^1 tf((g_2 y)(t))dt + x(0)\int_0^1 tf'(\lambda x(0) + (g_2 y)(t))dt = 0,$$

from which we get

$$x(0) = -b_2(x,y)\int_0^1 tf((g_2 y)(t))dt,$$

where

$$b_2(x,y) := \frac{1}{\int_0^1 tf'(\lambda x(0) + (g_2(y)(t))dt}$$

is such that $|b_2(x,y)| \le 2/\delta$. Hence, for all $t \in [0,1]$, the solution x satisfies the integral relation

$$x(t) = T(x, f(x)),$$

where the operator T is defined by the type

$$T(x,y) := -b_2(x,y)\int_0^1 rf((g_2 y)(r))dr + (g_2 y)(t), \quad t \in I$$

on $C(I,\mathbb{R}) \times C(I,\mathbb{R})$. From this equation and (6.2) it follows that

$$|T(x,y)(t)| \le \frac{2}{\delta} L_f \int_0^1 r|(g_2 y)(r)|dr + |(g_2 y)(t)|$$

and so, finally, we obtain

$$|T(x,y)(t)| \le K_{(6.1)-(1.21)}\|y\|,$$

where

$$K_{(6.1)-(1.21)} := \frac{2}{\delta} L_f \frac{11}{120} + \frac{1}{3}.$$

Thus we get $L_f K_{(6.1)-(1.21)} < 1$, because of (6.6).
Problem (6.1)-(1.22)*:* Assume that $\delta < L_f < -\delta + \sqrt{\delta^2 + 12\delta}$. Then we have

$$L_f \frac{L_f + 2\delta}{12\delta} < 1. \tag{6.9}$$

Assume that x is a solution of the problem. Hence it satisfies $x(0) = x''(0) = x''(1) = 0$. Integrating equation $x''' + f(x) = 0$, we obtain

$$x''(t) = -\int_0^t y(s)ds,$$

where $y(s) := f(x(s))$. Thus we have

$$\int_0^1 y(s)ds = 0. \tag{6.10}$$

Integrate two more times and get $x(t) = x'(0)t - (g_1 y)(t), \quad t \in [0,1]$, where

$$(g_1 y)(t) := \int_0^t \frac{(t-s)^2}{2} y(s)ds, \quad t \in I.$$

Define the function

$$(A_3 y)(v) := \int_0^1 f(vt - (g_1 y)(t))dt, \quad v \in \mathbb{R}$$

and we observe that it satisfies

$$(A_3 y)(0) + (A_3 y)'(\lambda x'(0))x'(0) = (A_3 y)(x'(0)) = 0,$$

because of (6.10), for some $\lambda \in [0,1]$. This implies that

$$\int_0^1 f(-(g_1 y)(t))dt + x'(0) \int_0^1 t f'(\lambda x'(0)t - (g_1 y)(t))dt) = 0.$$

Therefore, the value $x'(0)$ satisfies

$$x'(0) = -b_3(x,y) \int_0^1 f(-(g_1 y)(t))dt,$$

where the coefficient $b_3(x,y)$ is such that

$$b_3(x,y) := \frac{1}{\int_0^1 t f'(\lambda x'(0)t - (g_1 y)(t))dt}$$

and moreover

$$|b_3(x,y)| \le \frac{2}{\delta},$$

because of (6.2). Hence, for all $t \in [0,1]$, the solution x satisfies the operator equation $x(t) = T(x, f(x))(t)$, where T is defined on the space $C^{(1)}(I,\mathbb{R}) \times C(I,\mathbb{R})$ by the type

$$T(x,y)(t) := -b_3(x,y)t \int_0^1 f(-(g_1 y)(r))dr - (g_1 y)(t).$$

From here and (6.2) we obtain

$$|T(x,y)(t)| \le |b_3(x,y)| L \int_0^1 |(g_1 y)(r)|dr + |(g_1 y)|(t),$$

and so, finally,

$$|T(x,y)(t)| \le K_{(6.1)-(1.22)} \|y\|,$$

where
$$K_{(6.1)-(1.22)} := \frac{L_f}{12\delta} + \frac{1}{6}.$$
The latter implies that $L_f K_{(6.1)-(1.22)} < 1$, due to (6.9).

Problem (6.1)-(1.23)*:* Let $\delta < L_f < \frac{1}{3}(-\delta + \sqrt{\delta^2 + 36\delta})$. Then we have
$$L_f < \frac{12\delta}{3L_f + 2\delta}. \tag{6.11}$$
Assume that there is a solution $x(t)$, $t \in [0,1]$ of the problem such that $x(1) = x''(0) = x''(1) = 0$. Integrating equation $x''' + f(x) = 0$, we obtain
$$x''(t) = -\int_0^t y(s)ds,$$
where $y := f(x)$. Thus we have
$$\int_0^1 y(s)ds = 0. \tag{6.12}$$
Taking into account the boundary conditions, we integrate and get
$$x(t) = -(1-t)x'(0) + (g_3 y)(t), \quad t \in [0,1],$$
where
$$(g_3 y)(t) := \int_t^1 \frac{(1-s)^2}{2} y(s)ds + \int_0^t \frac{(1-t)^2 + 2(t-s)(1-t)}{2} y(s)ds, \quad t \in I.$$
Define the function
$$(A_4 y)(v) := \int_0^1 f(-(1-t)v + (g_3 y)(t))dt, \quad v \in \mathbb{R},$$
for which we observe that
$$(A_4 y)(0) + (A_4 y)'(\lambda x'(0))x'(0) = (A_4 y)(x'(0)) = 0,$$
because of (6.12), for some $\lambda \in [0,1]$. This implies that
$$\int_0^1 f((g_3 y)(t))dt - x'(0) \int_0^1 (1-t) f'(-(1-t)\lambda x'(0) + (g_3 y)(t))dt = 0$$
and, so,
$$x'(0) = b_4(x,y) \int_0^1 f((g_3 y)(t))dt,$$
where
$$b_4(x,y) := \frac{1}{\int_0^1 (1-t) f'(-(1-t)\lambda x'(0) + (g_3 y)(t))dt}.$$
Due to (6.2), we have $|b_4(x,y)| \le \frac{2}{\delta}$. Hence, for all $t \in [0,1]$, the solution x satisfies the relation
$$x(t) = T(x, f(x))(t),$$
where the operator T is defined on $C^{(1)}(I,\mathbb{R}) \times C(I,\mathbb{R})$ by the type
$$T(x,y)(t) := -(1-t) b_4(x,y) \int_0^1 f((g_3 y)(r))dr + (g_3 y)(t), \quad t \in [0,1].$$
Then, as in the previous cases, due to (6.2), we obtain
$$|T(x,y)(t)| \le K_{(6.1)-(1.23)} \|y\|,$$

where

$$K_{(6.1)-(1.23)} := L_f(\frac{L_f}{4\delta} + \frac{1}{6}).$$

This implies that $L_f K_{(6.1)-(1.23)} < 1$, because of (6.11).

Problem (6.1)-(1.24)*:* Assume that $\delta < L_f < -2\delta + \sqrt{4\delta^2 + 24\delta}$. Then we have

$$L_f < \frac{24\delta}{L_f + 4\delta}. \tag{6.13}$$

Let $x(t)$, $t \in [0,1]$ be a solution of the problem such that $x'(0) = x''(0) = x''(1) = 0$. Integrating equation $x''' + f(x) = 0$, we obtain $x''(t) = -\int_0^t y(s)ds$, where, again, $y(s) := f(x(s))$. Thus we have

$$\int_0^1 f(x(s))ds = 0. \tag{6.14}$$

Integrate and get $x(t) = x(0) - (g_1 y)(t)$, $t \in [0,1]$, where, again,

$$(g_1 y)(t) := \int_0^t \frac{(t-s)^2}{2} y(s)ds, \quad t \in I.$$

Define the function

$$(A_5 y)(v) := \int_0^1 f(v - (g_1 y)(t))dt, \quad v \in \mathbb{R},$$

and observe that

$$(A_5 y)(0) + (A_5 y)'(\lambda x(0))x(0) = (A_5 y)(x(0)) = 0,$$

because of (6.14), for some $\lambda \in [0,1]$. This implies that

$$\int_0^1 f(-(g_1 y)(t))dt + x(0)\int_0^1 f'(\lambda x(0) - (g_1 y)(t))dt = 0$$

and so

$$x(0) = -b_5(x,y)\int_0^1 f(-(g_1 y)(t))dt,$$

where

$$b_5(x,y) := \frac{1}{\int_0^1 f'(\lambda x(0) - (g_1 y)(t))dt},$$

which, obviously, satisfies $|b_5| \le \frac{1}{\delta}$. Hence, for all $t \in [0,1]$, the solution x satisfies the relation

$$x(t) = T(x, f(x))(t),$$

where the operator T is defined on $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ by the type

$$T(x,y)(t) := -b_5(x,y)\int_0^1 f(-(g_1 y)(r))dr) - (g_1 y)(t).$$

Thus, because of (6.2), we have

$$|T(x,y)(t)| \le |b_5(x,y)| L_f \int_0^1 \int_0^r \frac{(r-s)^2}{2}\|y\|\, ds\, dr + \int_0^t \frac{(t-s)^2}{2}\|y\| ds,$$

and so

$$|T(x,y)(t)| \le K_{(6.1)-(1.24)}\|y\|,$$

where

$$K_{(6.1)-(1.24)} := L_f(\frac{L_f}{24\delta} + \frac{1}{6}),$$

because of (6.2). This gives $L_f K_{(6.1)-(1.24)} < 1$, because of (6.13).
Problem (6.1)-(1.25)*:* Let $0 < L_f < \frac{1}{5}(-4\delta + 2\sqrt{4\delta^2 + 30\delta})$. Then we have

$$L_f \frac{5L_f + 8\delta}{24\delta} < 1. \tag{6.15}$$

Assume that there is a solution $x(t)$, $t \in [0,1]$ of the problem such that $x'(1) = x''(0) = x''(1) = 0$. Integrating the equation $x''' + f(x) = 0$, we obtain $x''(t) = -\int_0^t y(s)ds$, where $y := f(x)$. Thus we have

$$\int_0^1 y(s)ds = 0. \tag{6.16}$$

Integrate, again, and obtain $x(t) = x(0) + (g_4 y)(t)$, $t \in [0,1]$, where

$$(g_4 y)(t) := t\int_t^1 (1-s)y(s)ds + \int_0^t (t - \frac{t^2}{2} - \frac{s^2}{2})y(s)ds, \quad t \in I.$$

Define the function

$$(A_6 y)(v) := \int_0^1 f(v + (g_4 y)(t))dt, \quad v \in \mathbb{R},$$

for which we observe that

$$(A_6 y)(0) + (A_6 y)'(\lambda x(0))x(0) = (A_6 y)(x(0)) = 0,$$

because of (6.16), for some $\lambda \in [0,1]$. This implies that

$$\int_0^1 f((g_4 y)(t))dt + x(0)\int_0^1 f'(\lambda x(0) + (g_4 y)(t))dt = 0$$

and so we have

$$x(0) = -b_6(x,y)\int_0^1 f((g_4 y)(t))dt,$$

where

$$b_6(x,y) := \frac{1}{\int_0^1 f'(\lambda x(0) + (g_4 y)(t))dt},$$

which satisfies $|b_6(x,y)| \le \frac{1}{\delta}$. Hence, for all $t \in [0,1]$, the solution x satisfies the relation

$$x(t) = T(x, f(x))(t),$$

where T is the operator defined on $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ by the type

$$T(x,y)(t) := -b_6(x,y)\int_0^1 f((g_4 y)(r))dr + (g_4 y)(t).$$

Due to (6.2), this gives that

$$|T(x,y)(t)| \le |b_6(x,y)| L_f \|y\| \int_0^1 (\frac{r}{2} - \frac{r^3}{6})dr + \|y\|(\frac{t}{2} - \frac{t^3}{6}),$$

and so

$$|T(x,y)(t)| \le K_{(6.1)-(1.25)}\|y\|,$$

where

$$K_{(6.1)-(1.25)} := L_f(\frac{5L_f}{24\delta} + \frac{1}{3}).$$

The latter gives $L_f K_{(6.1)-(1.25)} < 1$ because of (6.15).

$\square$

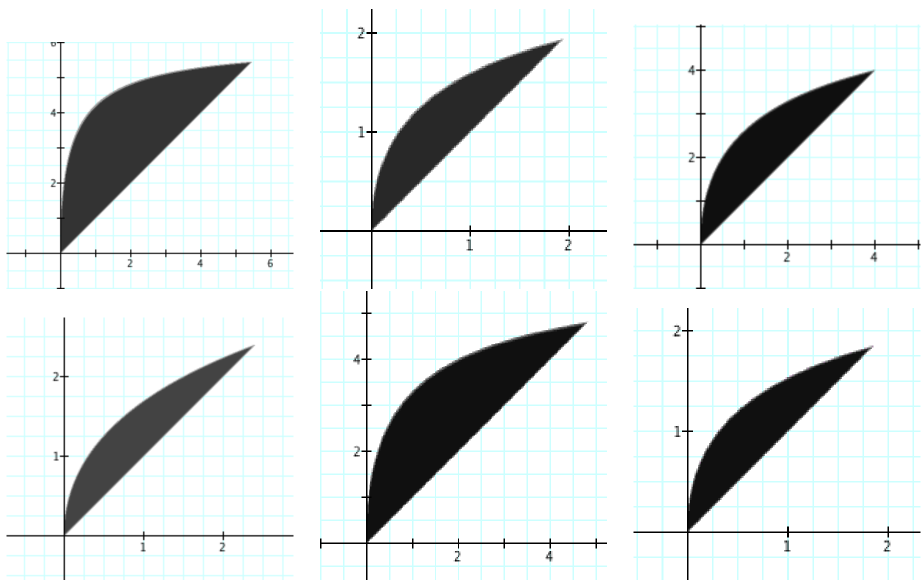

FIGURE 1. If the pair of parameters (δ, L) belongs to the shaded area in the $\delta - L$ plane, then there are no solutions of the BVPs (6.1)-(1.20), (6.1)-(1.21), (6.1)-(1.22), (6.1)-(1.23), (6.1)-(1.24), (6.1)-(1.25), respectively.

Application 6.2. Consider the differential equation

$$x''' + (\sigma + \frac{1}{4(1+x^2)})x = 0, \tag{6.17}$$

for a positive parameter σ. Setting $\delta := \sigma - \frac{1}{4}$ and $L := \sigma + \frac{1}{4}$, one can easily apply the tabular in Theorem 6.1 and conclude that the problems (6.17)-(1.20), $\cdots$, (6.17)-(1.25) do not have solutions when the parameter σ takes corresponding values, such as they are approximately given in the following tables:

	(6.17)-(1.20)	(6.17)-(1.21)	(6.17)-(1.22)
σ	$0.00231 < \sigma < 4.92951$	$0.00254 < \sigma < 2.3424$	$0.01223 < \sigma < 3.40443$

	(6.17)-(1.23)	(6.17)-(1.24)	(6.17)-(1.25)
σ	$0.04397 < \sigma < 1.70604$	$0.0059 < \sigma < 4.24411$	$0.03973 < \sigma < 1.21028$

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