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POSITIVE SOLUTIONS FOR A THIRD-ORDER THREE-POINT BOUNDARY-VALUE PROBLEM

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Abstract. In this article, we study the existence of positive solutions to a nonlinear third-order three point boundary value problem. The main tools are Krasnosel'skii fixed point theorem on cones, and the fixed point index theory.

1. Introduction

In this article, we are interested in the existence of single and multiple positive solutions to nonlinear third-order three-point boundary-value problem

$$u'''(t) + a(t)f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = 0, \quad u'(0) = u'(1) = \alpha u(\eta)\,, \tag{1.2}$$

where $\eta \in (0, 1)$, $\alpha \in [0, \frac{1}{\eta})$. We assume the following conditions hold in this article:

(H1) $f \in C([0, 1] \times [0, \infty), [0, \infty))$.
(H2) $a \in L^1[0, 1]$ is nonnegative and $a(t) \not\equiv 0$ on any subinterval of $[0, 1]$.

Third-order differential equation arise in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves o gravity driven flows and so on. Li in [4] by using Krasnosel'skii fixed point theorem on cone establish various results on the existence of positive solutions. Sun [7] use the Krasnosel'skii fixed point theorem and Schauder's fixed point theorem to obtain existence and nonexistence of positive solutions. In [5] Liu et al obtain results for the existence of at least one, two, three and infinitely many monotone positive solutions by using Krasnosel'skii and Leggett-Williams fixed point theorem. In [6] Luan et al obtain existence results under conditions that the nonlinear term satisfies Carathéodory condition, semipositone and lower unbounded by using the fixed point index theory. In [1], Bai the nonlinear term depends on u, u' and u'', prove the existence of at least one solution with the use of lower and upper solutions methods and Schauder fixed point theorem. Motivated by the above works, we obtain some sufficient conditions for the existence of at least one and two positive solutions for (1.1) and (1.2). The organization of this article is as follows. In section 2, we present some necessary definitions and preliminary results that will be used to prove our results.

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In section 3, we discuss the existence of at least one positive solution for (1.1) and (1.2). In section 4, we discuss the existence of multiple positive solutions for (1.1) and (1.2). Finally, we give some examples to illustrate our results in section 5.

2. Preliminaries

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called cone if

(1) if $x \in K$ and $\lambda > 0$, then $\lambda x \in K$
(2) it $x \in K$ and $-x \in K$, then $x = 0$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Remark 2.3. By the positive solution of (1.1), (1.2) we understand a function $u(t)$ wich is positive on $[0, 1]$ and satisfies the differential equation (1.1) and the boundary conditions (1.2).

We shall consider the Banach space $E = C[0, 1]$ equipped with standard norm

$$\|u\| = \max_{0\le t\le 1} |u(t)| .$$

The proof of existence of solution is based on an applications of the following theorems.

Theorem 2.4 ([2, 3])**.** *Let E be a Banach space and let $K \subseteq E$ be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$ and let*

$$T : K \cap (\overline{\Omega_2}\backslash\Omega_1) \to K$$

be completely continuous such that

(i) $\|Tu\| \le \|u\|$ *if* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$ *if* $u \in K \cap \partial\Omega_2$*; or*
(ii) $\|Tu\| \ge \|u\|$ *if* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$ *if* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega_2}\backslash\Omega_1)$

Theorem 2.5 ([2, 3])**.** *Let E be a Banach space and K be a cone of E. For $r > 0$, define $K_r = \{u \in K : \|u\| \le r\}$ and assume that $T : K_r \to K$ is a completely continuous operator such that $Tu \ne u$ for $u \in \partial K_r$*

(1) *If $\|Tu\| \le \|u\|$ for all $u \in \partial K_r$, then $i(T, K_r, K) = 1$*
(2) *If $\|Tu\| \ge \|u\|$ for all $u \in \partial K_r$, then $i(T, K_r, K) = 0$.*

Consider the three-point boundary-value problem

$$u''' + h(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = 0, \quad u'(0) = u'(1) = \alpha u(\eta)\,, \tag{2.2}$$

where $\eta \in (0, 1)$, $\alpha \in [0, 1/\eta)$.

Lemma 2.6. *Let $\alpha\eta \ne 1$, $h \in L^1[0, 1]$. Then the three-point boundary-value problem (2.1) and (2.2) has a unique solution*

$$u(t) = \int_0^1 G(t, s)h(s)ds,$$

where $G(t,s) = g(t,s) + \frac{\alpha t}{1-\alpha\eta} g(\eta,s)$, *and*

$$g(x,y) = \begin{cases} \frac{1}{2}(2x - x^2 - y)y & 0 \le y \le x \le 1 \\ \frac{1}{2}x^2(1-y) & 0 \le x \le y \le 1\,. \end{cases} \tag{2.3}$$

Proof. From (2.1), $u''' = -h(t)$. Applying the method of variation of parameter, we obtain

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 h(s)ds + At^2 + Bt + C, \tag{2.4}$$

where $A, B, C \in \mathbb{R}$. From (2.2), $C = 0$. Since $u'(0) = u'(1)$,

$$B = -\int_0^1 (1-s)h(s)ds + 2A + B.$$

Therefore,

$$A = \frac{1}{2}\int_0^1 (1-s)h(s)ds\,.$$

Since $u'(0) = \alpha u(\eta)$, we obtain:

$$\begin{gathered} B = -\frac{\alpha}{2}\int_0^\eta (\eta-s)^2 h(s)ds + \frac{\alpha\eta^2}{2}\int_0^1 (1-s)h(s)ds + B\alpha\eta, \\ (1-\alpha\eta)B = -\frac{\alpha}{2}\int_0^\eta (\eta-s)^2 h(s)ds + \frac{\alpha\eta^2}{2}\int_0^1 (1-s)h(s)ds, \\ B = -\frac{\alpha}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 h(s)ds + \frac{\alpha\eta^2}{2(1-\alpha\eta)}\int_0^1 (1-s)h(s)ds\,. \end{gathered}$$

Replacing these expressions in (2.4),

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 h(s)ds + \frac{t^2}{2}\int_0^1 (1-s)h(s)ds - \frac{\alpha t}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 h(s)ds \\ &\quad + \frac{\alpha t\eta^2}{2(1-\alpha\eta)}\int_0^1 (1-s)h(s)ds \\ &= -\frac{1}{2}\int_0^t (t-s)^2 h(s)ds + \frac{1}{2}\int_0^t t^2(1-s)h(s)ds + \frac{1}{2}\int_t^1 t^2(1-s)h(s)ds \\ &\quad - \frac{\alpha t}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 h(s)ds + \frac{\alpha t\eta^2}{2(1-\alpha\eta)}\int_0^1 (1-s)h(s)ds \\ &= \frac{1}{2}\Big[\int_0^t (2t - t^2 - s)sh(s)ds + \int_t^1 t^2(1-s)h(s)ds\Big] \\ &\quad + \frac{\alpha t}{1-\alpha\eta}\frac{1}{2}\Big[\int_0^1 \eta^2(1-s)h(s)ds - \int_0^\eta (\eta-s)^2 h(s)ds\Big] \\ &= \int_0^1 g(s,t)h(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 g(\eta,s)h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds\,. \end{aligned}$$

$\square$

Lemma 2.7. *Let $\sigma \in (0,1]$ be fixed. Then*
$$\frac{1}{2}\gamma s(1-s) \le g(t,s) \le \frac{1}{2}s(1-s), \quad \forall (t,s) \in [\sigma,1]\times[0,1],$$
where $\gamma = \sigma^2$.

Proof. If $s \le t$, from (2.3),
$$\begin{aligned}
g(t,s) &= \frac{1}{2}(2t-t^2-s)s \\
&= \frac{1}{2}(-(t^2-2t)-s)s \\
&= \frac{1}{2}(-[(t-1)^2-1]-s)s \\
&= \frac{1}{2}[1-(1-t)^2-s]s \\
&= \frac{1}{2}[(1-s)-(1-t)^2]s\,.
\end{aligned}$$
Then
$$g(t,s) \le \frac{1}{2}s(1-s)\,.$$
On the other hand,
$$\begin{aligned}
g(t,s) &= \frac{1}{2}(2t-t^2-s)s \\
&= \frac{1}{2}ts(1-s) + \frac{1}{2}[(1-t)(t-s)s] \\
&\ge \frac{1}{2}ts(1-s) \\
&\ge \frac{1}{2}t^2s(1-s)\,.
\end{aligned}$$
If $t \le s$, from (2.3),
$$\begin{aligned}
\frac{1}{2}t^2(1-s)s &\le g(t,s) \\
&= \frac{1}{2}t^2(1-s) \\
&\le \frac{1}{2}s^2(1-s) \\
&\le \frac{1}{2}s(1-s)\,.
\end{aligned}$$
Therefore
$$\frac{1}{2}t^2(1-s)s \le g(t,s) \le \frac{1}{2}(1-s)s \quad \forall (t,s) \in [0,1]\times[0,1]\,. \tag{2.5}$$
For $t \in [\sigma,1]$, we have
$$\frac{1}{2}\sigma^2(1-s)s \le g(t,s) \le \frac{1}{2}(1-s)s \quad \forall (t,s) \in [\sigma,1]\times[0,1]\,.$$
□

Remark 2.8. For $t=1$ in (2.5), we have
$$\frac{1}{2}(1-s)s = g(1,s)\,. \tag{2.6}$$

Lemma 2.9. *Let $h(t) \in C^+[0,1]$. The unique solution $u(t)$ of (2.1), (2.2) is nonnegative and satisfies*
$$\min_{\sigma\le t\le 1} u(t) \ge \gamma\|u\|\,.$$

Proof. From Lemma 2.6 and Lemma 2.7, $u(t)$ is nonnegative. For $t\in[0,1]$, from Lemma 2.6 and Lemma 2.7, we have that
$$\begin{aligned} u(t) &= \int_0^1 g(t,s)h(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 g(\eta,s)h(s)ds \\ &\le \frac{1}{2}\int_0^1 s(1-s)h(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)h(s)ds\,. \end{aligned}$$
Then
$$\|u\| \le \frac{1}{2}\int_0^1 s(1-s)h(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)h(s)ds\,. \tag{2.7}$$
On the other hand, Lemma 2.7 imply that, for any $t\in[\sigma,1]$,
$$\begin{aligned} u(t) &= \int_0^1 g(t,s)h(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 g(\eta,s)h(s)ds \\ &\ge \frac{1}{2}\gamma\int_0^1 s(1-s)h(s)ds + \frac{\alpha t^2}{1-\alpha\eta}\int_0^1 g(\eta,s)h(s)ds \\ &\ge \frac{1}{2}\gamma\int_0^1 s(1-s)h(s)ds + \frac{\alpha \sigma^2}{1-\alpha\eta}\int_0^1 g(\eta,s)h(s)ds \\ &= \gamma\Big[\frac{1}{2}\int_0^1 s(1-s)h(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)h(s)ds\Big] \\ &\ge \gamma\|u\|\,. \end{aligned}$$
Therefore
$$\min_{\sigma\le t\le 1} u(t) \ge \gamma\|u\|\,.$$
□

We introduce the notation
$$f_a := \liminf_{u\to a}\min_{0\le t\le 1}\frac{f(t,u)}{u}, \quad f^b := \limsup_{u\to b}\max_{0\le t\le 1}\frac{f(t,u)}{u},$$
where $a,b = 0^+,\infty$,
$$\begin{aligned} N &= \int_\sigma^1 \frac{\gamma}{2}s(1-s)a(s)ds + \frac{\alpha\gamma}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)ds, \\ M &= \int_0^1 \frac{1}{2}s(1-s)a(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)ds\,. \end{aligned}$$
Define the cone
$$K = \{u\in C[0,1] : u(t)\ge 0,\ \min_{\sigma\le t\le 1} u(t)\ge\gamma\|u\|\}$$
and the operator $T: K\to E$ by
$$Tu(t) = \int_0^1 g(t,s)a(s)f(s,u)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \tag{2.8}$$

Remark 2.10. By Lemma 2.6, problem (1.1), (1.2) has a positive solution $u(t)$ if and only if u is a fixed point of T.

Lemma 2.11. *The operator defined in* (2.8), *is completely continuous and satisfies* $T(K) \subseteq K$.

Proof. By Lemma 2.9, $T(K) \subseteq K$. T is completely continuous by an application of Arzela-Ascoli theorem. ☐

In what follow, we will use the following conditions

(a) $f^0 = 0$ and $f_\infty = \infty$;
(b) $f_0 = \infty$ and $f^\infty = 0$;
(c) $f_0 = \infty$ and $f_\infty = \infty$;
(d) $f^0 = 0$ and $f^\infty = 0$;
(e) $0 \leq f^0 < R$ and $r < f_\infty \leq \infty$;
(f) $r < f_0 \leq \infty$ and $0 \leq f^\infty < R$;
(g) $\exists \rho > 0 : f(t,u) < R\rho,\ 0 < u \leq \rho,\ t \in [0,1]$;
(h) $\exists \rho > 0 : f(t,u) > r\rho,\ \rho < u \leq \frac{\rho}{\gamma},\ t \in [\sigma,1]$.

Remark 2.12. We note that (a) corresponds to the superlinear case and (b) corresponds to the sublinear case. In conditions (e) and (f), $r = N^{-1}$ and $R = M^{-1}$. It is obvious that $r > R > 0$.

3. Existence of Positive Solutions

Theorem 3.1. *Assume that the conditions on* a, f *and* (a) *hold. Then* (1.1), (1.2) *has at least one positive solution.*

Proof. Since $f^0 = 0$, $\exists\ H_1 > 0$ such that $f(t,u) \leq \varepsilon u$, for all $t \in [0,1]$, $0 < u \leq H_1$, where $\varepsilon > 0$. Then for $u \in K \cap \partial\Omega_1$, with $\Omega_1 = \{u \in X : \|u\| < H_1\}$, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 g(t,s)a(s)f(s,u)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&\leq \int_0^1 \frac{1}{2}s(1-s)a(s)f(s,u)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&\leq \int_0^1 \frac{1}{2}s(1-s)a(s)\varepsilon u ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)\varepsilon u ds \\
&\leq \varepsilon\Big[\int_0^1 \frac{1}{2}s(1-s)a(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)ds\Big]\|u\|.
\end{aligned}$$

If $\varepsilon M \leq 1$, then $Tu(t) \leq \|u\|$. Therefore,

$$\|Tu\| \leq \|u\|.$$

On the other hand, since $f_\infty = \infty$, there exists $\bar{H}_2 > 0$ such that $f(t,u) \geq \delta u$, for all $t \in [\sigma,1]$ with $\bar{H}_2 \leq u$ and $\delta > 0$. Then for $u \in K \cap \partial\Omega_2$, where $\Omega_2 = \{u \in X : \|u\| < H_2\}$ with $H_2 = \max\{2H_1, \frac{\bar{H}_2}{\gamma}\}$. Then $u \in K \cap \partial\Omega_2$ implies that $\min_{\sigma \leq t \leq 1} u(t) \geq \gamma\|u\| = \gamma H_2 > \bar{H}_2$. So, by (2.6), we obtain

$$\begin{aligned}
(Tu)(1) &= \int_0^1 g(1,s)a(s)f(s,u)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&\geq \int_\sigma^1 \frac{1}{2}s(1-s)a(s)f(s,u)ds + \frac{\alpha}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)f(s,u)ds \\
&\geq \int_\sigma^1 \frac{1}{2}s(1-s)a(s)\delta u(s)ds + \frac{\alpha}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)\delta u(s)ds
\end{aligned}$$

$$\geq \delta\Big[\int_\sigma^1 \frac{\gamma}{2}s(1-s)a(s)ds + \frac{\alpha\gamma}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)ds\Big]\|u\|\,.$$

If $\delta N \geq 1$, then

$$Tu(1) \geq \|u\| \tag{3.1}$$

which implies that

$$\|Tu\| \geq \|u\|\,. \tag{3.2}$$

Therefore, by Theorem 2.4, the operator T has at least one fixed point, which is a positive solution of (1.1), (1.2). ☐

Theorem 3.2. *Assume that the conditions on a, f and* (b) *hold. Then* (1.1)*,* (1.2) *has at least one positive solution.*

Proof. Since $f_0 = \infty$, there exists $H_1 > 0$ such that $f(t,u) \geq \xi u$, for all $t \in [\sigma, 1]$, $0 < u \leq H_1$ where $\xi > 0$; thus, for $u \in K \cap \partial\Omega_1$, with $\Omega_1 = \{u \in X : \|u\| < H_1\}$, by (2.6), we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 g(1,s)a(s)f(s,u)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&\geq \int_\sigma^1 \frac{1}{2}s(1-s)a(s)f(s,u)ds + \frac{\alpha}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)f(s,u)ds \\
&\geq \int_\sigma^1 \frac{1}{2}s(1-s)a(s)\delta u(s)ds + \frac{\alpha}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)\delta u(s)ds \\
&\geq \xi\left[\int_\sigma^1 \frac{\gamma}{2}s(1-s)a(s)ds + \frac{\alpha\gamma}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)ds\right]\|u\|\,.
\end{aligned}$$

If $\xi N \geq 1$, then $Tu(1) \geq \|u\|$. Therefore

$$\|Tu\| \geq \|u\|\,.$$

On the other hand, since $f^\infty = 0$, there exists $\bar{H}_2 > 0$ such that $f(t,u) \leq \lambda u$, for all $t \in [0,1]$ with $\bar{H}_2 \leq u$ and $\lambda > 0$.

We consider two cases:

Case 1. Suppose f is bounded. Let L such that $f(t,u) \leq L$ and $\Omega_2 = \{u \in X : \|u\| < H_2\}$ where $H_2 = \max\{2H_1, LM\}$. If $u \in K \cap \partial\Omega_2$, then by Lemma 2.7, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 g(t,s)a(s)f(s,u)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&\leq \int_0^1 \frac{1}{2}s(1-s)a(s)f(s,u)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&\leq \int_0^1 \frac{1}{2}s(1-s)a(s)Lds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)Lds \\
&\leq L\Big[\int_0^1 \frac{1}{2}s(1-s)a(s) + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)ds\Big] \\
&\leq H_2 = \|u\|
\end{aligned}$$

and consequently, $\|Tu\| \leq \|u\|$.

Case 2. Suppose f is unbounded, then from (H1) there is $H_2 > \max\{2H_1, \bar{H}_2\}$ such that $f(t,u) \le f(t,H_2)$ with $0 < u \le H_2$ and let $\Omega_2 = \{u \in X : \|u\| < H_2\}$. If $u \in K \cap \partial\Omega_2$ and $\lambda M \le 1$, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 g(t,s)a(s)f(s,u)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&\le \int_0^1 \frac{1}{2}s(1-s)a(s)f(s,H_2)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,H_2)ds \\
&\le \int_0^1 \frac{1}{2}s(1-s)a(s)\lambda H_2 ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)\lambda H_2 ds \\
&\le \lambda\Big[\int_0^1 \frac{1}{2}s(1-s)a(s) + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)ds\Big] H_2 \\
&\le H_2 = \|u\|\,.
\end{aligned}$$

Thus, $\|Tu\| \le \|u\|$.

Therefore by Theorem 2.4, the operator T has at least one fixed point, which is a positive solution of (1.1), (1.2). □

Theorem 3.3. *Assume that the conditions on a, f and* (e) *hold. Then* (1.1), (1.2) *has at least one positive solution.*

Proof. Since $0 \le f^0 < R$, there exists $H_1 > 0$ and $0 < \varepsilon_1 < R$ such that $f(t,u) \le (R-\varepsilon_1)u$, $0 \le t \le 1$, $0 < u \le H_1$. Let $\Omega_1 = \{u \in X : \|u\| < H_1\}$. So for any $u \in K \cap \partial\Omega_1$,

$$\begin{aligned}
Tu(t) &= \int_0^1 g(t,s)a(s)f(s,u)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&\le \int_0^1 \frac{1}{2}s(1-s)a(s)(R-\varepsilon_1)uds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)(R-\varepsilon_1)uds \\
&\le (R-\varepsilon_1)\Big[\int_0^1 \frac{1}{2}s(1-s)a(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)ds\Big]\|u\| \\
&= (R-\varepsilon_1)M\|u\| < \|u\|\,.
\end{aligned}$$

Thus $\|Tu\| < \|u\|$.

Since $r < f_\infty \le \infty$, there exist $\bar{H}_2 > 0$ and $\varepsilon_2 > 0$ such that $f(t,u) \ge (r+\varepsilon_2)u$ for $u \ge \bar{H}_2$ and $\sigma \le t \le 1$. Let $H_2 = \max\{2H_1, \frac{\bar{H}_2}{\gamma}\}$ and $\Omega_2 = \{u \in X : \|u\| < H_2\}$. Then $u \in K \cap \partial\Omega_2$ implies $\min_{\sigma \le t \le 1} u(t) \ge \gamma\|u\| = \gamma H_2 > \bar{H}_2$. So, by (2.6) we obtain

$$\begin{aligned}
Tu(1) &= \int_0^1 g(1,s)a(s)f(s,u)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&\ge \int_\sigma^1 \frac{1}{2}s(1-s)a(s)(r+\varepsilon_2)uds + \frac{\alpha}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)(r+\varepsilon_2)uds \\
&\ge (r+\varepsilon_2)\Big[\int_\sigma^1 \frac{\gamma}{2}s(1-s)a(s)ds + \frac{\alpha\gamma}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)ds\Big]\|u\| \\
&= (r+\varepsilon_2)N\|u\| > \|u\|\,.
\end{aligned}$$

Thus, $\|Tu\| > \|u\|$.

Therefore, by Theorem 2.4, the operator T has at least one fixed point, which is a positive solution of (1.1), (1.2). □

Theorem 3.4. *Assume that the conditions on a, f and* (f) *hold. Then* (1.1), (1.2) *has at least one positive solution.*

4. Multiplicity results

Theorem 4.1. *Assume that the conditions on a, f,* (c) *and* (g) *hold. Then* (1.1), (1.2) *has at least two positive solutions.*

Proof. Since $f_0 = \infty$, $\exists H_1 > 0$ where $0 < H_1 < \rho$ such that $f(t,u) > ru$ with $0 < u \le H_1$ and $t \in [\sigma, 1]$. Let $\Omega_1 = \{u \in X : \|u\| < H_1\}$. Then for any $u \in K \cap \partial\Omega_1$,

$$\begin{aligned}
Tu(1) &= \int_0^1 g(1,s)a(s)f(s,u)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&> \int_\sigma^1 \frac{1}{2}s(1-s)a(s)ruds + \frac{\alpha}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)ruds \\
&> r\left[\int_\sigma^1 \frac{\gamma}{2}s(1-s)a(s)ds + \frac{\gamma\alpha}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)ds\right]\|u\| \\
&= rN\|u\| = \|u\|
\end{aligned}$$

Thus, $\|Tu\| > \|u\|$. Therefore, by Theorem 2.5

$$i(T, K_{H_1}, K) = 0\,.$$

Since $f_\infty = \infty$, there exists $\bar{H_2} > \rho$ such that $f(t,u) > ru$ with $u \ge \bar{H_2}$, $t \in [\sigma, 1]$. Let $H_2 = \frac{\bar{H_2}}{\gamma}$ and $\Omega_2 = \{u \in X : \|u\| < H_2\}$. Then for $u \in K \cap \partial\Omega_2$, we have $\min_{\sigma \le t \le 1} u(t) \ge \gamma\|u\| = \gamma H_2 = \bar{H_2}$. Hence,

$$\begin{aligned}
Tu(1) &= \int_0^1 g(1,s)a(s)f(s,u)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&> \int_\sigma^1 \frac{1}{2}s(1-s)a(s)ruds + \frac{\alpha}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)ruds \\
&> r\Big[\int_\sigma^1 \frac{\gamma}{2}s(1-s)a(s)ds + \frac{\gamma\alpha}{1-\alpha\eta}\int_\sigma^1 g(\eta,s)a(s)ds\Big]\|u\| \\
&= rN\|u\| = \|u\|\,.
\end{aligned}$$

Thus, $\|Tu\| > \|u\|$. Therefore, by Theorem 2.5

$$i(T, K_{H_2}, K) = 0\,.$$

On the other hand, let $\Omega_3 = \{u \in X : \|u\| < \rho\}$. For any $u \in K \cap \partial\Omega_3$, we get from (e) that $f(t,u) < R\rho$ for $0 \le t \le 1$, then

$$\begin{aligned}
Tu(t) &= \int_0^1 g(t,s)a(s)f(s,u)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)f(s,u)ds \\
&< \int_0^1 \frac{1}{2}s(1-s)a(s)R\rho\,ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)R\rho\,ds \\
&< R\Big[\int_0^1 \frac{1}{2}s(1-s)a(s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 g(\eta,s)a(s)ds\Big]\rho \\
&< RM\rho \le \|u\|\,.
\end{aligned}$$

Thus, $\|Tu\| < \|u\|$. Therefore, by Theorem 2.5,

$$i(T, K_\rho, K) = 1\,.$$

Hence,

$$i(T, K_{H_2} \setminus \overline{K_\rho}, K) = i(T, K_{H_2}, K) - i(T, K_\rho, K) = 0 - 1 = -1$$
$$i(T, K_\rho \setminus \overline{K_{H_1}}, K) = i(T, K_\rho, K) - i(T, K_{H_1}, K) = 1 - 0 = 1$$

Therefore, there exist at least two positive solutions $u_1 \in K \cap (\bar{\Omega_3} \backslash \Omega_1)$ and $u_2 \in K \cap (\bar{\Omega_2} \backslash \Omega_3)$ of (1.1),(1.2) in K, such that

$$0 < \|u_1\| < \rho < \|u_2\|\,. \tag{4.1}$$

□

Theorem 4.2. *Assume that the conditions on a, f,* (d) *and* (h) *hold. Then* (1.1), (1.2) *has at least two positive solutions.*

5. Examples

Example 5.1 (Superlinear and Sublinear Case)**.**

(a) If $f(t,u) = u^\alpha$, $\alpha > 1$, the conclusions of Theorem 3.1, hold.

(b) If $f(t,u) = 1 + u^\alpha$, $\alpha \in (0,1)$ the conclusions of Theorem 3.2, hold.

Example 5.2. Let $f(t,u) = \lambda t \ln(1+u) + u^2$, fix $\lambda > 0$, sufficiently small. Clearly $f^0 = \lambda$ and $f_\infty = \infty$. By Theorem 3.3, (1.1) and (1.2) have at least one positive solution.

Example 5.3. Let $f(t,u) = u^2 e^{-u} + \mu \sin u$, fix $\mu > 0$ sufficiently large. Then $f_0 = \mu$ and $f^\infty = 0$. By Theorem 3.4, (1.1) and (1.2) have at least one positive solution.

Example 5.4. Consider the boundary-value problem

$$u'''(t) + u^b + u^c = 0, \quad 0 < t < 1, \tag{5.1}$$

$$u(0) = 0, \quad u'(0) = u'(1) = \frac{1}{4} u(\frac{1}{2}), \tag{5.2}$$

where $f(t,u) = f(u) = u^b + u^c$, $a(t) = 1$, $b \in (0,1)$ and $c > 1$. Then $f_0 = \infty$ and $f_\infty = \infty$. By a simple calculation, $M = 2/21$ then $R = 21/2$.

On the other hand, we could choose $\rho = 1$, then $f(t,u) \le 2 < \frac{21}{2}1 = R\rho$ for $(t,u) \in [0,1] \times [0,\rho]$. By Theorem 4.1, (5.1) and (5.2) have at least two positive solutions.

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