

NUMERICAL APPROACH FOR A RELAXED MINIMIZATION PROBLEM ARISING IN TOMOGRAPHIC RECONSTRUCTION

ALI SROUR, MOUHAMMAD JRADEH

ABSTRACT. The purpose of this article is to develop a numerical scheme for a system of optimality conditions for a smooth minimization problem that arises in tomographic reconstruction of binary axially symmetric objects. The density function of the object with the Lagrange multipliers is seen as a saddle point of an associated Lagrangian, then we use the Usawa scheme mixed with descent gradient method to give the corresponding numerical scheme.

1. INTRODUCTION

The applied mathematics is nowadays dedicating a great attention to the tomographic reconstruction, especially through the most eminent strategy entitled the variational method. In this article, we consider the problem

$$\begin{aligned} \text{minimize } F_o(u) &:= \frac{1}{2} \|Hu - g\|_{L^2(\Omega)}^2 + \lambda J(u), \\ u &\in \mathcal{D}, \end{aligned} \tag{1.1}$$

where

$$\mathcal{D} = \{u \in BV(\Omega) : u(u - 1) = 0 \text{ a.e. in } \Omega\},$$

and $BV(\Omega)$ denotes the space of the functions of bounded variation space defined by

$$BV(\Omega) = \{u \in L^1(\Omega) : J(u) < \infty\},$$

with

$$J(u) = \sup \left\{ \int_{\Omega} u(x) \operatorname{div}(\phi(x)) dx : \phi \in C_c^1(\Omega, \mathbb{R}^2), \|\phi\|_{\infty} \leq 1 \right\}.$$

Here $C_c^1(\Omega, \mathbb{R}^2)$ denotes the space of the C^1 functions with compact support in Ω with value in \mathbb{R}^2 . In the following, we shall denote $\|\cdot\|$ the $L^2(\Omega)$ -norm. In the same way, $(\cdot, \cdot)_2$ denotes the $L^2(\Omega)$ - scalar product, $(\cdot, \cdot)_{H^1}$ denotes the $H^1(\Omega)$ -scalar product and $\langle \cdot, \cdot \rangle_{V', V}$, the duality product between V' and V , where V is a Banach space and V' is the dual space of V . H is a projection operator which is detailed in the next section, and g is the density of the observed image.

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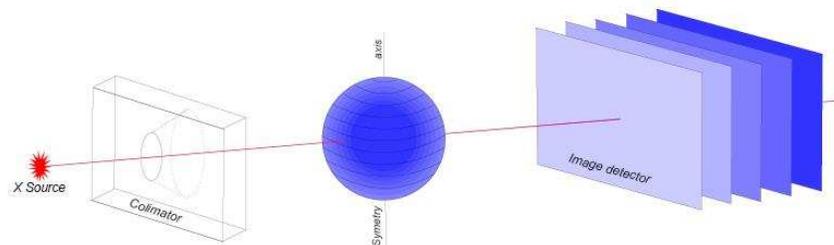


FIGURE 1. Experimental step [1]

Problem (1.1) is a non-regular minimization problem which has been obtained by Abraham et al [1], in order to reconstruct the density of a binary axially symmetric object by using the variational method. The problem (1.1) has two main difficulties, the first one is the lack of differentiability and the second is the non-convexity constraint. Considering these difficulties, we could not apply the classical optimization methods to prove the optimality system condition associated to (1.1).

In [5], the first author and Bergounioux have relaxed the binary constraint and regularized (1.1) to obtain a smooth minimization problem (3.2) without a convexity constraint and then prove the optimality system associated to (3.2). In this article, we will prove that through these modifications, we obtain an appropriate numerical result for the reconstruction of our symmetrical object.

The outline of this article is as follows: in the second section, we introduce and analyze the tomographic model; the setting of the problem is stated in section 3. In fourth section, we use a Lagrangien method based on a Uzawa method and the descent gradient method to develop a stable numerical scheme. The fifth section is dedicated to present our numerical scheme. Finally we discuss it by using some numerical test.

2. OUR MODEL OF TOMOGRAPHIC RECONSTRUCTION

In [1], the authors tried to use the tomographic technology to reconstruct the volume of a three-dimensional-axially-symmetrical object from a two-dimensional projection; that is, if an object is exposed to a beam of electrons, every point of this object is characterized by its attenuation coefficient. The tomographic reconstruction method consists of gathering these 2D projections of an X-ray radiography through an object and using these projections to rebuild the structure of this object in 3D (Figure 1).

Frequently, by using a sufficient number of projections, we can build the density of the object (see for instance [6]), however in [1] the authors used a single X-ray to rebuild the density of a binary axially-symmetrical object composed of one homogeneous material (drawn in black), and of some holes (in white) (see Figure 2(a)). With this hypothesis, it is sufficient to consider a plane section, and after making a rotation around the axle of symmetry, we obtained the 3D object.

Our work example, shown in Figure 2(a) represents a symmetric object containing all standard difficulties that may appear, such as:

- Several disconnected holes.

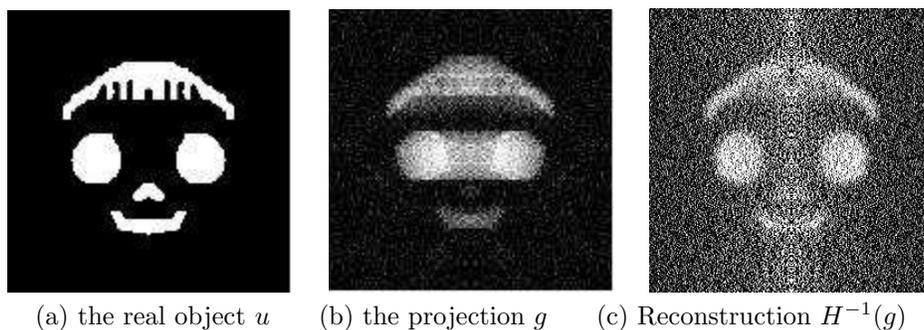


FIGURE 2. Instability of reconstruction by using H^{-1}

- Small holes located on the symmetry axis (where details are expected to be difficult to recover because the noise variance is maximal around the symmetric axis after reconstruction).
- Smaller details on the boundary of the top holes serve as a test for lower bound detection.

Assuming that u is the density of the initial object, and g is the density of the observed image. The problem of reconstruction consists of finding u such that $Hu = g$ where H is the projection operator from $L^2(\Omega)$ to $L^2(\Omega)$ given for a symmetric axial object by

$$Hu(y, z) = \int_{|y|}^{+\infty} u(r, z) \frac{r}{\sqrt{r^2 - y^2}} dr. \quad (2.1)$$

Here, Ω is a regular open bounded in \mathbb{R}^2 which represents the domain of image ($\Omega = [0, a] \times (-a, a)$, $a > 0$).

In [1, Lemma 1], the author proved that H is a continuous operator from $L^2(\Omega)$ to $L^2(\Omega)$. Now, to find u , we will use the inverse operator H^{-1} given in [1] by:

$$u(r, z) = H^{-1}g(r, z) = -\frac{1}{\pi} \int_r^a \frac{\frac{\partial g}{\partial y}(y, z)}{\sqrt{y^2 - r^2}} dy, \quad (2.2)$$

for all $(r, z) \in \Omega$. Since the operator H^{-1} contains a derivative term, it cannot be extended as a continuous linear operator from $L^p(\Omega)$ to $L^q(\Omega)$. Then, applying H^{-1} provides a deficient and imperfect reconstruction of the original image; see Figure 2.

Because of the experiment step, there is an additional noise perturbation to the given image g . This perturbation assumed to be an addition of Gaussian white noise, denoted τ , of zero mean, and of standard deviation σ_τ . Other perturbations, such as blur due to the detector response, and X-ray source spot size, or blur motion, are not taken into account in our study. With these assumptions, the projection of an object u is:

$$g = Hu + \tau.$$

3. VARIATIONAL METHOD: THE RELAXED PROBLEM

To avoid the instability of the inverse projection method, we will use the variational method technique of finding the density u of the initial object, it consists to find u as a minimum of the problem (1.1).

Aubert and Kornprobst [4], Acar and Vogel et al [2, 10] studied closed problems. In our model, we faced two supplementary difficulties, the first one comes from the fact that, the domain \mathcal{D} is not convex, and its interior is empty for most usual topologies. The second difficulty is that, the total variation $J(u)$ of a function u in $BV(\Omega)$ is not Frchet differentiable. According to these difficulties, we cannot apply the classical theory of optimization to obtain an optimality system of (1.1).

To study numerically this problem, Bergounioux and the first author have modified problem (1.1) through numerically-accepted modifications [5]. Firstly, to preserve the differentiability of J , we replace the total variation $J(u)$ by the $L^2(\Omega)$ -norm of the gradient of $u \in H^1(\Omega)$ and considering the $H^1(\Omega)$ as an underlying space instead of $BV(\Omega)$. For the second difficulty, we relaxed the domain \mathcal{D} by considering the domain

$$\mathcal{D}_\alpha := \{u \in H^1(\Omega) : 0 \leq u \leq 1, (u, 1 - u)_2 \leq \alpha\}, \quad (3.1)$$

where $\alpha > 0$, this relaxation of the binary constraint is motivated and justified numerically, indeed, it is not possible to ensure $(u, 1 - u)_2 = 0$ during computation but rather $(u, 1 - u)_2 \leq \alpha$ where α may be chosen small as wanted.

After these modifications, we consider the smooth relaxed problem

$$\begin{aligned} \text{minimize } F(u) &:= \frac{1}{2} \|Hu - g\|^2 + \frac{\lambda}{2} \|\nabla u\|^2, \\ u &\in \mathcal{D}_\alpha \end{aligned} \quad (3.2)$$

In [5], we proved that the problem (3.2) has at least one solution in $H^1(\Omega)$, however since the constraint \mathcal{D}_α is also not convex, it is not possible to find the ‘‘admissible’’ directions to compute derivatives, then we use general mathematical programming problems results and optimal control in Banach spaces (Zowe and Kurcyusz [7], and Trltzsch [9, 8]) to derive optimality conditions for (3.2).

To apply the method of mathematical programming to (3.2), we introduce a virtual control variable, we can write problem (3.2) as

$$\begin{aligned} \text{minimize } F(u) \\ (u, v) \in \mathcal{C}_\alpha, \end{aligned} \quad (3.3)$$

where $(u_\alpha, v_\alpha = 1 - u_\alpha)$ denoted the solution of (3.2) in $H^1(\Omega) \times L^2(\Omega)$. Using the mathematical programming method and after a step of penalization, we derive the following optimality conditions.

Theorem 3.1. *Assume u_α is a solution to (3.2) and $v_\alpha = 1 - u_\alpha$. There exists a Lagrange multiplier $(q_\alpha, r_\alpha) \in \mathcal{M}(\Omega) \times \mathbb{R}^+$ such that for all $u \in H^1(\Omega) \cap L^\infty(\Omega)$ with $u \geq 0$,*

$$(H^*(Hu_\alpha - g) + r_\alpha v_\alpha, u - u_\alpha)_2 + \lambda(\nabla u_\alpha, \nabla(u - u_\alpha))_2 + \langle q_\alpha, u - u_\alpha \rangle_{\mathcal{M}, L^\infty} \geq 0, \quad (3.4)$$

$$\langle q_\alpha, v - v_\alpha \rangle_{\mathcal{M}, L^\infty} + r_\alpha(u_\alpha, v - v_\alpha)_2 \geq 0, \quad \forall v \in \mathcal{V}_{\text{ad}} \cap L^\infty(\Omega), \quad (3.5)$$

$$r_\alpha[(u_\alpha, v_\alpha)_2 - \alpha] = 0, \quad (3.6)$$

where

$$\mathcal{V}_{\text{ad}} = \{v \in H^1(\Omega) : v \geq 0 \text{ a.e.}\}.$$

Now, we will use the above optimality system to establish a numerical scheme for (3.2) by considering the couple (solution, Lagrangian multipliers) as a saddle-point of a Lagrangian function associated to (3.2).

4. SADDLE-POINT FORMULATION

Let L^α be the Lagrangian function associated with (3.2), defined on $H^1(\Omega) \times L^2(\Omega) \times \mathcal{M}(\Omega) \times \mathbb{R}$ by

$$L^\alpha(u, v, q, r) = \frac{1}{2} \|Hu - g\|^2 + \frac{\lambda}{2} \|\nabla u\|^2 + \langle q, u + v - 1 \rangle_{\mathcal{M}, L^\infty} + r[(u, v)_2 - \alpha]$$

For the rest of this article, we denote

$$F(u) = \frac{1}{2} \|Hu - g\|^2 + \frac{\lambda}{2} \|\nabla u\|^2.$$

Theorem 4.1. *Let (u_α, v_α) be a solution of (3.2), then $(u_\alpha, v_\alpha, q_\alpha, r_\alpha)$ satisfies*

$$L^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha) \geq L^\alpha(u_\alpha, v_\alpha, q, r) \quad \forall (q, r) \in \mathcal{M} \times \mathbb{R}^+. \quad (4.1)$$

and

$$\nabla_{u,v} L^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha)(u - u_\alpha, v - v_\alpha) \geq 0, \quad (4.2)$$

for all $(u, v) \in \mathcal{U}_{\text{ad}} \cap L^\infty \times \mathcal{V}_{\text{ad}} \cap L^\infty$.

Proof. The first assertion comes from the fact that for all $(q, r) \in \mathcal{M} \times \mathbb{R}^+$,

$$L^\alpha(u_\alpha, v_\alpha, q, r) = F(u_\alpha) + r[(u_\alpha, v_\alpha) - \alpha] \leq F(u_\alpha),$$

and

$$F(u_\alpha) = L^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha).$$

Moreover, adding (3.4) and (3.5), we obtain exactly the second part of the theorem. \square

We denote

$$\mathcal{A} = \mathcal{U}_{\text{ad}} \cap L^\infty \times \mathcal{V}_{\text{ad}} \cap L^\infty \times \mathcal{M} \times \mathbb{R}^+.$$

Because of the bilinear term $(u, v)_2 - \alpha$, the Lagrangian L^α is not convex and the theorem 4.1 is not sufficient to ensure the existence of a saddle point of L^α . But, it is easy to see that this theorem is still valid if we replace L^α by the linearized Lagrangian function \mathcal{L}^α ,

$$\mathcal{L}^\alpha(u, v, q, r) = F(u) + (q, u + v - 1)_{\mathcal{M}, L^\infty} + r[(u, v_\alpha)_2 + (u_\alpha, v)_2 - 2\alpha].$$

More precisely, we have the following statement.

Theorem 4.2. *The couple $(u_\alpha, v_\alpha, q_\alpha, r_\alpha)$ (solution, Lagrange multiplier) is a saddle point of the linearized Lagrangian function \mathcal{L}^α on \mathcal{A} :*

$$\mathcal{L}^\alpha(u_\alpha, v_\alpha, q, r) \leq \mathcal{L}^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha) \leq \mathcal{L}^\alpha(u, v, q_\alpha, r_\alpha) \quad \text{for all } (u, v, q, r) \in \mathcal{A}.$$

Proof. We get first the left hand part of the above inequality since for any $(q, r) \in \mathcal{M} \in \mathbb{R}^+$,

$$\mathcal{L}^\alpha(u_\alpha, v_\alpha, q, r) = F(u_\alpha) + 2r[(u_\alpha, v_\alpha)_2 - \alpha] \leq F(u_\alpha) = \mathcal{L}^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha).$$

The right hand part comes from

$$\nabla_{u,v} \mathcal{L}^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha) = \nabla_{u,v} L^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha),$$

and the convexity of \mathcal{L}^α . \square

In the case where the bilinear constraint is inactive $(u_\alpha, v_\alpha) < \alpha$ we obtain $r_\alpha = 0$. In this case, it is easy to see that $\nabla_{u,v} \mathcal{L}^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha)$ is then equal that $\nabla_{u,v} L^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha)$ and the above theorem yields

$$L^\alpha(u_\alpha, v_\alpha, q_\alpha, r_\alpha) \leq L^\alpha(u, v, q_\alpha, r_\alpha),$$

and then $(u_\alpha, v_\alpha, q_\alpha, r_\alpha)$ is a saddle point of L^α on \mathcal{A} .

Remark 4.3. As in our initial problem (3.2), we look for the couple (u_α, v_α) such that $u_\alpha \cdot v_\alpha = u_\alpha(1 - u_\alpha)$ is close to 0, it will be interesting to study numerically the case when the constraint $(u, v)_2$ is inactive i.e:

$$(u, v)_2 < \alpha.$$

5. NUMERICAL SCHEME AND DISCRETIZATION

The basic method to compute a saddle point is the Uzawa algorithm and the descent gradient method [3]. In this method, we look for the couple (solution, Lagrangian multiplier) as the following way.

Algorithm (A).

Step 0 Initialization. Set $n = 0$, choose $u^0 \in \mathcal{U}_{\text{ad}}, q^0 \in L^2(\Omega), v^{-1} \in \mathcal{V}_{\text{ad}}$.

Step 1 Compute:

$$\begin{aligned} u^n &= \operatorname{argmin} L^\alpha(u, v^{n-1}, q^n, r^n) \text{ on } \mathcal{U}_{\text{ad}}, \\ v^n &= \operatorname{argmin} L^\alpha(u^n, v, q^n, r^n) \text{ on } \mathcal{V}_{\text{ad}}. \end{aligned}$$

Step 2 Compute q^n :

$$q^{n+1} = q^n + \rho_1(u^n + v^n - 1), \quad \rho_1 \geq 0.$$

Step 3 Compute r^n :

$$r^{n+1} = r^n + \rho_2[(u^n, v^n)_2 - \alpha]_+, \quad \rho_2 \geq 0.$$

Step 4 Criterion of stopping: Stop the process as soon as $\|u^{n+1} - u^n\|_\infty < \text{tol}$ and $\|q^{n+1} - q^n\|_\infty < \text{tol}$, where tol is a positive real number.

To calculate the first step, we use the gradient descent method: we look for u^n and v^n as follows

$$\begin{aligned} u^{n+1} &= [u^n - \mu_n \nabla_u L^\alpha(u^n, v^n, r^n, q^n)]_+, \\ v^{n+1} &= [v^n - \delta_n \nabla_v L^\alpha(u^n, v^n, r^n, q^n)]_+, \end{aligned}$$

where μ_n and δ_n are two positive real numbers.

In addition, as the study of a measure is too hard from a numerical point of view, we consider in step 2, the term q_α as a function in $L^2(\Omega)$. With gradient descent method, the numerical scheme becomes

$$\begin{aligned} u^{n+1} &= [u^n - \mu_n(H^*Hu^n + H^*g_{i,j} - \lambda\Delta u^n + q^n + r^n v^{n-1})]_+, \\ v^n &= [v^{n-1} - \delta_n(v^{n-1} + r^n u^n)]_+, \\ q^{n+1} &= q^n + \rho_1(u^n + v^n - 1), \\ r^{n+1} &= r^n + \rho_2[(u^n, v^n)_2 - \alpha]_+, \end{aligned} \quad (5.1)$$

where $\mu_n, \delta_n, \rho_1, \rho_2$ are positive real numbers. The discretization process is the standard Finite Difference Method. For the projection operator H , we use its explicit formula to find the matrix associated to this operator as claimed in the following theorem.

Theorem 5.1. *The matrix associated to the projection operator is given by*

$$h_{i,j} = \frac{2}{N}(\sqrt{j^2 - (i-1)^2} - \sqrt{(j-1)^2 - (i-1)^2}) \quad \text{for } j \geq i \quad (5.2)$$

and

$$h_{i,j} = 0 \quad \text{for } j < i. \quad (5.3)$$

Proof. For all function $u \in L^\infty(\Omega)$, $\Omega = (-a, +a) \times (-a, +a)$ where a is a positive real, the projection operator H is given by

$$Hu(y, z) = 2 \int_{|y|}^a u(r, z) \frac{r}{\sqrt{r^2 - y^2}} dr.$$

In a discrete space, we suppose that for all $j = 1 : N$, and for all $i = 1 : N$, $u(\cdot, y_j)$ is constant in the interval $[ih, (i+1)h]$, then

$$\begin{aligned} Hu(ih, jh) &= 2 \int_{ih}^{Nh} u(r, hj) \frac{r}{\sqrt{r^2 - (ih)^2}} dr \\ &= 2 \sum_{k=i}^{N-1} \int_{kh}^{(k+1)h} u_{k,j} \frac{r}{\sqrt{r^2 - (ih)^2}} \\ &\quad \text{(where } u_{k,j} \text{ is the value of } u \text{ on } [kh, (k+1)h]) \\ &= 2 \sum_{k=i}^{N-1} u_{k,j} \int_{kh}^{(k+1)h} \frac{r}{\sqrt{r^2 - (ih)^2}} \\ &= 2 \sum_{k=i}^{N-1} [\sqrt{r^2 - (ih)^2}]_{kh}^{(k+1)h} \\ &= 2 \sum_{k=i}^{N-1} u_{k,j} [\sqrt{(k+1)^2 h^2 - (ih)^2} - \sqrt{(kh)^2 - (ih)^2}] \\ &= 2 \sum_{k=i}^{N-1} u_{k,j} h [\sqrt{(k+1)^2 - (i)^2} - \sqrt{(k)^2 - (i)^2}]. \end{aligned}$$

By taking $h = 1/N$ and using the previous formula, we conclude the proof. \square

The Laplacian term, can be discretized as

$$\Delta u^n(x_i, y_j) = \frac{u_{i+1,j}^n + u_{i-1,j}^n - 2u_{i,j}^n}{h^2} + \frac{u_{i,j+1}^n + u_{i,j-1}^n - 2u_{i,j}^n}{h^2} \quad (5.4)$$

Replacing (5.4) in (5.1), we obtain

$$\begin{aligned} u_{i,j}^{n+1} &= [u_{i,j}^n - \mu_n [H^* H u_{i,j}^n + H^* g_{i,j} - \lambda \left(\frac{u_{i+1,j}^n + u_{i-1,j}^n - 2u_{i,j}^n}{h^2} \right. \\ &\quad \left. + \frac{u_{i,j+1}^n + u_{i,j-1}^n - 2u_{i,j}^n}{h^2} \right) + q_{i,j}^n + r^n v_{i,j}^n]_+, \\ v_{i,j}^n &= [v_{i,j}^{n-1} - \delta_n (q_{i,j}^n + r^n u_{i,j}^n)]_+, \\ q_{i,j}^{n+1} &= q_{i,j}^n + \rho_1 (u_{i,j}^n + v_{i,j}^n - 1), \\ r^{n+1} &= r^n + \rho_2 [(u^n, v^n)_{2,i,j} - \alpha]_+, \end{aligned}$$

where $H^* H u_{i,j}^n$ is the discretization of $H^* H u^n$ and $H^* g_{i,j}$ is that of $H^* g$.

The discretized image is represented by a $N \times N$ array identified with a N^2 vector. Due to the symmetry, it suffices to deal with half image of size $N \times N/2$. In our study, the projected image(observed data) is perturbed with a Gaussian noise τ with standard deviation σ_τ ,

$$\tau(x) = \frac{1}{\sqrt{2\pi}\sigma_\tau} e^{-\frac{|x|^2}{2\sigma_\tau^2}}.$$

With this perturbation, the observed image is given by:

$$g = H u_{\text{orig}} + \tau. \quad (5.5)$$

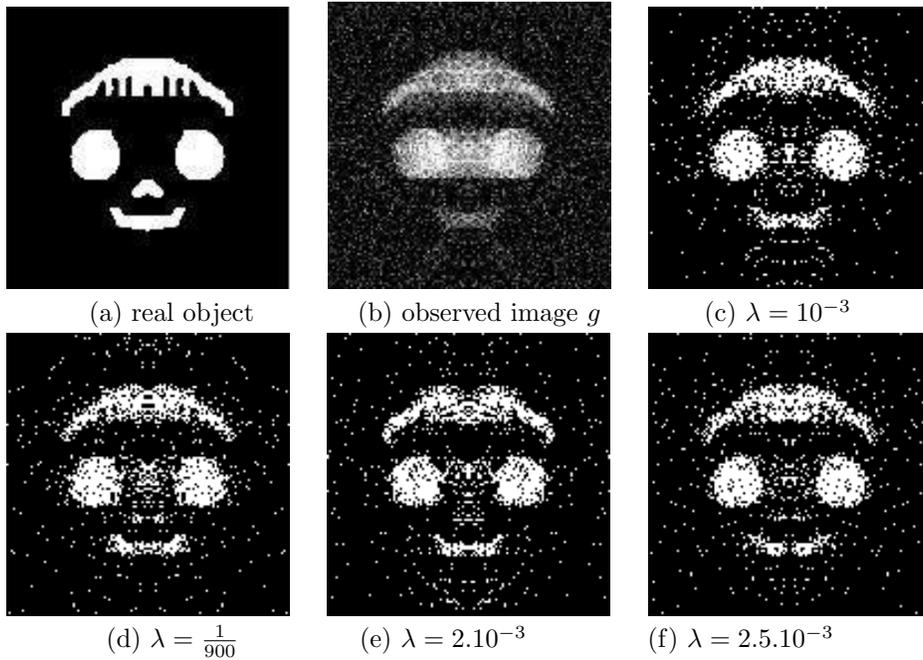


FIGURE 3. Reconstruction for some small values of λ with $\alpha = 10^{-2}$ and $\text{tol} = 0.001$

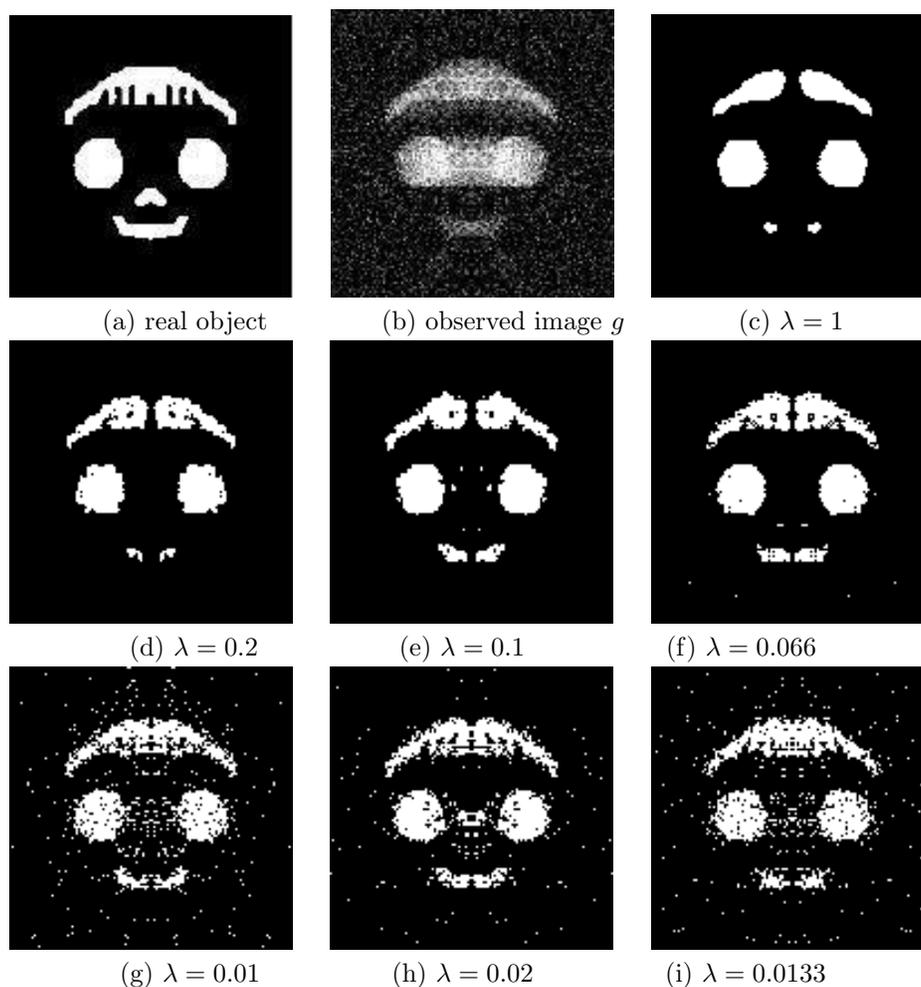


FIGURE 4. Reconstruction for different values of λ with $\alpha = 10^{-2}$ and $\text{tol} = 0.001$

6. INTERPRETATION AND SENSITIVITY OF NUMERICAL RESULTS

The algorithm (\mathcal{A}) depends on two parameters λ and α . Next we show that there is a strong sensibility of the results with respect to λ and weak with respect to α .

6.1. Sensitivity with respect to λ . The aim of the constant λ before the regularization term $\|u\|^2$ is to increase or decrease the smoothness of the function u , more precisely if λ is too small, we recover some information on the original image, but we do not succeed in abolishing efficiently the Gaussian noise with the standard deviation σ_τ . These theoretical predictions are confirmed numerically as shown in Figure 3.

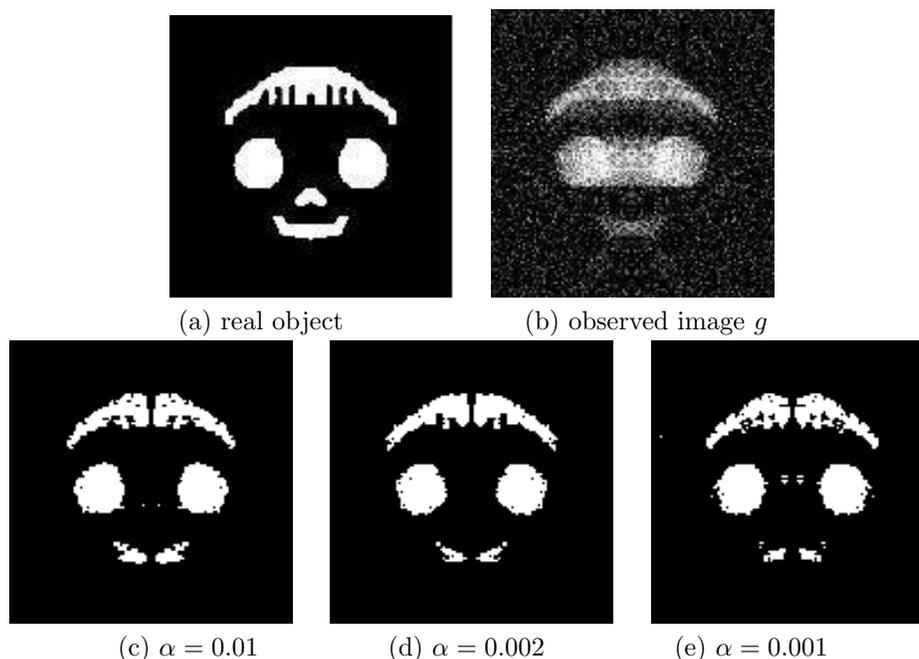


FIGURE 5. Reconstruction for different values of α and with $\lambda = 0.1$

However, when λ is large, the solution becomes regular and this may imply an important loss of information (the image of the mouth for example turns to an arc of circle) (see Figure 4).

So the best λ will be not too small in order to “erase” the noise and not too large to keep all the important parts of the original image.

6.2. Sensitivity with respect to α . The algorithm (\mathcal{A}) has a low sensibility with respect to the parameter α , this weak dependence is due to the fact that the integral $(u, v)_2$ converges rapidly to zero, and by consequence, the inactive assumption $(u, v)_2 < \alpha$ is always verified from a small number of iterations. The Figure 5 confirms this claim.

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ALI SROUR

DEPARTMENT OF MATHEMATICS, LEBANESE UNIVERSITY, HAWCH AL-OUMARA, ZAHLEH, LEBANON
E-mail address: `ali.sroure.lb@gmail.com`

MOUHAMMAD JRADEH

DEPARTMENT OF MATHEMATICS, LEBANESE UNIVERSITY, HAWCH AL-OUMARA, ZAHLEH, LEBANON
E-mail address: `mouhamad.jradeh@gmail.com`