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ALMOST AUTOMORPHY FOR BOUNDED SOLUTIONS TO SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

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Abstract. By introducing the method of decomposition of almost automorphic sequence, we give some results on the almost automorphy of bounded solutions for the second-order neutral differential equations with piecewise constant argument

$$(x(t) + px(t-1))'' = qx([t]) + f(t), \quad t \in \mathbb{R},$$

where $[\cdot]$ denotes the greatest integer function, p, q are nonzero constants, and $f(t)$ is almost automorphic. Some examples illustrate our results.

1. Introduction

Differential equations with piecewise constant argument (EPCA) describe hybrid dynamical systems (a combination of continuous and discrete dynamics). These equations have the structure of continuous dynamical systems within intervals and the solution is continuous, and so they combine the properties of both differential and difference equations. The study of EPCA was initiated by Cooke, Busenberg, Wiener and Shah in the early 1980's [3, 4, 20, 24]. Then there are many results concerning EPCA (see e.g. [5, 2, 9, 13, 22] and references therein). However, there are only a few works on the almost automorphy of solutions of EPCA. To the best of our knowledge, only Minh et al [18] in 2006 and Dimbour [8] in 2011 studied in this line. They give sufficient conditions for the almost automorphy of bounded solutions to some first-order EPCA.

The purpose of this paper is to give some results for the almost automorphy of bounded solutions to second-order EPCA

$$(x(t) + px(t-1))'' = qx([t]) + f(t), \tag{1.1}$$

where p and q are nonzero constants, $f : \mathbb{R} \to \mathbb{R}$, and $[\cdot]$ denotes the greatest integer function. Some results on the existence and the spectrum inclusion of almost periodic solutions to (1.1) were obtained in [19, 11, 12, 23]. For the studies of equations of almost automorphy we refer the readers to [16, 17, 21, 6, 7, 15, 25] and the references therein.

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The standard method to deal with EPCA such as (1.1) is as follows. First, get the solution of the corresponding difference system which is given by a series in the form

$$u(n)=\sum_{m\le n-1}\lambda^{n-m-1}k(m)\quad\text{or}\quad u(n)=-\sum_{m\ge n}\lambda^{n-m-1}k(m),\tag{1.2}$$

where λ is an eigenvalue of some matrix of the difference system. The convergence of the series is guaranteed by $|\lambda|\neq 1$ which was always assumed. Then construct the solutions of the differential equation inductively by

$$x(t)=\begin{cases}-\sum_{m=1}^{\infty}(-p)^{-m}\omega(t+m), & |p|>1,\\ \sum_{m=0}^{\infty}(-p)^{m}\omega(t-m), & |p|<1,\end{cases}\tag{1.3}$$

where $|p|\neq 1$ and $w(t)$ is a function in terms of $u(n)$ and $f(t)$ (see e.g. [1, 26, 27]). In this paper, we present a result on the almost automorphy of bounded solutions to (1.1) for this case (see Theorem 3.5).

However, for the case when $|p|=1$ or $|\lambda|=1$, the problem becomes much different, the series in (1.2) and (1.3) may not converge. This is the main difficult in the study of (1.1) for this case. To overcome this difficult, a valid method – *decomposition of almost periodic sequence* – was introduced in [11, 14]. Motivated by this decomposition, we introduce the *decomposition of almost automorphic sequence* in this paper. By using this decomposition method, we give some results on the the almost automorphy of bounded solutions to (1.1) for the case $|p|=1$ or $|\lambda|=1$ (see Theorem 3.6 -3.8).

This article is organized as follows. In section 2, some preliminary results and notation are presented. In section 3, we first transform the corresponding difference system of (1.1) to some vector forms in subsection 3.1. Then we state our main results in subsection 3.2. Section 4 is devoted to the proof of our main results. Finally, some examples are given to illustrate the main results in section 5.

2. Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, integers, real and complex numbers, respectively. $\mathbb{X}$ stands for the N-dimensional Euclidean space $\mathbb{R}^N$ or $\mathbb{C}^N$ endowed with Euclidean norm $|\cdot|$. Moreover, let $BC(\mathbb{R},\mathbb{X})$ be the space of bounded continuous functions $f:\mathbb{R}\to\mathbb{X}$ and let $C(\mathbb{R},\mathbb{X})$ be the space of continuous functions from $\mathbb{R}$ to $\mathbb{X}$.

2.1. **Almost automorphic function.**

Definition 2.1. A measurable bounded function $f:\mathbb{R}\to\mathbb{X}$ is said to be almost automorphic if for any sequence of real numbers $\{s_n'\}$, there exists a subsequence $\{s_n\}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty}f(t+s_n-s_m)=f(t)$$

for any $t\in\mathbb{R}$. Denote by $AA(\mathbb{X})$ the set of all such functions.

This limit means that

$$g(t)=\lim_{n\to\infty}f(t+s_n)$$

is well defined for each $t\in\mathbb{R}$ and

$$f(t)=\lim_{n\to\infty}g(t-s_n)$$

for each $t\in\mathbb{R}$.

It is clear that the function g in Definition 2.1 is bounded and measurable. Some properties of the almost automorphic functions are listed below.

Proposition 2.2 ([16, 17]). *Let $f, f_1, f_2 \in AA(\mathbb{X})$. Then the following statements hold:*

(i) $\alpha f_1 + \beta f_2 \in AA(\mathbb{X})$ *for* $\alpha, \beta \in \mathbb{R}$.
(ii) $f_\tau := f(\cdot + \tau) \in AA(\mathbb{X})$ *for every fixed* $\tau \in \mathbb{R}$.
(iii) $\breve{f} = f(-\cdot) \in AA(\mathbb{X})$.
(iv) *The range R_f of f is precompact, so f is bounded.*
(v) *If $\{f_n\} \subset AA(\mathbb{X})$ such that $f_n \to f$ uniformly on $\mathbb{R}$, then $f \in AA(\mathbb{X})$.*

By Proposition 2.2 (v), $AA(\mathbb{X})$ is a Banach space equipped with the sup norm $\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$.

2.2. **Almost automorphic sequence.** Let $l^\infty(\mathbb{X})$ be the space of all bounded (two-sided) sequences $x : \mathbb{Z} \to \mathbb{X}$ with sup-norm, and c_0 be the Banach space of all numerical sequences $\{a_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} a_n = 0$, equipped with sup-norm.

Definition 2.3. A sequence $x \in l^\infty(\mathbb{X})$ is said to be almost automorphic if for any sequence of integers $\{k_n'\}$, there exists a subsequence $\{k_n\}$ such that

$$\lim_{m\to\infty} \lim_{n\to\infty} x_{p+k_n-k_m} = x_p$$

for any $p \in \mathbb{Z}$. Denote by $aa(\mathbb{X})$ the set of all such sequences.

This limit means that

$$y_p = \lim_{n\to\infty} x_{p+k_n}$$

is well defined for each $p \in \mathbb{Z}$ and

$$x_p = \lim_{n\to\infty} y_{p-k_n}$$

for each $p \in \mathbb{Z}$.

It is obvious that $aa(\mathbb{X})$ is a closed subspace of $l^\infty(\mathbb{X})$, the range of an almost automorphic sequence is precompact, and $\{x(n)\} \in aa(\mathbb{R})$ if $x \in AA(\mathbb{R})$.

Lemma 2.4 ([10]). *Let B be a bounded linear operator in $\mathbb{X}$ with $\sigma_\Gamma(B)$ (the part of the spectrum of B on the unit circle of the complex plane) being countable, and let $\mathbb{X}$ not contain any subspace isomorphic to c_0. Assume further that $x = \{x_n\} \in l^\infty(\mathbb{X})$ satisfies*

$$x_{n+1} = Bx_n + y_n, \quad n \in \mathbb{Z},$$

where $\{y_n\} \in aa(\mathbb{X})$. Then $x \in aa(\mathbb{X})$.

Let $2^{aa(\mathbb{X})} = \{U : U \subset aa(\mathbb{X})\}$ and $\mathbb{E}$ be the set of all sequences in $\mathbb{X}$. For the decomposition of almost automorphic sequence, we define three functions $D_\gamma : aa(\mathbb{X}) \to 2^{aa(\mathbb{X})}$ for $\gamma = 1$ or -1 , $\phi_\theta : aa(\mathbb{X}) \to \mathbb{E}$ and $\psi_\theta : aa(\mathbb{X}) \to \mathbb{E}$ for $\theta \in (0, \pi)$ by

$$D_\gamma\{a(n)\} = \{\{b(n)\} \in aa(\mathbb{X}) : a(n) = b(n+1) + \gamma b(n), n \in \mathbb{Z}\},$$

$$\phi_\theta\{a(n)\}(n) = \begin{cases} \sum_{m=0}^{n-1} a(m)\cos(n-m-1)\theta, & n > 0,\\ 0, & n = 0,\\ -\sum_{m=-1}^{n} a(m)\cos(n-m-1)\theta, & n < 0, \end{cases}$$

and

$$\psi_\theta\{a(n)\}(n) = \begin{cases} \sum_{m=0}^{n-1} a(m)\sin(n-m-1)\theta, & n>0, \\ 0, & n=0, \\ -\sum_{m=-1}^{n} a(m)\sin(n-m-1)\theta, & n<0, \end{cases}$$

respectively, for $\{a(n)\} \in aa(\mathbb{X})$. Let $\{a(n)\} = \{(a_1(n), a_2(n), \dots, a_N(n))^T\} \in aa(\mathbb{X})$. It is clear that $D_\gamma\{a(n)\} \neq \emptyset$ if and only if $D_\gamma\{a_i(n)\} \neq \emptyset, i = 1, 2, \dots, N$. We note that $\alpha U + \beta V = \{\{c(n)\} : c(n) = \alpha a(n) + \beta b(n), n \in \mathbb{Z}, \{a(n)\} \in U, \{b(n)\} \in V\}$ for $\alpha, \beta \in \mathbb{R}$, $U, V \subset aa(\mathbb{X})$, and D_γ^k is the k-th power of D_γ for $k \in \mathbb{N}$.

Now we give some basic properties of the decomposition of almost automorphic sequences.

Proposition 2.5. *Let $\{a(n)\}, \{b(n)\} \in aa(\mathbb{X})$. Then the following statements hold:*

(i) *Let $\mu = \phi_\theta$ or $\psi_\theta (0 < \theta < \pi), \alpha \in \mathbb{X} \setminus \{0\}$. Then*

$$D_\gamma\{\alpha a(n)\} = \alpha D_\gamma\{a(n)\}, \quad D_\gamma\{a(n)\} + D_\gamma\{b(n)\} \subset D_\gamma\{a(n)+b(n)\},$$
$$\mu\{\alpha a(n)\} = \alpha\mu\{a(n)\}, \quad \mu\{a(n)\} + \mu\{b(n)\} = \mu\{a(n)+b(n)\}.$$

(ii) *$D_\gamma\{a(n)\} \neq \emptyset$ implies that $D_\gamma\{Aa(n)\} \neq \emptyset$ for any real or complex matrix A.*

(iii) *$D_\gamma\{(-\gamma)^n c\} \neq \emptyset$ for $c \in \mathbb{X}$ if and only if $c = 0$.*

(iv) *If $\{b(n)\} \in D_\gamma^k\{a(n)\}$, $k \in \mathbb{N}$,*

$$D_\gamma^k\{a(n)\} = \{\{b(n) + (-\gamma)^n c\} : c \in \mathbb{X}\}. \tag{2.1}$$

Furthermore, there is at most one $\{b(n)\} \in D_\gamma^k\{a(n)\}$ such that $D_\gamma\{b(n)\} \neq \emptyset$.

Proof. Statement (i) follows immediately from the definitions of the functions D_γ, ϕ_θ and ψ_θ, and (ii) follows form (i).

(iii) It is obvious that $D_\gamma\{0\} \neq \emptyset$. Meanwhile, suppose that there exists $\{b(n)\} \in D\{(-\gamma)^n c\}$. Then $(-\gamma)^n c = b(n+1) + \gamma b(n), n \in \mathbb{Z}$. This implies that $b(n) = (-\gamma)^{n-1} nc + (-\gamma)^n b(0)$, which yields that $c = 0$ since $\{b(n)\}$ is bounded. So (iii) holds.

(iv) Suppose that $\{b(n)\} \in D_\gamma\{a(n)\}$. It is easy to see that $\{b(n) + (-\gamma)^n c\} \in D_\gamma\{a(n)\}$ for any $c \in \mathbb{X}$. On the other hand, if $\{\bar{b}(n)\} \in D_\gamma\{a(n)\}$, then $a(n) = b(n+1) + \gamma b(n) = \bar{b}(n+1) + \gamma\bar{b}(n)$ for $n \in \mathbb{Z}$. This implies that $\bar{b}(n) = b(n) + (-\gamma)^n(\bar{b}(0) - b(0)), n \in \mathbb{Z}$, that is, $\{\bar{b}(n)\} \in \{\{b(n) + (-\gamma)^n c\} : c \in \mathbb{X}\}$. Hence (2.1) holds for $k = 1$.

Let $D_\gamma\{a(n)\} = \{\{b(n) + (-\gamma)^n c\} : c \in \mathbb{X}\}$, and assume that $D_\gamma\{b(n) + (-\gamma)^n c\} \neq \emptyset$ for some $c \in X$. Then by (i), we have $\emptyset \neq D_\gamma\{b(n) + (-\gamma)^n c\} - D_\gamma\{b(n)\} \subset D_\gamma\{(-\gamma)^n c\}$. This implies that $c = 0$ by (iii). So (iv) holds for $k = 1$.

Suppose that (iv) holds for $k = l$. Let $\{d(n)\} \in D_\gamma^l\{a(n)\}$ be the only one such that $D_\gamma\{d(n)\} \neq \emptyset$. If $\{e(n)\} \in D_\gamma^{l+1}\{a(n)\}$, $\{e(n)\} \in D_\gamma\{d(n)\}$. By the same argument as in the proof of (iv) for $k = 1$, we can prove that $D_\gamma\{d(n)\} = \{\{e(n) + (-1)^n c\} : c \in \mathbb{X}\}$ and there exists at most one $\{e'(n)\} \in D_\gamma\{d(n)\}$ such that $D_\gamma\{e'(n)\} \neq \emptyset$. So (iv) holds for $k = l + 1$ since $D_\gamma\{d(n)\} = D_\gamma^{l+1}\{a(n)\}$, and then (iv) holds. $\square$

Remark 2.6. Let $\{a(n)\} \in aa(\mathbb{R}^N)$. Denote by $S_r = D_\gamma\{a(n)\}$ for $\{a(n)\}$ regarded as a sequence in $\mathbb{R}^N$ and $S_c = D_\gamma\{a(n)\}$ for $\{a(n)\}$ regarded as a sequence in $\mathbb{C}^N$. Then $S_r \subset S_c$. On the other hand, if $\{b(n)\} \in S_c$, by Proposition 2.5 (iv) we have $S_c = \{\{b(n) + (-\gamma)^n c\} : c \in \mathbb{C}^N\}$, and it is easy to see that $S_r = \{\{Re(b(n)) + (-\gamma)^n c\} : c \in \mathbb{R}^N\}$, where $Re(b(n))$ is the real part of $b(n)$. So $S_c \neq \emptyset$ if and only if $S_r \neq \emptyset$.

3. Statements of main results

3.1. Transform of the corresponding difference system of (1.1).

Definition 3.1. A function $x(t)$ on $\mathbb{R}$ is said to be a solution of (1.1) if

(1) $x(t)$ is continuous on $\mathbb{R}$.
(2) The one sided second derivatives of $x(t) + px(t-1)$ exist everywhere, with the possible exception of the points $n(n \in \mathbb{Z})$, where one-sided second derivatives exist.
(3) $x(t)$ satisfies (1.1) for all $t \in \mathbb{R}$, $t \neq n \in \mathbb{Z}$.

We can rewrite (1.1) as the system

$$\begin{gathered}(x(t) + px(t-1))' = y(t),\\ y'(t) = qx([t]) + f(t).\end{gathered} \tag{3.1}$$

The solution $(x(t), y(t))$ of (3.1) can be defined similar to Definition 3.1. Denote $\omega(t) = x(t) + px(t-1)$, it is easy to see that

$$\omega(t) = x(n) + px(n-1) + y(n)(t-n) + \frac{q}{2}x(n)(t-n)^2 + \int_n^t \int_n^s f(r)\,dr\,ds \tag{3.2}$$

for $t \in [n, n+1]$. As in [19, 11, 12], let

$$f_n^{(1)} = \int_n^{n+1} \int_n^s f(r)\,dr\,ds, \quad f_n^{(2)} = \int_n^{n+1} f(s)ds.$$

By a process of integrating (3.1), we can see that, if $(x(t), y(t))$ is a solution of (3.1), $\{(x(n), y(n))\}$ is a solution of the difference system

$$\begin{gathered}x(n+1) = ax(n) + y(n) + px(n-1) + f_n^{(1)},\\ y(n+1) = qx(n) + y(n) + f_n^{(2)},\end{gathered}$$

where $a = 1 - p + q/2$, and this can be also expressed equivalently as

$$v_{n+1} = Av_n + h_n, \quad n \in \mathbb{Z}, \tag{3.3}$$

with $v_n = (x(n), y(n), x(n-1))^{\mathrm{T}}$, $h_n = (f_n^{(1)}, f_n^{(2)}, 0)^{\mathrm{T}}$ and

$$A = \begin{pmatrix} a & 1 & p \\ q & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In the following, $\lambda_i, i = 1, 2, 3$ denote the three eigenvalues of A. Then there exists a nonsingular matrix $P = (p_{ij})_{3\times 3}$ such that

$$PAP^{-1} = \Lambda, \tag{3.4}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 1 & \lambda_3 \end{pmatrix} \quad or \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_2 & 0 \\ 0 & 1 & \lambda_3 \end{pmatrix},$$

with $\lambda_i \neq \lambda_j$ if $i \neq j$, $\lambda_1 \neq \lambda_2 = \lambda_3$ or $\lambda_1 = \lambda_2 = \lambda_3$, and (3.3) can be written in the form

$$u_{n+1} = \Lambda u_n + k_n, \tag{3.5}$$

where $u_n = (u_1(n), u_2(n), u_3(n))^T = Pv_n$, $k_n = (k_1(n), k_2(n), k_3(n))^T = Ph_n$ for $n \in \mathbb{Z}$.

For the eigenvalues of A, we have the following results.

Lemma 3.2 ([11, Lemma 1.1])**.** *The following three statements are equivalent.*

(1) *One of the eigenvalues of A has absolute value 1.*
(2) *-1 is an eigenvalue of A.*
(3) *$p = 1$.*

Furthermore, if one of the above three statements holds, we have

(i) *If $q < -8$ or $q > 0$, $1 + (q \pm \sqrt{q^2 + 8q})/4 \in \mathbb{R}$ are the other two eigenvalues of A, which have absolute values different from* 1.
(ii) *If $-8 < q < 0$, the other two eigenvalues of A are $e^{i\theta}$ and $e^{-i\theta}$ with $0 < \theta < \pi$.*
(iii) *If $q = -8$, all the eigenvalues of A are -1.*

We assume that $f \in AA(\mathbb{R})$ throughout the paper without any further mention.

Lemma 3.3. *$\{f_n^{(1)}\}, \{f_n^{(2)}\}$ belongs to $aa(\mathbb{R})$.*

Proof. $\{f_n^{(2)}\} \in aa(\mathbb{R})$ is given by [18, Lemma 3.2]. Since $f(t)$ is almost automorphic, for any sequence $\{n'_k\}$, there exists a subsequence $\{n_k\}$ and a measurable function $g(t)$ such that

$$\lim_{k\to\infty} f(t + n_k) = g(t), \quad \lim_{k\to\infty} g(t - n_k) = f(t), \quad t \in \mathbb{R}.$$

Consequently, it follows from the *Lebesgue dominated convergence theorem* that, for each $n \in \mathbb{Z}$,

$$\begin{aligned}
f^{(1)}_{n+n_k} &= \int_{n+n_k}^{n+1+n_k} \int_{n+n_k}^{s} f(r)\,dr\,ds = \int_n^{n+1} \int_n^s f(r + n_k)\,dr\,ds \\
&\to \int_n^{n+1} \int_n^s g(r)\,dr\,ds \triangleq \bar{g}_n, \\
\bar{g}_{n-n_k} &= \int_{n-n_k}^{n+1-n_k} \int_{n-n_k}^{s} g(r)\,dr\,ds = \int_n^{n+1} \int_n^s g(r - n_k)\,dr\,ds \\
&\to \int_n^{n+1} \int_n^s f(r)\,dr\,ds = f^{(1)}_n,
\end{aligned}$$

as $k \to \infty$. So $\{f_n^{(1)}\} \in aa(\mathbb{R})$. $\square$

Lemma 3.4. *If $x(t)$ is a bounded solution of* (1.1), *$\{v_n\} = \{(x(n), y(n), x(n-1))^T\}$ is an almost automorphic solution of* (3.3).

Proof. By the boundedness of $x(t)$ and $f(t)$, it is easy to see that $x(t) + px(t-1)$ and $(x(t) + px(t-1))''$ are bounded. This implies that $y(t) = (x(t) + px(t-1))'$ is bounded (say, by the well known Landau-Hadamand inequality). Then $\{v_n\}$ is a bounded solution of (3.3). By Lemma 3.3, $\{h_n\} = \{(f_n^{(1)}, f_n^{(2)}, 0)^T\} \in aa(\mathbb{R}^3)$. Meanwhile, it is clear that $\mathbb{R}^3$ does not contain any subspace isomorphic to c_0, and the bounded linear operator A on $\mathbb{R}^3$ has finite spectrum. So Lemma 2.4 implies that $\{v_n\} \in aa(\mathbb{R}^3)$. $\square$

3.2. **Statements of the main results.** The following assumptions will be used later:

- (H1-) there exists $g \in AA(\mathbb{R})$ such that $\{g(n+\delta)\} \in D_{-1}\{f(n+\delta)\}$ for $\delta \in [0,1)$.
- (H1+) there exists $g \in AA(\mathbb{R})$ such that $\{g(n+\delta)\} \in D_1\{f(n+\delta)\}$ for $\delta \in [0,1)$.
- (H2) $D_1^2\{f_n^{(i)}\} \neq \emptyset, i = 1, 2$.
- (H3) $D_1^4\{f_n^{(i)}\} \neq \emptyset$, $i = 1, 2$.
- (H4) $D_1^2\{f_n^{(i)}\} \neq \emptyset$, $\mu\{f_n^{(i)}\} \in aa(\mathbb{X})$ and $D_1\{\mu\{f_n^{(i)}\}\} \neq \emptyset, i = 1, 2$, where $\mu = \phi_\theta$ or ψ_θ, $0 < \theta < \pi$.

Now we are in a position to give the main results in this article, which will be proved in the next section.

Theorem 3.5. *If $|p| \neq 1$, every bounded solution of* (1.1) *is almost automorphic.*

Theorem 3.6. *If $p = -1$ and* (H1-) *holds, every bounded solution of* (1.1) *is almost automorphic.*

Theorem 3.7. *Suppose that $p = 1$ and* (H1+) *holds. Let $x(t)$ be a bounded solution of* (1.1) *and $\{v_n\} = \{(x(n), y(n), x(n-1))^T\}$ be as that in Lemma 3.4. Then $x \in AA(\mathbb{R})$ if*

$$D_1\{v_n\} \neq \emptyset. \tag{3.6}$$

If $p = 1$, for the solutions $\{v_n\}$ of (3.3) satisfying condition (3.6) we have the following result.

Theorem 3.8. *If $p = 1$, the following statements are true:*

- (I) *If $q < -8$ or $q > 0$, and* (H2) *holds, then* (3.3) *has a unique solution $\{v_n\}$ such that* (3.6) *holds.*
- (II) *If $q = -8$ and* (H3) *holds, then* (3.3) *has a unique solution $\{v_n\}$ such that* (3.6) *holds.*
- (III) *If $-8 < q < 0$ and* (H4) *is satisfied for θ given in Lemma 3.2* (ii), *then* (3.3) *has a solution $\{v_n\} = \{(x_n, y_n, x_{n-1})^T\}$ such that* (3.6) *holds. Moreover, the set of solutions of* (3.3) *such that* (3.6) *holds is that*

$$\begin{aligned} \mathcal{S} &= \{\{\bar{v}_n\} = \{(\bar{x}_n, \bar{y}_n, \bar{x}_{n-1})^T\} \\ &= \{v_n + A^n C : C = (c_1, c_2, c_3)^T \in \mathbb{R}^3, -2c_1 + c_2 + 2c_3 = 0\}. \end{aligned}$$

4. Proofs of main results

Lemma 4.1. *Let $\varsigma \in \mathbb{Z}, \rho \in \mathbb{R}, \{x_m\}, \{y_m\}, \{z_m\} \in aa(\mathbb{R}), g \in AA(\mathbb{R})$ and*

$$\begin{aligned} &\Omega_\varsigma\{\rho, x_m, y_m, z_m, g\}(t) \\ &= x_{n+\varsigma} + \rho x_{n+\varsigma-1} + y_{n+\varsigma}(t-n) + \frac{q}{2} z_{n+\varsigma}(t-n)^2 + \int_n^t \int_n^s g(r+\varsigma)\, dr\, ds \end{aligned} \tag{4.1}$$

for $t \in [n, n+1), n \in \mathbb{Z}$. Then $\Omega_\varsigma\{\rho, x_m, y_m, z_m, g\} \in AA(\mathbb{R})$. Moreover, denote $\Omega_\varsigma\{\rho, x_m, y_m, z_m, g\}$ by Ω_ς for simplicity, we have

$$\Omega_\varsigma(t) = \Omega_{\varsigma+1}(t-1), \quad t \in \mathbb{R}. \tag{4.2}$$

Proof. Equality (4.2) follows from (4.1) directly. To prove $\Omega_\varsigma \in AA(\mathbb{R})$, the proof for the case $\varsigma \neq 0$ is similar to the case $\varsigma = 0$. So we only consider the case $\varsigma = 0$, and this is completed by the following two steps.

Step 1. For any $\{n_k'\} \subset \mathbb{Z}$, there exists a subsequence $\{n_k\}$ of $\{n_k'\}$, two sequences $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$ and a function $\tilde{g} : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{gathered}
\lim_{k\to\infty} x_{n+n_k} = \tilde{x}_n, \quad \lim_{k\to\infty} \tilde{x}_{n-n_k} = x_n, \quad n \in \mathbb{Z}, \\
\lim_{k\to\infty} y_{n+n_k} = \tilde{y}_n, \quad \lim_{k\to\infty} \tilde{y}_{n-n_k} = y_n, \quad n \in \mathbb{Z}, \\
\lim_{k\to\infty} z_{n+n_k} = \tilde{z}_n, \quad \lim_{k\to\infty} \tilde{z}_{n-n_k} = z_n, \quad n \in \mathbb{Z}, \\
\lim_{k\to\infty} g(t+n_k) = \tilde{g}(t), \quad \lim_{k\to\infty} \tilde{g}(t-n_k) = g(t), \quad t \in \mathbb{R}.
\end{gathered} \tag{4.3}$$

Let

$$\tilde{\omega}(t) := \tilde{x}_n + \rho\tilde{x}_{n-1} + \tilde{y}_n(t-n) + \frac{q}{2}\tilde{z}_n(t-n)^2 + \int_n^t \int_n^s \tilde{g}(r)\,dr\,ds \tag{4.4}$$

for $t \in [n, n+1), n \in \mathbb{Z}$. Noticing that g and $\tilde{g}$ are bounded measurable, by (4.3) and (4.4),

$$\begin{aligned}
&|\Omega_0(t+n_k) - \tilde{\omega}(t)| \\
&\le |x_{n+n_k} - \tilde{x}_n| + \rho|x_{n+n_k-1} - \tilde{x}_{n-1}| + |y_{n+n_k} - \tilde{y}_n|(t-n) \\
&\quad + \frac{q}{2}|z_{n+n_k} - \tilde{z}_n|(t-n)^2 + \Big| \int_{n+n_k}^{t+n_k} \int_{n+n_k}^{s} g(r)\,dr\,ds - \int_n^t \int_n^s \tilde{g}(r)\,dr\,ds \Big| \\
&\le |x_{n+n_k} - \tilde{x}_n| + \rho|x_{n+n_k-1} - \tilde{x}_{n-1}| + |y_{n+n_k} - \tilde{y}_n| \\
&\quad + \frac{q}{2}|z_{n+n_k} - \tilde{z}_n| + \int_n^t \int_n^s |g(r+n_k) - \tilde{g}(r)|\,dr\,ds \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Similarly, we can show that $\lim_{k\to\infty} \tilde{\omega}(t-n_k) = \Omega_0(t)$ for each $t \in \mathbb{R}$.

Step 2. We consider the general case where $\{s_k'\}_{k\in\mathbb{Z}}$ may not be an integer sequence. Let $n_k' = [s_k']$ and $t_k' = s_k' - n_k' \in [0,1)$ for each k. Then by *Step 1*, there exist subsequences $\{t_k\}, \{s_k\}$ and $\{n_k\}$ of $\{t_k'\}, \{s_k'\}$ and $\{n_k'\}$, respectively, such that $t_k = s_k - n_k, k \in \mathbb{Z}$, $\lim_{k\to\infty} t_k = \bar{t} \in [0,1]$, (4.3) holds and for each $t \in \mathbb{R}$,

$$\lim_{k\to\infty} \Omega_0(t+\bar{t}+n_k) = \tilde{\omega}(t+\bar{t}), \quad \lim_{k\to\infty} \tilde{\omega}(t+\bar{t}-n_k) = \Omega_0(t+\bar{t}), \tag{4.5}$$

where $\tilde{\omega}$ is given by (4.4). Let $\tilde{\omega}_1 = \tilde{\omega}(\cdot + t_0)$. Then it is sufficient to prove that

$$\lim_{k\to\infty} \Omega_0(t+s_k) = \tilde{\omega}_1(t), \quad \lim_{k\to\infty} \tilde{\omega}_1(t-s_k) = \Omega_0(t), \quad \text{for each } t \in \mathbb{R}. \tag{4.6}$$

Now there are two cases to be considered: $\bar{t} + t > [\bar{t} + t]$ and $\bar{t} + t = [\bar{t} + t]$. Assume that $\bar{t} + t > [\bar{t} + t]$. Then $[t + \bar{t}] = [t + t_k]$ for sufficiently large k. Noticing the boundedness of $g(t), \{x_n\}$ and $\{y_n\}$, for sufficiently large k,

$$\begin{aligned}
&|\Omega_0(t+s_k) - \Omega_0(t+\bar{t}+n_k)| \\
&= |\Omega_0(t+t_k+n_k) - \Omega_0(t+\bar{t}+n_k)| \\
&\le |y_{[t+\bar{t}]+n_k}||t_k - \bar{t}| + \frac{q}{2}|z_{[t+\bar{t}]+n_k}||(t+t_k-[t+t_k])^2 - (t+\bar{t}-[t+\bar{t}])^2| \\
&\quad + \Big| \int_{[t+t_k]+n_k}^{t+t_k+n_k} \int_{[t+t_k]+n_k}^{s} g(r)\,dr\,ds - \int_{[t+\bar{t}]+n_k}^{t+\bar{t}+n_k} \int_{[t+\bar{t}]+n_k}^{s} g(r)\,dr\,ds \Big| \\
&\le |y_{[t+\bar{t}]+n_k}||t_k - \bar{t}| + \frac{q}{2}|z_{[t+\bar{t}]+n_k}|\,|(2t+t_k+\bar{t}-2[t+\bar{t}])(t_k-\bar{t})|
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t+\bar t}^{t+t_k}\int_{[t+\bar t]}^{s}|g(n_k+r)|\,dr\,ds\\
&\to 0 \quad \text{as } k\to\infty.
\end{aligned}
$$

This together with (4.5) implies that $\lim_{k\to\infty}\Omega_0(t+s_k)=\tilde\omega_1(t)$.

Assume that $t+\bar t=[t+\bar t]$, that is $t+\bar t\in\mathbb{Z}$. If $t+t_k\ge t+\bar t$, (4.6) can be proved by an argument similar to the above one, and we omit the details. If $t+t_k<t+\bar t$, $[t+t_k]=t+\bar t-1$ for sufficiently large k and $t+t_k-[t+t_k]\to 1$ as $k\to\infty$. Notice also that

$$
\Omega_0(m)=x_{m-1}+px_{m-2}+y_{m-1}+\frac{q}{2}z_{m-1}+\int_{m-1}^{m}\int_{m-1}^{s}g(r)\,dr\,ds
$$

for any $m\in\mathbb{Z}$. Then for sufficiently large k,

$$
\begin{aligned}
&|\Omega_0(t+s_k)-\Omega_0(t+\bar t+n_k)|\\
&=|\Omega_0(t+t_k+n_k)-\Omega_0(t+\bar t+n_k)|\\
&\le |y_{t+\bar t-1+n_k}||t_k-\bar t|+\frac{q}{2}|z_{t+\bar t-1+n_k}||(t_k-\bar t+1)^2-1|\\
&\quad+\Big|\int_{[t+t_k]+n_k}^{t+t_k+n_k}\int_{[t+t_k]+n_k}^{s}g(r)\,dr\,ds-\int_{t+\bar t-1+n_k}^{t+\bar t+n_k}\int_{t+\bar t-1+n_k}^{s}g(r)\,dr\,ds\Big|\\
&\le |y_{t+\bar t-1+n_k}||t_k-\bar t|+\frac{q}{2}|z_{t+\bar t-1+n_k}|\,|(t_k-\bar t+1)^2-1|\\
&\quad+\int_{t+\bar t}^{t+t_k}\int_{t+\bar t-1}^{s}|g(r+n_k)|\,dr\,ds\\
&\to 0 \quad \text{as } k\to\infty.
\end{aligned}
$$

This together with (4.5) leads to $\lim_{k\to\infty}\Omega_0(t+s_k)=\tilde\omega_1(t)$.

Similarly, we can prove that $\lim_{k\to\infty}\tilde\omega_1(t-s_k)=\Omega_0(t)$ for each $t\in\mathbb{R}$, and then (4.6) is true. $\square$

Proof of Theorem 3.5. Let $x(t)$ be a bounded solution of (1.1), and $\omega(t)=x(t)+px(t-1)$ be given by (3.2). Then $\omega=\Omega_0\{p,x(m),y(m),x(m),f\}$. By Lemma 4.1, $\omega\in AA(\mathbb{R})$. It is easy to obtain that

$$
x(t)=\begin{cases}-\sum_{m=1}^{\infty}(-p)^{-m}\omega(t+m), & |p|>1,\\ \sum_{m=0}^{\infty}(-p)^{m}\omega(t-m), & |p|<1.\end{cases}
$$

Therefore $x\in AA(\mathbb{R})$ by Proposition 2.2. This completes the proof. $\square$

Proof of Theorem 3.6. Let $x(t)$ be a bounded solution of (1.1). Then $\{v_n\}=\{(x(n),y(n),x(n-1))^T\}\in aa(\mathbb{R}^3)$ is a solution of (3.3) by Lemma 3.4. We first prove that

$$
D_{-1}\{v_n(n)\}\neq\emptyset. \tag{4.7}
$$

By Lemma 3.3, $\{h_n\}=\{(f_n^{(1)},f_n^{(2)},0)^T\}\in aa(\mathbb{R}^3)$. So $\{k_n\}=\{Ph_n\}\in aa(\mathbb{C}^3)$. By (H1-), it is easy to see that $D_{-1}\{f_n^{(j)}\}\neq\emptyset$, $j=1,2$. Then $D_{-1}\{h_n\}\neq\emptyset$. It follows from Proposition 2.5 (ii) that

$$
D_{-1}\{k_n\}=D_{-1}\{Ph_n\}\neq\emptyset. \tag{4.8}
$$

Since $p=-1$, it follows from Lemma 3.2 that $|\lambda_i|\neq 1$ for $i=1,2,3$. Let α,β be two constants defined such that, if all the eigenvalues of A are simple, $\alpha=\beta=0$; if one of the eigenvalues of A is double, without loss of generality $\lambda_2=\lambda_3$, then

$\alpha = 0$, $\beta = 1$; if the eigenvalue of A is triple, $\alpha = \beta = 1$. Then it is easy to see that the solution $\{u_n\} = \{(u_1(n), u_2(n), u_3(n))^T\}$ of (3.5) can be uniquely given by

$$u_1(n) = \begin{cases} \sum_{m\le n-1} \lambda_1^{n-m-1} k_1(m), & |\lambda_1| < 1, \\ -\sum_{m\ge n} \lambda_1^{n-m-1} k_1(m), & |\lambda_1| > 1, \end{cases}$$

$$u_2(n) = \begin{cases} \sum_{m\le n-1} \lambda_2^{n-m-1} (k_2(m) + \alpha u_1(m)), & |\lambda_2| < 1, \\ -\sum_{m\ge n} \lambda_2^{n-m-1} (k_2(m) + \alpha u_1(m)), & |\lambda_2| > 1, \end{cases}$$

$$u_3(n) = \begin{cases} \sum_{m\le n-1} \lambda_3^{n-m-1} (k_3(m) + \beta u_2(m)), & |\lambda_3| < 1, \\ -\sum_{m\ge n} \lambda_3^{n-m-1} (k_3(m) + \beta u_2(m)), & |\lambda_3| > 1. \end{cases}$$

Moreover, it is easy to verify that $\{u_n\} \in aa(\mathbb{C}^3)$ by Proposition 2.2 and $D_{-1}\{u_n\} \neq \emptyset$ by (4.8). Since the solution $\{u_n\}$ of (3.5) is unique, the solution $\{v_n\} = \{P^{-1}u_n\}$ of (3.3) is also unique. Then by Proposition 2.5 (ii), $D_{-1}\{v_n\} = D_{-1}\{P^{-1}u_n\} \neq \emptyset$, and (4.7) holds.

By (4.7) and Remark 2.6, we can choose

$$\{(x^*(m), y^*(m), x^*(m-1))^T\} \in D_{-1}\{v_m\} \subset aa(\mathbb{R}^3),$$

and let $\Omega_\varsigma = \Omega_\varsigma\{0, x(m-1), y^*(m), x^*(m), g\}$ with g given by (H_1^-). By (3.2), (4.1), (4.2) and (H_1^-), for $t \in [n, n+1), n \in \mathbb{Z}$,

$$\begin{aligned} x(t) - x(t-1) &= \omega(t) = \Omega_0\{-1, x(m), y(m), x(m), f\}(t) \\ &= x(n) - x(n-1) + y(n)(t-n) + \frac{q}{2} x(n)(t-n)^2 + \int_n^t \int_n^s f(r)\,dr\,ds \\ &= x(n) - x(n-1) + (y^*(n+1) - y^*(n))(t-n) \\ &\quad + \frac{q}{2}(x^*(n+1) - x^*(n))(t-n)^2 + \int_n^t \int_n^s (g(1+r) - g(r))\,dr\,ds \\ &= \Omega_1(t) - \Omega_0(t) = \Omega_1(t) - \Omega_1(t-1). \end{aligned}$$

Then

$$x(t) = \Omega_1(t) + x(t-n) - \Omega_1(t-n), \quad t \in [n, n+1), n \in \mathbb{Z}.$$

Notice that $(x(t-n) - \Omega_1(t-n)), t \in [n, n+1), n \in \mathbb{Z}$ is a periodic function, then it is in $AA(\mathbb{R})$. Meanwhile, $\Omega_1 \in AA(\mathbb{R})$ by Lemma 4.1. Therefore $x \in AA(\mathbb{R})$ by Proposition 2.2 (i). This completes the proof. □

Proof of Theorem 3.7. Let $x(t)$ be the bounded solution of (1.1). If (3.6) holds, we can choose $\{(x^*(m), y^*(m), x^*(m-1))^T\} \in D_{-1}\{v_m\}$, and let $\Omega_\varsigma = \Omega_\varsigma\{0, x(m-1), y^*(m), x^*(m), g\}$ with g given by (H_1^+). By (3.2), (4.1), (4.2) and (H_1^+), for $t \in [n, n+1), n \in \mathbb{Z}$,

$$\begin{aligned} &x(t) + x(t-1) = \omega(t) \\ &= \Omega_0\{1, x(m), y(m), x(m), f\}(t) \\ &= x(n) + x(n-1) + y(n)(t-n) + \frac{q}{2} x(n)(t-n)^2 + \int_n^t \int_n^s f(r)\,dr\,ds \\ &= x(n) + x(n-1) + (y^*(n+1) + y^*(n))(t-n) + \frac{q}{2}(x^*(n+1) + x^*(n))(t-n)^2 \\ &\quad + \int_n^t \int_n^s (g(1+r) + g(r))\,dr\,ds \end{aligned}$$

$$= \Omega_1(t) + \Omega_0(t) = \Omega_1(t) + \Omega_1(t-1).$$

Then

$$x(t) = \Omega_1(t) + (-1)^n(x(t-n) - \Omega_1(t-n)), \quad t \in [n, n+1), n \in \mathbb{Z}.$$

Now by an argument similar to the end part of the proof of Theorem 3.6, we can get that $x \in AA(\mathbb{R})$. This completes the proof. $\square$

Proof of Theorem 3.8. By (H2) or (H3) or (H4), as to obtain (4.8), we can obtain that

$$D_1\{k_n\} = D_1\{Ph_n\} \neq \emptyset. \tag{4.9}$$

(I) By Lemma 3.2 (i), we may assume that $|\lambda_2| > 1$, $|\lambda_3| < 1$ if $q > 0$, and $|\lambda_2| < 1$, $|\lambda_3| > 1$ if $q < -8$. By (H2) and Proposition 2.5 (iv), we let $\{g_n^{(i)}\} \in aa(\mathbb{R})$ be the only one in $D_1\{f_n^{(i)}\}$ such that

$$D_1\{g_n^{(i)}\} \neq \emptyset, \quad i = 1, 2. \tag{4.10}$$

Define $\{u_n\} = \{(u_1(n), u_2(n), u_3(n))^T\}$ by

$$u_1(n) = p_{11}g_n^{(1)} + p_{12}g_n^{(2)},$$

$$u_2(n) = \begin{cases} -\sum_{m\geq n} \lambda_2^{n-m-1} k_2(m), & q > 0, \\ \sum_{m\leq n} \lambda_2^{n-m} k_2(m-1), & q < -8, \end{cases} \tag{4.11}$$

$$u_3(n) = \begin{cases} \sum_{m\leq n} \lambda_3^{n-m} k_3(m-1), & q > 0, \\ -\sum_{m\geq n} \lambda_3^{n-m-1} k_3(m), & q < -8, \end{cases} \tag{4.12}$$

with p_{11}, p_{12} two elements of the first row of matrix P given in (3.4). Then it is easy to verify that $\{u_n\} \in aa(\mathbb{R}^3)$ and is a solution of (3.5). Moreover, $D_1\{u_n\} \neq \emptyset$ by (4.9) and (4.10).

Assume that $\{\bar{u}_n\} = \{(\bar{u}_1(n), \bar{u}_2(n), \bar{u}_3(n))^T\}$ is a solution of (3.5) such that $D_1\{\bar{u}_n\} \neq \emptyset$. Notice that the last two components $u_2(n)$ and $u_3(n)$ of solution u_n for (3.5) is uniquely determined by (4.11) and (4.12), respectively. Then $\bar{u}_2(n) = u_2(n)$ and $\bar{u}_3(n) = u_3(n)$. By (3.5), we can get that $u_1(n+1)+u_1(n) = \bar{u}_1(n+1)+\bar{u}_1(n) = k_1(n), n \in \mathbb{Z}$. This means that $\bar{u}_1(n) = u_1(n) + (-1)^n(\bar{u}_1(0) - u_1(0))$, $n \in \mathbb{Z}$. Then by Proposition 2.5 (i) and (iii), $D_1\{\bar{u}_1(n)\} \neq \emptyset$ if and only if $u_1(0) - \bar{u}_1(0) = 0$. This implies that $D_1\{\bar{u}_1(n)\} \neq \emptyset$ if and only if $u_1(n) = \bar{u}_1(n), n \in \mathbb{Z}$. That is (3.5) has a unique solution $\{u_n\}$ satisfying $D_1\{u_n\} \neq \emptyset$. Therefore (I) holds, since $v_n = P^{-1}u_n, n \in \mathbb{Z}$.

(II) By Lemma 3.2 (iii), all the three eigenvalues of A are -1. By Proposition 2.5 (iv) and (H_4), let $\{a_j^{(i)}(n)\}$ be the only one in $D_1^j\{f_n^{(i)}\}$ such that

$$D_1\{a_j^{(i)}\} \neq \emptyset \quad i = 1, 2, j = 1, 2, 3. \tag{4.13}$$

Then $\{a_j^{(i)}(n)\} \in D_1^j\{f_n^{(i)}\} = D_1\{a_{j-1}^{(i)}\}$ for $i = 1, 2, j = 2, 3$. By a fundamental calculation as in [11, Section 3], P can be chosen as

$$P = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define $\{u_n\} = \{(u_1(n), u_2(n), u_3(n))^T\}$ by

$$u_1(n) = -2a_1^{(1)}(n) + a_1^{(2)}(n),$$

$$u_2(n) = -2a_2^{(1)}(n) + a_2^{(2)}(n) + a_1^{(1)}(n),$$
$$u_3(n) = -2a_3^{(1)}(n) + a_3^{(2)}(n) + a_2^{(1)}(n).$$

It is easy to verify that $\{u_n\} \in aa(\mathbb{R}^3)$ and is a solution of (3.5). Moreover, $D_1\{u_n\} \neq \emptyset$ by (4.13).

Assume that $\{\bar{u}_n\} = \{(\bar{u}_1(n), \bar{u}_2(n), \bar{u}_3(n))^T\}$ is a solution of (3.5) such that $D_1\{\bar{u}_n\} \neq \emptyset$. By an argument similar to prove $u_1(n) = \bar{u}_1(n), n \in \mathbb{Z}$ in the proof of (I), we can prove that $u_i(n) = \bar{u}_i(n), n \in \mathbb{Z}$ for $i = 1, 2, 3$. That is, (3.5) has a unique solution $\{u_n\}$ satisfying $D_1\{u_n\} \neq \emptyset$. This implies that (II) holds since $v_n = P^{-1}u_n, n \in \mathbb{Z}$.

(III) It follows from Lemma 3.2 (ii) that the eigenvalues of A are -1, $e^{i\theta}$ and $e^{-i\theta}$. By (H3) and Proposition 2.5 (iv), we can get a sequence $\{g_n^{(i)}\} \in aa(\mathbb{R})$ which is the only one in $D_1\{f_n^{(i)}\}$ such that (4.10) holds, $i = 1, 2$. By a fundamental calculation as in [11, Section 3], P and P^{-1} can be chosen as

$$P = \begin{pmatrix} -2 & 1 & 2 \\ e^{i\theta} - 1 & 1 & 1 - e^{-i\theta} \\ e^{-i\theta} - 1 & 1 & 1 - e^{i\theta} \end{pmatrix}, \tag{4.14}$$

$$P^{-1} = \begin{pmatrix} -1/(2\cos\theta + 2) & -\sigma e^{i\theta} & \sigma \\ (\cos\theta - 1)/(\cos\theta + 1) & 1/(\cos\theta + 1) & 1/(\cos\theta + 1) \\ 1/(2\cos\theta + 2) & -\sigma & \sigma e^{i\theta} \end{pmatrix} \tag{4.15}$$

with $\sigma = 1/\big((e^{i\theta} + 1)(e^{-i\theta} - e^{i\theta})\big)$. Define $\{u_n\} = \{(u_1(n), u_2(n), u_3(n))^T\}$ by

$$u_1(n) = -2g_n^{(1)} + g_n^{(2)},$$
$$u_2(n) = (e^{i\theta} - 1)(\phi_\theta\{f_n^{(1)}\}(n) + i\psi_\theta\{f_n^{(1)}\}(n)) + \phi_\theta\{f_n^{(2)}\}(n) + i\psi_\theta\{f_n^{(2)}\}(n),$$
$$u_3(n) = (e^{i\theta} - 1)(\phi_\theta\{f_n^{(1)}\}(n) - i\psi_\theta\{f_n^{(1)}\}(n)) + \phi_\theta\{f_n^{(2)}\}(n) - i\psi_\theta\{f_n^{(2)}\}(n).$$

It is easy to verify that $\{u_n\} \in aa(\mathbb{C}^3)$ and is a solution of (3.5). Moreover, $D_1\{u_n\} \neq \emptyset$ by (H4) and (4.10). Then (3.3) has a solution $\{v_n\} = \{(x_n, y_n, z_n)^T\} = \{P^{-1}u_n\} \in aa(\mathbb{C}^3)$ such that (3.6) holds. Moreover, by (4.14), (4.15) and a long-winded but fundamental calculation, we can verify that $v_n \in \mathbb{R}^3, n \in \mathbb{Z}$.

Now we prove that $\mathcal{S}$ is the set of all the solutions $\{\bar{v}_n\}$ of (3.3) such $D_1\{\bar{v}_n\} \neq \emptyset$.

If $\{\bar{v}_n\} = \{(\bar{x}_n, \bar{y}_n, \bar{x}_{n-1})^T\} \in aa(\mathbb{R}^3)$ is a solution of (3.3) such that $D_1\{\bar{v}_n\} \neq \emptyset$, $D_1\{P(\bar{v}_n - v_n)\} \neq \emptyset$ by Proposition 2.5 (i), and it follows from (3.3) that $\bar{v}_n = v_n + A^nC, n \in \mathbb{Z}$ with $C = (c_1, c_2, c_3)^T = \bar{v}_0 - v_0 \in \mathbb{R}^3$. Noticing (4.14), the first component of vector $P(\bar{v}_n - v_n) = PA^nC = \Lambda^nPC$ is $(-2c_1 + c_2 + 2c_3)(-1)^n$. Then $D_1\{(-2c_1 + c_2 + 2c_3)(-1)^n\} \neq \emptyset$, which implies that $-2c_1 + c_2 + 2c_3 = 0$ by Proposition 2.5 (iii). That is, $\{\bar{v}_n\} = \{v_n + A^nC\} \in S$.

On the other hand, assume that $\{\bar{v}_n\} = \{\bar{x}_n, \bar{y}_n, \bar{x}_{n-1})^T\} = \{v_n + A^nC\} \in S$. Then $-2c_1 + c_2 + 2c_3 = 0$. It is easy to verify that $\{\bar{v}_n\}$ is a solution of (3.3). Let $\beta_j = p_{j1}c_1 + p_{j2}c_2 + p_{j3}c_3$, $j = 2, 3$. By (4.14),

$$PA^nC = \Lambda^nPC = (0, \beta_2e^{i\theta n}, \beta_3e^{-i\theta n})^T$$

for $n \in \mathbb{Z}$. Since $0 < \theta < \pi$, $\cos(\theta/2) \neq 0$. Define $u^*(n) = (u_1^*(n), u_2^*(n), u_3^*(n))^T$ by

$$u_1^*(n) = 0,$$
$$u_2^*(n) = \frac{\beta_2}{2\cos(\theta/2)}\left(e^{i(n-1/2)\theta} + (-1)^{n-1}e^{-i(1/2)\theta}\right),$$

$$u_3^*(n) = \frac{\beta_3}{2\cos(\theta/2)}\left(e^{-i(n-1/2)\theta} + (-1)^{n-1}e^{i(1/2)\theta}\right),$$

for $n \in \mathbb{Z}$. It is not difficult to check that $\{u^*(n)\} \in aa(\mathbb{C}^3)$ and $PA^nC = u^*(n+1) + u^*(n), n \in \mathbb{Z}$. That is $D_1\{PA^nC\} \neq \emptyset$, and then $D_1\{A^nC\} \neq \emptyset$ by Proposition 2.5 (ii). Now by (3.6) and Proposition 2.5 (i), $D_1\{\bar{v}_n\} = D_1\{v_n + A^nC\} \neq \emptyset$. This completes the proof. □

5. Examples

In this section, we illustrate our main results. For this purpose, it suffice to use a function $f \in AA(\mathbb{R})$ such that:

(1) if $p = -1$, (H1-) holds;
(2) if $p = 1$, (H1+) holds. Moreover, (H2) holds if $q < -8$ or $q > 0$; (H3) holds if $q = -8$; (H4) holds if $-8 < q < 0$.

By the same argument for the proof of [11, Proposition 2.2] and [12, Proposition 1.1], we can get the following result (we omit the details of the proof).

Proposition 5.1. *Suppose that $\alpha, \beta \in \mathbb{R}$. Then the following statements hold.*

(i) *For $\alpha \neq 2k\pi, k \in \mathbb{Z}$,*
$$\Big\{\frac{\sin(\alpha(n-1/2)+\beta)}{2\sin(\alpha/2)}\Big\} \in D_{-1}\{\cos(\alpha n + \beta)\},$$
$$\Big\{-\frac{\cos(\alpha(n-1/2)+\beta)}{2\sin(\alpha/2)}\Big\} \in D_{-1}\{\sin(\alpha n + \beta)\}.$$
(ii) *For $\alpha \neq (2k+1)\pi, k, l \in \mathbb{Z}, l > 0$,*
$$\Big\{\frac{\cos(\alpha(n-l/2)+\beta)}{(2\cos(\alpha/2))^l}\Big\} \in D_1^l\{\cos(\alpha n + \beta)\},$$
$$\Big\{\frac{\sin(\alpha(n-l/2)+\beta)}{(2\cos(\alpha/2))^l}\Big\} \in D_1^l\{\sin(\alpha n + \beta)\}.$$
(iii) *Assume that $\alpha \neq (2k+1)\pi$ and $\alpha \neq 2k\pi \pm \theta$ for $0 < \theta < \pi, k \in \mathbb{Z}$, then $\mu\{\cos(\alpha n+\beta)\} \in aa(\mathbb{R})$, $\mu\{\sin(\alpha n+\beta)\} \in aa(\mathbb{R})$, $D_1\{\mu\{\cos(\alpha n+\beta)\}\} \neq \emptyset$ and $D_1\{\mu\{\sin(\alpha n + \beta)\}\} \neq \emptyset$ for $\mu = \phi_\theta$ or ψ_θ.*

Example 5.2. Let $\alpha_j, \alpha_j', \beta_j, \beta_j', \gamma_j, \gamma_j' \in \mathbb{R}$ and
$$f(t) = \sum_{j=1}^n \big(\gamma_j \cos(\alpha_j t + \beta_j) + \gamma_j' \sin(\alpha_j' t + \beta_j')\big).$$
Then $f \in AA(\mathbb{R})$.

(1) Assume that $p = -1$ and $\alpha_j, \alpha_j' \neq 2k\pi, k \in \mathbb{Z}$, $j = 1, 2, \dots, n$. Let
$$f^*(t) = \sum_{j=1}^n \Big(\gamma_j \frac{\sin(\alpha_j t + \beta_j - \alpha_j/2)}{2\sin(\alpha_j/2)} - \gamma_j' \frac{\cos(\alpha_j' t + \beta_j' - \alpha_j'/2)}{2\sin(\alpha_j'/2)}\Big).$$
Then $f^* \in AA(\mathbb{R})$, and by Proposition 2.5 (i) and 5.1 (i), we can check easily that $\{f^*(n+\delta)\} \in D_{-1}\{f(n+\delta)\}$ for each $\delta \in [0, 1)$. Hence condition (H1-) holds.

(2) Assume that $p = 1$, $\alpha_j, \alpha_j' \neq (2k+1)\pi, k \in \mathbb{Z}$, $j = 1, 2, \dots, n$. Let
$$f_l^*(t) = \sum_{j=1}^n \Big(\gamma_j \frac{\cos(\alpha_j t + \beta_j - l\alpha_j/2)}{(2\cos(\alpha_j/2))^l} + \gamma_j' \frac{\sin(\alpha_j' t + \beta_j' - l\alpha_j'/2)}{(2\cos(\alpha_j'/2))^l}\Big).$$

for $l > 0$. It follows from Proposition 2.5 (i) and 5.1 (ii) that $\{f_l^*(n+\delta)\} \in D_1^l\{f(n+\delta)\}$ for $\delta \in [0,1), l > 0$, and then we can check easily that conditions (H1+), (H2) and (H3) are satisfied.

Assume that $-8 < q < 0$, and in addition, $\alpha_j, \alpha_j' \neq 2k\pi \pm \theta, k \in \mathbb{Z}, j = 1, 2, \dots, n$. Here $\theta (0 < \theta < \pi)$ is given in Lemma 3.2 (ii). It follows from Proposition 2.5 (i) and Proposition 5.1 (iii) that $D_1\{\mu\{f(n)\}\} \neq \emptyset$ for $\mu = \phi_\theta$ or ψ_θ, and we can check easily that $\mu\{f_n^{(i)}\} \in aa(\mathbb{R})$ and $D_1\{\mu\{f_n^{(i)}\}\} \neq \emptyset$ for $i = 1, 2, \mu = \phi_\theta$ or ψ_θ. So condition (H4) is satisfied.

It is easy to see that the function $f(t)$ in Example 5.2 is almost periodic. For the case when $f(t)$ is almost automorphic but not almost periodic, an example is given below.

Example 5.3. Denote $\alpha(t) = \beta/(1 + \cos t \cos \pi t)$ with $\beta \neq 0$ a real constant.

(1) Assume $p = -1$. Let

$$f(t) = \sin \alpha(t+1) - \sin \alpha(t), \quad \hat{f}(t) = \sin \alpha(t).$$

Obviously, $f, \hat{f} \in AA(\mathbb{R})$ are not almost periodic, and $\{\hat{f}(n+\delta)\} \in D_{-1}\{f(n+\delta)\}$ for each $\delta \in [0,1)$. That is (H1-) holds.

(2) Assume $p = 1$. Let

$$\begin{gathered}
f_1(t) = \sin \alpha(t) + 4 \sin \alpha(t+1) + 6 \sin \alpha(t+2) + 4 \sin \alpha(t+3) + \sin \alpha(t+4), \\
f_2(t) = \sin \alpha(t) + 3 \sin \alpha(t+1) + 3 \sin \alpha(t+2) + \sin \alpha(t+3), \\
f_3(t) = \sin \alpha(t) + 2 \sin \alpha(t+1) + \sin \alpha(t+2), \\
f_4(t) = \sin \alpha(t) + \sin \alpha(t+1), \\
f_5(t) = \sin \alpha(t).
\end{gathered}$$

Obviously, $f_i \in AA(\mathbb{R})$, $i = 1, 2, 3, 4, 5$ are not almost periodic, and $\{f_i(n+\delta)\} \in D_1\{f_{i-1}(n+\delta)\}$ for $\delta \in [0,1), i = 2, 3, 4, 5$. Then it is easy to verify that (H1+) holds for $f = f_1, f_2, f_3$ or f_4; (H2) holds for $f = f_1, f_2$ or f_3; (H3) holds for $f = f_1$.

If $-8 < q < 0$, let $f = f_3$ and $\beta = 2(1 - \cos 1)\pi/(1 + \cos 1)$. Then it is easy to check that $f_4(0) = 0$ and $\{\psi_\theta\{f_4(n)\}(n)\}, \{\phi_\theta\{f_4(n)\}(n)\} \in aa(\mathbb{R})$. Moreover, for $n > 0$,

$$\begin{aligned}
&\psi_\theta\{f_4(n)\}(n+1) + \psi_\theta\{f_4(n)\}(n) \\
&= \sum_{m=0}^{n} f_4(m) \sin(n-m)\theta + \sum_{m=0}^{n-1} f_4(m) \sin(n-m-1)\theta \\
&= f_4(0) \sin n\theta + \sum_{m=0}^{n-1} (f_4(m+1) + f_4(m)) \sin(n-m-1)\theta \\
&= \sum_{m=0}^{n-1} f_3(m) \sin(n-m-1)\theta \\
&= \psi_\theta\{f_3(n)\}(n).
\end{aligned}$$

Similarly, we can prove that $\psi_\theta\{f_4(n)\}(n+1) + \psi_\theta\{f_4(n)\}(n) = \psi_\theta\{f_3(n)\}(n)$ also holds for $n \leq 0$. This leads to $D_1\{\psi_\theta\{f_n^{(i)}\}\} \neq \emptyset$. Furthermore, we can prove similarly that $D_1\{\phi_\theta\{f_n^{(i)}\}\} \neq \emptyset$. So (H4) is satisfied for $f = f_3$.

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