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QUENCHING FOR SINGULAR AND DEGENERATE QUASILINEAR DIFFUSION EQUATIONS

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Abstract. This article concerns the quenching phenomenon of the solution to the Dirichlet problem of a singular and degenerate quasilinear diffusion equation. It is shown that there exists a critical length for the special domain in the sense that the solution exists globally in time if the length of the special domain is less than this number while the solution quenches if the length is greater than this number. Furthermore, we also study the quenching properties for the quenching solution, including the location of the quenching points and the blowing up of the derivative of the solution with respect to the time at the quenching time.

1. Introduction

In the paper, we consider the problem

$$x^q\frac{\partial u}{\partial t}-\frac{\partial^2 u^m}{\partial x^2}=f(u^m),\quad (x,t)\in(0,a)\times(0,T), \tag{1.1}$$

$$u(0,t)=0=u(a,t),\quad t\in(0,T), \tag{1.2}$$

$$u(x,0)=0,\quad x\in(0,a), \tag{1.3}$$

where $a>0$, $q\in\mathbb{R}$, $m\geq 1$ and $f\in C^2([0,c^m))$ with $c>0$ satisfies

$$f(0)>0,\quad f'(0)>0,\quad f''(s)\geq 0\quad \text{for } 0<s<c^m,\quad \lim_{s\to c^m}f(s)=+\infty.$$

At $x=0$, (1.1) is singular if $q>0$ and degenerate if $q<0$. Furthermore, (1.1) is degenerate at the points where $u=0$ in the quasilinear case $m>1$. If $q=0$ and $m>1$, (1.1) is the famous porous medium equation, which arises from many physical and biological models [19]. If $q>0$ and $m=1$, (1.1) can be used to describe the Ockendon model for the flow in a channel of a fluid whose viscosity is temperature-dependent [10, 14].

Due to the properties of f, the solution u to (1.1)–(1.3) may quench at a finite time. That is to say, there exists a finite time $T>0$ such that

$$\lim_{t\to T^-}\sup_{(0,a)}u(\cdot,t)=c.$$

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Quenching phenomena were introduced by Kawarada [12] in 1975 for the problem (1.1)–(1.3) in the case $q = 0$, $m = 1$ and

$$f(s) = \frac{1}{1-s}, \quad 0 \le s < 1,$$

where Kawarada proved the existence of the critical length (which is $2\sqrt{2}$). That is to say, the solution exists globally in time if a is less than the critical length, while it quenches if a is greater than the critical length. For the quenching case, Kawarada also showed that $a/2$ is the quenching point and the derivative of the solution with respect to the time blows up at the quenching time. Since then, there are many interesting results on quenching phenomena for semilinear uniformly parabolic equations (see, e.g., [1, 2, 8, 15, 16, 17] and the references therein) and for singular or degenerate semilinear parabolic equations (see, e.g., [3, 4, 5, 6, 11, 13] and the references therein). Among these, Chan and Kong [3] considered (1.1)–(1.3) in the semilinear case $m = 1$, where the authors showed the existence of the critical length, the location of the quenching points and the blowing up of the derivative of the solution with respect to the time at the quenching time.

Recently, there are few results on quenching phenomena for quasilinear diffusion equations [7, 9, 18, 20, 21]. [7] and [21] showed some sufficient conditions for quenching solutions to the Dirichlet problems of porous medium equations. In [9] and [20], the authors considered quenching phenomena for the one-dimensional homogeneous porous medium equation and p-Laplacian equation with singular boundary flux, respectively. It is shown that the solution quenches at the singular boundary and the quenching rate was estimated. Winkler [18] studied the following problem for a strongly degenerate diffusion equation with strong absorption

$$\begin{gathered}
\frac{\partial u}{\partial t} - u^p \frac{\partial^2 u}{\partial x^2} = -u^{-\beta} \chi_{\{u>0\}}, \quad (x,t) \in (0,a) \times (0,T), \\
u(0,t) = 0 = u(a,t), \quad t \in (0,T), \\
u(x,0) = u_0(x), \quad x \in (0,a),
\end{gathered}$$

where $a > 0$, $p > 1$, $-1 < \beta < p - 1$, $0 \le u_0 \in C([0,a])$ and χ is the characteristic function. Due to $p > 1$, this equation in non-divergence form cannot be transformed into the porous medium equation. Winkler [18] ruled out the possibility of quenching in infinite time under certain assumptions on p, β and a.

In this article, we study the quenching phenomenon of the solution to the problem (1.1)–(1.3). The equation (1.1) is quasilinear in the case $m > 1$. Furthermore, there are two kinds of singularity or degeneracy in (1.1): one is the degeneracy at $u = 0$ in the case $m > 1$, the other is the singularity ($q > 0$) or degeneracy ($q < 0$) at $x = 0$. Therefore, the classical solution to the problem (1.1)–(1.3) may not exist and the weak solution should be considered. By precise estimates near the parabolic boundary, it is shown that the problem (1.1)–(1.3) admits a continuous solution before the quenching time. By constructing suitable super and sub solutions, we prove the existence of the critical length. For the quenching solution, we also study the location of the quenching points and the blowing up of the derivative of the solution with respect to the time at the quenching time by energy estimates and many kinds of comparison principles. Due to the quasilinearity and the two kinds of singularity or degeneracy in (1.1), we have to overcome some technical difficulties when doing estimates, constructing super and sub solutions, and using comparison principles.

This paper is arranged as follows. The well-posedness of the problem (1.1)–(1.3) is shown in §2. The existence of the critical length is proved in §3. Subsequently, in §4 we study the quenching properties for the quenching solution, including the location of the quenching points and the blowing up of the derivative of the solution with respect to the time at the quenching time.

2. Well-posedness

Solutions to (1.1), and super and sub solutions, are defined as follows.

Definition 2.1. *A nonnegative function $u \in L^\infty((0,a)\times(0,T))$ is said to be a super (sub) solution to* (1.1) *in* $(0,T)$ *for some* $0<T\le+\infty$, *if for any* $0<\tilde T<T$, $\sup_{(0,a)\times(0,\tilde T)} u<c$ *and*

$$-\int_0^{\tilde T}\int_0^a x^q u\frac{\partial\varphi}{\partial t}\,dx\,dt-\int_0^{\tilde T}\int_0^a u^m\frac{\partial^2\varphi}{\partial x^2}\,dx\,dt\ge(\le)\int_0^{\tilde T}\int_0^a f(u^m)\varphi\,dx\,dt$$

for each $0\le\varphi\in C_0^2([0,a]\times[0,\tilde T])$. *Furthermore, u is said to be a solution to* (1.1), *if it is both a super solution and a sub solution.*

The following comparison principle can be established by a duality argument. The proof is similar to [19, Theorem 1.3.1] and it is omitted.

Theorem 2.2. *Assume that $\hat u$ and $\check u$ are a super solution and a sub solution to* (1.1) *in* $(0,T)$ *for some* $0<T\le+\infty$, *respectively. Furthermore,* $\hat u,\check u\in C([0,a]\times[0,T))$. *If*

$$\hat u(\cdot,0)\ge\check u(\cdot,0)\ \text{in }(0,a),\quad \hat u(0,\cdot)\ge\check u(0,\cdot),\quad \hat u(a,\cdot)\ge\check u(a,\cdot)\ \text{in }(0,T),$$

then $\hat u\ge\check u$ *in* $(0,a)\times(0,T)$.

Let us turn to the existence of local solution.

Theorem 2.3. *The problem* (1.1)–(1.3) *admits uniquely a solution u in* $(0,T)$ *for some* $T>0$. *Furthermore,* $u\in C^{2,1}((0,a)\times(0,T])\cap C([0,a]\times[0,T])$, $\frac{\partial u}{\partial t},\frac{\partial u}{\partial x}\in C^{2,1}((0,a)\times(0,T])$ *and*

$$u(x,t)>0,\quad \frac{\partial u}{\partial t}(x,t)>0,\quad (x,t)\in(0,a)\times(0,T),$$

$$\int_0^a\Big(\frac{\partial u^m}{\partial x}(x,t)\Big)^2dx\le 2ma\int_0^{\max_{(0,a)}u(\cdot,t)}s^{m-1}f(s^m)ds,\quad t\in(0,T).$$

Proof. Fix $0<c_0<c$. For an integer $n\ge\frac{(2\mathrm{e})^{1/m}}{c_0}$ and $k\ge1$, the classical theory yields that the problem

$$\Big(x+\frac1k\Big)^q\frac{\partial u_{n,k}}{\partial t}-\frac{\partial^2u_{n,k}^m}{\partial x^2}=f(u_{n,k}^m),\quad (x,t)\in(0,a)\times(0,T),\tag{2.1}$$

$$u_{n,k}(0,t)=u_{n,k}(a,t)=\frac1n,\quad t\in(0,T),\tag{2.2}$$

$$u_{n,k}(x,0)=\frac1n,\quad x\in(0,a)\tag{2.3}$$

admits a unique solution $u_{n,k}\in C^{2,1}((0,a)\times[0,T])\cap C([0,a]\times[0,T])$ locally in time. Set

$$\bar u_{n,k}(x,t)=\frac{c_0}{2^{1/m}}\Big(1-\Big(\frac xa\Big)^{3/2}+\mathrm{e}^{t/T_0-1}\Big)^{1/m},\quad (x,t)\in[0,a]\times[0,T_0],$$

where

$$T_0 = \frac{c_0 \min\left\{\delta^q, (a+1)^q\right\}}{2me f(c_0^m)}, \quad \delta = \min\left\{\frac{a}{2}, \frac{9c_0^{2m}}{64a^3 f^2(c_0^m)}\right\}.$$

Then

$$\frac{c_0}{(2\mathrm{e})^{1/m}} \le \bar{u}_{n,k} \le c_0, \quad (x,t) \in [0,a] \times [0,T_0]$$

and

$$\begin{aligned}\Big(x + \frac{1}{k}\Big)^q \frac{\partial \bar{u}_{n,k}}{\partial t} - \frac{\partial^2 \bar{u}_{n,k}^m}{\partial x^2} &= \frac{c_0^m \left(x + \frac{1}{k}\right)^q}{2m\bar{u}_{n,k}^{m-1} T_0} \mathrm{e}^{t/T_0 - 1} + \frac{3c_0^m}{8a^{3/2} x^{1/2}} \\ &\ge \begin{cases} \frac{3c_0^m}{8a^{3/2}\delta^{1/2}} \ge f(c_0^m), & (x,t) \in (0,\delta) \times (0,T_0), \\ \frac{c_0 \min\left\{\delta^q, (a+1)^q\right\}}{2meT_0}, & (x,t) \in (\delta, a) \times (0, T_0). \end{cases}\end{aligned}$$

Note that $\frac{c_0 \min\left\{\delta^q, (a+1)^q\right\}}{2meT_0} \ge f(c_0^m)$. Therefore, $\bar{u}_{n,k}$ is a supersolution to (2.1)–(2.3) in $(0,T_0)$. The classical comparison principle yields that $u_{n,k}$ exists in $(0,T_0)$ and

$$\frac{1}{n} \le u_{n,k}(x,t) \le c_0, \quad (x,t) \in (0,a) \times (0,T_0). \tag{2.4}$$

Since $\frac{\partial u_{n,k}}{\partial t}$ solves

$$\begin{gathered}\Big(x + \frac{1}{k}\Big)^q \frac{\partial}{\partial t}\Big(\frac{\partial u_{n,k}}{\partial t}\Big) - m\frac{\partial^2}{\partial x^2}\Big(u_{n,k}^{m-1}(x,t)\frac{\partial u_{n,k}}{\partial t}\Big) \\ -mf'(u_{n,k}^m(x,t))u_{n,k}^{m-1}(x,t)\frac{\partial u_{n,k}}{\partial t} = 0, \quad (x,t) \in (0,a) \times (0,T_0), \\ \frac{\partial u_{n,k}}{\partial t}(0,t) = \frac{\partial u_{n,k}}{\partial t}(a,t) = 0, \quad t \in (0,T_0), \\ \frac{\partial u_{n,k}}{\partial t}(x,0) \ge 0, \quad x \in (0,a),\end{gathered}$$

the classical comparison principle leads to

$$\frac{\partial u_{n,k}}{\partial t}(x,t) \ge 0, \quad (x,t) \in (0,a) \times (0,T_0). \tag{2.5}$$

Set

$$\bar{v}_{n,k}(x,t) = \Big(\Big(\frac{1}{n}\Big)^m + \frac{1}{2} f(c_0^m) x(a-x)\Big)^{1/m}, \quad (x,t) \in [0,a] \times [0,T_0],$$

Then, $\bar{v}_{n,k}$ is a supersolution of the problem

$$\begin{gathered}\Big(x + \frac{1}{k}\Big)^q \frac{\partial v_{n,k}}{\partial t} - \frac{\partial^2 v_{n,k}^m}{\partial x^2} = f(c_0^m), \quad (x,t) \in (0,a) \times (0,T_0), \\ v_{n,k}(0,t) = v_{n,k}(a,t) = \frac{1}{n}, \quad t \in (0,T_0), \\ v_{n,k}(x,0) = \frac{1}{n}, \quad x \in (0,a).\end{gathered}$$

Furthermore, (2.4) shows that $v_{n,k}$ is a supersolution of the problem (2.1)–(2.3) in $(0,T_0)$. It follows from the classical comparison principle that

$$u_{n,k}(x,t) \le v_{n,k}(x,t) \le \bar{v}_{n,k}(x,t), \quad (x,t) \in (0,a) \times (0,T_0). \tag{2.6}$$

For a fixed $\tilde{x} \in (0, a)$, denote $\delta = \frac{1}{2}\min\{\tilde{x}, a - \tilde{x}\}$. Set

$$\underline{w}_{n,k}(x,t) = \frac{1}{n} + (x - \tilde{x} + \delta)(\tilde{x} + \delta - x)\min\{t, T_1\},$$
$$(x,t) \in [\tilde{x} - \delta, \tilde{x} + \delta] \times [0, T_0],$$
$$\bar{z}_{n,k}(x,t) = \frac{1}{n} + \Big(c_0 - \frac{1}{n}\Big)\delta^{-2}(x - \tilde{x})^2 + \frac{t}{T_2}, \quad (x,t) \in [\tilde{x} - \delta, \tilde{x} + \delta] \times [0, T_2],$$

where $0 < T_1, T_2 < T_0$. Then, there exist sufficiently small $T_1, T_2 > 0$, which are independent of n and k, such that

$$\Big(x + \frac{1}{k}\Big)^q \frac{\partial \underline{w}_{n,k}}{\partial t} - \frac{\partial^2 \underline{w}^m_{n,k}}{\partial x^2} \le f(0), \quad (x,t) \in (\tilde{x} - \delta, \tilde{x} + \delta) \times (0, T_0),$$
$$\Big(x + \frac{1}{k}\Big)^q \frac{\partial \bar{z}_{n,k}}{\partial t} - \frac{\partial^2 \bar{z}^m_{n,k}}{\partial x^2} \ge f(c_0^m), \quad (x,t) \in (\tilde{x} - \delta, \tilde{x} + \delta) \times (0, T_2).$$

Therefore, $\underline{w}_{n,k}$ is a subsolution of the problem

$$\Big(x + \frac{1}{k}\Big)^q \frac{\partial w_{n,k}}{\partial t} - \frac{\partial^2 w^m_{n,k}}{\partial x^2} = f(0), \quad (x,t) \in (\tilde{x} - \delta, \tilde{x} + \delta) \times (0, T_0), \tag{2.7}$$
$$w_{n,k}(\tilde{x} - \delta, t) = w_{n,k}(\tilde{x} + \delta, t) = \frac{1}{n}, \quad t \in (0, T_0), \tag{2.8}$$
$$w_{n,k}(x, 0) = \frac{1}{n}, \quad x \in (\tilde{x} - \delta, \tilde{x} + \delta), \tag{2.9}$$

$\bar{z}_{n,k}$ is a supersolution to the problem

$$\Big(x + \frac{1}{k}\Big)^q \frac{\partial z_{n,k}}{\partial t} - \frac{\partial^2 z^m_{n,k}}{\partial x^2} = f(c_0^m), \quad (x,t) \in (\tilde{x} - \delta, \tilde{x} + \delta) \times (0, T_2), \tag{2.10}$$
$$z_{n,k}(\tilde{x} - \delta, t) = z_{n,k}(\tilde{x} + \delta, t) = c_0, \quad t \in (0, T_2), \tag{2.11}$$
$$z_{n,k}(x, 0) = \frac{1}{n}, \quad x \in (\tilde{x} - \delta, \tilde{x} + \delta). \tag{2.12}$$

Further, (2.4) shows that $u_{n,k}$ is a supersolution to the problem (2.7)–(2.9) in $(0, T_0)$ and a subsolution to the problem (2.10)–(2.12) in $(0, T_2)$. It follows from the classical comparison principle that

$$u_{n,k}(x,t) \ge w_{n,k}(x,t) \ge \underline{w}_{n,k}(x,t), \quad (x,t) \in (\tilde{x} - \delta, \tilde{x} + \delta) \times (0, T_0), \tag{2.13}$$
$$u_{n,k}(x,t) \le z_{n,k}(x,t) \le \bar{z}_{n,k}(x,t), \quad (x,t) \in (\tilde{x} - \delta, \tilde{x} + \delta) \times (0, T_2). \tag{2.14}$$

Set

$$E_{n,k}(t) = \frac{1}{2}\int_0^a \Big(\frac{\partial u^m_{n,k}}{\partial x}(x,t)\Big)^2 dx - m\int_0^a \int_0^{u_{n,k}(x,t)} s^{m-1} f(s^m)\, ds\, dx,$$

for $t \in [0, T_0]$. Integrating by parts, one gets

$$\begin{aligned}
E'_{n,k}(t) &= \int_0^a \frac{\partial u^m_{n,k}}{\partial x}(x,t)\frac{\partial^2 u^m_{n,k}}{\partial t \partial x}(x,t)dx - m\int_0^a \frac{\partial u_{n,k}}{\partial t} u^{m-1}_{n,k}(x,t) f(u^m_{n,k}(x,t))dx \\
&= \frac{\partial u^m_{n,k}}{\partial x}(a,t)\frac{\partial u^m_{n,k}}{\partial t}(a,t) - \frac{\partial u^m_{n,k}}{\partial x}(0,t)\frac{\partial u^m_{n,k}}{\partial t}(0,t) \\
&\quad - m\int_0^a u^{m-1}_{n,k}(x,t)\frac{\partial u_{n,k}}{\partial t}(x,t)\Big(\frac{\partial^2 u^m_{n,k}}{\partial x^2}(x,t) + f(u^m_{n,k}(x,t))\Big)dx \\
&= -m\int_0^a u^{m-1}_{n,k}(x,t)\Big(\frac{\partial u_{n,k}}{\partial t}(x,t)\Big)^2 dx \le 0, \quad t \in (0, T_0).
\end{aligned}$$

Therefore,

$$E_{n,k}(t) \le E_{n,k}(0) \le 0, \quad t \in (0, T_0),$$

which leads to

$$\begin{aligned} \int_0^a \Big(\frac{\partial u_{n,k}}{\partial x}(x,t)\Big)^2 dx &\le 2m \int_0^a \int_0^{u_{n,k}(x,t)} s^{m-1} f(s^m)\, ds\, dx \\ &\le 2ma \int_0^{\max_{(0,a)} u_{n,k}(\cdot,t)} s^{m-1} f(s^m) ds, \quad t \in (0, T_0). \end{aligned} \tag{2.15}$$

From the classical comparison principle and (2.5), we have

$$u_{n_2,k}(x,t) \le u_{n_1,k}(x,t), \quad (x,t) \in [0,a] \times [0,T_0], \quad n_2 \ge n_1 \ge \frac{(2\mathrm{e})^{1/m}}{c_0},$$

for $k \ge 1$, and

$$\begin{cases} u_{n,k_2}(x,t) \le u_{n,k_1}(x,t), & \text{if } q \ge 0, \\ u_{n,k_2}(x,t) \ge u_{n,k_1}(x,t), & \text{if } q \le 0, \end{cases}$$

for $(x,t) \in (0,a) \times (0,T_0)$, $n \ge (2\mathrm{e})^{1/m}/c_0$, $k_2 \ge k_1 \ge 1$. Let

$$u(x,t) = \lim_{k\to\infty} \lim_{n\to\infty} u_{n,k}(x,t), \quad (x,t) \in [0,a] \times [0,T_0].$$

Due to (2.4), (2.5), (2.6), (2.13) and (2.14), the function u satisfies

$$0 \le u(x,t) \le c_0, \quad (x,t) \in (0,a) \times (0,T_0), \tag{2.16}$$

$$u(x,\cdot) \text{ is increasing in } (0,T_0), \quad x \in (0,a), \tag{2.17}$$

$$0 \le u(x,t) \le \Big(\frac{1}{2} f(c_0^m) x(a-x)\Big)^{1/m}, \quad (x,t) \in (0,a) \times (0,T_0), \tag{2.18}$$

$$u(x,t) \ge (x - \tilde{x} + \delta)(\tilde{x} + \delta - x) \min\{t, T_1\}, \quad (x,t) \in (\tilde{x}-\delta, \tilde{x}+\delta) \times (0,T_0), \tag{2.19}$$

$$u(x,t) \le c_0 \delta^{-2} (x-\tilde{x})^2 + \frac{t}{T_2}, \quad (x,t) \in (\tilde{x}-\delta, \tilde{x}+\delta) \times (0,T_2). \tag{2.20}$$

It is not hard to show that u is a solution of (1.1)–(1.3) in $(0,T_0)$. Furthermore, (2.19) yields that

$$u(x,t) > 0, \quad (x,t) \in (0,a) \times (0,T_0). \tag{2.21}$$

Therefore, $u \in C^{2,1}((0,a) \times (0,T_0])$, which together with (2.16)–(2.21) and $f \in C^2([0,c^m))$, implies that $u \in C([0,a] \times [0,T_0])$ satisfies (1.2) and (1.3), $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x} \in C^{2,1}((0,a) \times (0,T_0])$ and

$$\frac{\partial u}{\partial t}(x,t) \ge 0, \quad (x,t) \in (0,a) \times (0,T_0). \tag{2.22}$$

Noting $\frac{\partial u}{\partial t} \in C^{2,1}((0,a) \times (0,T_0])$ with (2.22) solves

$$x^q \frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big) - m\frac{\partial^2}{\partial x^2}\Big(u^{m-1}(x,t)\frac{\partial u}{\partial t}\Big) - mf'(u^m(x,t))u^{m-1}(x,t)\frac{\partial u}{\partial t} = 0, \tag{2.23}$$

for $(x,t) \in (0,a) \times (0,T_0)$. From the classical strong maximum principle and (2.21) we obtain

$$\frac{\partial u}{\partial t}(x,t) > 0, \quad (x,t) \in (0,a) \times (0,T_0). \tag{2.24}$$

Indeed, if (2.24) is wrong, then there exists $(x_0,t_0)\in(0,a)\times(0,T_0)$ such that $\frac{\partial u}{\partial t}(x_0,t_0)=0$. For any $0<\varepsilon<\min\{x_0,a-x_0\}$ and any $0<\tau<t_0$, (2.22) shows that

$$\frac{\partial u}{\partial t}(x_0,t_0)=0=\min_{(\varepsilon,a-\varepsilon)\times(\tau,t_0)}\frac{\partial u}{\partial t}.$$

Since (2.23) is uniformly parabolic in $(\varepsilon,a-\varepsilon)\times(\tau,t_0)$ due to (2.21), from the classical strong maximum principle, we have

$$\frac{\partial u}{\partial t}(x,t)=0,\quad (x,t)\in(\varepsilon,a-\varepsilon)\times(\tau,t_0).$$

Then, it follows from the arbitrariness of $\varepsilon\in(0,\min\{x_0,a-x_0\})$ and $\tau\in(0,t_0)$ that

$$\frac{\partial u}{\partial t}(x,t)=0,\quad (x,t)\in(0,a)\times(0,t_0),$$

which contradicts (1.3) and (2.21). Finally, (2.15) leads to

$$\int_0^a\Big(\frac{\partial u^m}{\partial x}(x,t)\Big)^2dx\le 2ma\int_0^{\max_{(0,a)}u(\cdot,t)}s^{m-1}f(s^m)ds,\quad t\in(0,T_0).$$

□

Denote

$$T_*=\sup\big\{T>0:\ \text{the problem (1.1)–(1.3) admits a solution in } (0,T)\big\}.$$

We call T_* the life span of the solution to problem (1.1)–(1.3).

Remark 2.4. By the standard extension process, one can show that Problem (1.1)–(1.3) admits uniquely a solution u in $(0,T_*)$. Furthermore, $u\in C^{2,1}((0,a)\times(0,T_*))\cap C([0,a]\times[0,T_*))$, $\frac{\partial u}{\partial t},\frac{\partial u}{\partial x}\in C^{2,1}((0,a)\times(0,T_*))$ and

$$u(x,t)>0,\quad \frac{\partial u}{\partial t}(x,t)>0,\quad (x,t)\in(0,a)\times(0,T_*),$$
$$\int_0^a\Big(\frac{\partial u^m}{\partial x}(x,t)\Big)^2dx\le 2ma\int_0^{\max_{(0,a)}u(\cdot,t)}s^{m-1}f(s^m)ds,\quad t\in(0,T_*).$$

3. Critical length

Assume that u is the solution to (1.1)–(1.3) and T_* is its life span. If $T_*=+\infty$, then u exists globally in time. If $T_*<+\infty$, then u must quench at a finite time, i.e.

$$\lim_{t\to T_*^-}\sup_{(0,a)}u(\cdot,t)=c.$$

Let us study the relation between T_* and a in this section. For convenience, we denote u_a by the solution to (1.1)–(1.3), and $T_*(a)$ its life span.

Lemma 3.1. *If a is positive and sufficiently small, then $T_*(a)=+\infty$ and*

$$\sup_{(0,a)\times(0,+\infty)}u_a<c.$$

Proof. Fix $0 < c_0 < c$ and $0 < a \le \Big(\frac{8c_0^m}{f(c_0^m)}\Big)^{1/2}$. Set

$$\bar{u}_a(x,t) = \Big(\frac{f(c_0^m)}{2}x(a-x)\Big)^{1/m}, \quad (x,t) \in [0,a] \times [0,+\infty).$$

Then, $\bar{u}_a$ satisfies

$$0 \le \bar{u}_a(x,t) \le \Big(\frac{f(c_0^m)a^2}{8}\Big)^{1/m} \le c_0, \quad (x,t) \in [0,a] \times [0,+\infty),$$
$$x^q \frac{\partial \bar{u}_a}{\partial t} - \frac{\partial^2 \bar{u}_a^m}{\partial x^2} = f(c_0^m) \ge f(\bar{u}_a^m), \quad (x,t) \in (0,a) \times (0,+\infty).$$

The comparison principle (Theorem 2.2) shows that

$$u_a(x,t) \le \bar{u}_a(x,t) \le c_0, \quad (x,t) \in [0,a] \times [0,+\infty).$$

□

Lemma 3.2. *If $a > 0$ is sufficiently large, then $T_*(a) < +\infty$.*

Proof. Set

$$\underline{u}_a(x,t) = \frac{t}{T}\Big(\frac{f(0)}{4}(x-a/2)(a-x)\Big)^{1/m}, \quad (x,t) \in [a/2,a] \times [0,T]$$

with

$$T = 2\max\{(a/2)^q, a^q\}\Big(\frac{a^2}{64 f^{m-1}(0)}\Big)^{1/m}.$$

Then, $\underline{u}_a$ satisfies

$$\begin{aligned} x^q \frac{\partial \underline{u}_a}{\partial t} - \frac{\partial^2 \underline{u}_a^m}{\partial x^2} &= \frac{x^q}{T}\Big(\frac{f(0)}{4}(x-a/2)(a-x)\Big)^{1/m} + \frac{f(0)}{2}\Big(\frac{t}{T}\Big)^m \\ &\le f(0) \le f(\underline{u}_a^m), \quad (x,t) \in (a/2,a) \times (0,T). \end{aligned}$$

The comparison principle (Theorem 2.2) shows that

$$u_a(x,t) \ge \underline{u}_a(x,t), \quad (x,t) \in [a/2,a] \times [0,T].$$

Particularly,

$$u_a(3a/4,T) \ge \Big(\frac{f(0)a^2}{64}\Big)^{1/m},$$

which yields $T_*(a) < +\infty$ if $a \ge 8c^{m/2} f^{-1/2}(0)$. □

Lemma 3.3. *For any $0 < a_1 < a_2$,*

$$u_{a_1}(x,t) < u_{a_2}(x,t), \quad (x,t) \in (0,a_1) \times (0,T_*(a_2)).$$

Proof. For any $0 < a_1 < a_2$, Remark 2.4 shows that

$$u_{a_2}(a_1,t) > 0, \quad t \in (0,T_*(a_2)).$$

Then, it follows from the comparison principle (Theorem 2.2) that

$$u_{a_1}(x,t) \le u_{a_2}(x,t), \quad (x,t) \in (0,a_1) \times (0,T_*(a_2)).$$

Set

$$w(x,t) = u_{a_1}(x,t) - u_{a_2}(x,t), \quad (x,t) \in [0,a_1] \times [0,T_*(a_2)).$$

By Remark 2.4, $w \in C^{2,1}((0,a_1) \times (0,T_*(a_2))) \cap C([0,a_1] \times [0,T_*(a_2)])$ and solves

$$x^q \frac{\partial w}{\partial t} - m\frac{\partial^2}{\partial x^2}\Big(w \int_0^1 (\sigma u_{a_1}(x,t) + (1-\sigma)u_{a_2}(x,t))^{m-1} d\sigma\Big)$$

$$\begin{aligned}&= mw\int_0^1 f'(\sigma u_{a_1}^m(x,t)+(1-\sigma)u_{a_2}^m(x,t))d\sigma\\&\quad\times\int_0^1(\sigma u_{a_1}(x,t)+(1-\sigma)u_{a_2}(x,t))^{m-1}d\sigma,\quad (x,t)\in(0,a_1)\times(0,T_*(a_2)),\end{aligned}$$

where $u_{a_1}, u_{a_2} \in C^{2,1}((0,a_1)\times(0,T_*(a_2)))\cap C([0,a_1]\times[0,T_*(a_2)])$ with

$$u_{a_1}(x,t)>0,\quad u_{a_2}(x,t)>0,\quad (x,t)\in(0,a_1)\times(0,T_*(a_2)).$$

The classical strong maximum principle (a similar discussion to (2.24) in Theorem 2.3) leads to

$$w(x,t)<0,\quad (x,t)\in(0,a_1)\times(0,T_*(a_2)),$$

i.e.

$$u_{a_1}(x,t)<u_{a_2}(x,t),\quad (x,t)\in(0,a_1)\times(0,T_*(a_2)).$$

□

Lemma 3.4. *There exists at most one $a>0$ such that u_a quenches at the infinite time, i.e. $T_*(a)=+\infty$ and $\sup_{(0,a)\times(0,+\infty)} u_a = c$.*

Proof. Assume that u_{a_0} quenches at the infinite time for some $a_0>0$. For $a>a_0$, set

$$\underline{u}_a(x,t)=\lambda^{2/m}u_{a_0}(\lambda^{-1}x,\lambda^{-2/m-q}t),\quad (x,t)\in[0,a]\times[0,+\infty),\quad \lambda=\frac{a}{a_0}.$$

Then, $\lambda>1$, and $\underline{u}_a$ solves

$$x^q\frac{\partial \underline{u}_a}{\partial t}-\frac{\partial^2\underline{u}_a^m}{\partial x^2}=f(\lambda^{-2}\underline{u}_a^m),\quad (x,t)\in(0,a)\times(0,+\infty).$$

Therefore, $\underline{u}_a$ is a subsolution to (1.1)–(1.3). Since

$$\lim_{t\to+\infty}\sup_{(0,a)}\underline{u}_a(\cdot,t)=\lambda^{2/m}c>c,$$

u_a must quench at a finite time. □

Theorem 3.5. *There exists $a_*>0$ such that*

(i) *$T_*(a)=+\infty$ and $\sup_{(0,a)\times(0,+\infty)} u_a<c$ if $0<a<a_*$,*
(ii) *$T_*(a)<+\infty$ if $a>a_*$.*

Proof. Set

$$S=\big\{a>0: T_*(a)=+\infty \text{ and } \sup_{(0,a)\times(0,+\infty)} u_a<c\big\}.$$

By Lemmas 3.1 and 3.2, this set is bounded. Denote

$$a_*=\sup S.$$

By Lemma 3.3, $a\in S$ for each $0<a<a_*$. For $a>a_*$, the definition of S shows that $T_*(a)<+\infty$ or u_a quenches at the infinite time. Let us prove that the latter case is impossible by contradiction. Otherwise, assume that u_{a_0} quenches at the infinite time for some $a_0>a_*$. From the definition of S and Lemma 3.3, $u_{\tilde a}$ must quench at the infinite time for each $a_*<\tilde a<a_0$, which contradicts Lemma 3.4. □

Remark 3.6. Using Lemma 3.3, it is not difficult to show that

$$u_{a_*}(x,t)=\lim_{a\to a_*}u_a(x,t),\quad (x,t)\in(0,a_*)\times(0,T_*(a_*)).$$

Therefore, $T_*(a_*)=+\infty$. However, it is unknown whether u_{a_*} quenches or not at the infinite time.

4. Quenching properties

Assume that u is the solution of (1.1)–(1.3). According to Theorem 3.5, u quenches at a finite time if and only if $a > a_*$. In this section, we investigate the location of the quenching points and the blowing up of $\frac{\partial u}{\partial t}$.

Definition 4.1. Assume that the solution u to (1.1)–(1.3) quenches at $0 < T_* < +\infty$. A point $x \in [0, a]$ is said to be a quenching point if there exist two sequences $\{t_n\}_{n=1}^{\infty} \subset (0, T_*)$ and $\{x_n\}_{n=1}^{\infty} \subset (0, a)$ such that

$$\lim_{n\to\infty} t_n = T_*, \quad \lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} u(x_n, t_n) = c.$$

Theorem 4.2. *Assume that $a > a_*$. Then*

(i) *there is no quenching point in $(a/2, a)$ if $q > 0$,*
(ii) *there is no quenching point in $(0, a/2)$ if $q < 0$.*

Proof. We prove the case $q > 0$ only; the other case can be proved similarly. By Remark 2.4,

$$u(x,t) > 0, \quad \frac{\partial u}{\partial t}(x,t) > 0, \quad (x,t) \in (0,a) \times (0,T_*). \tag{4.1}$$

Set

$$v(x,t) = u(a - x, t), \quad (x,t) \in [0, a/2] \times [0, T_*).$$

Then, v is a solution of the equation

$$(a-x)^q \frac{\partial v}{\partial t} - \frac{\partial^2 v^m}{\partial x^2} = f(v^m), \quad (x,t) \in (0, a/2) \times (0, T_*). \tag{4.2}$$

By (4.1), u is a supersolution to (4.2). Similar to the proof of Lemma 3.3, one can show that

$$u(x,t) > v(x,t), \quad (x,t) \in (0, a/2) \times (0, T_*). \tag{4.3}$$

Set

$$w(x,t) = u^m(x,t) - v^m(x,t), \quad (x,t) \in [0, a/2] \times [0, T_*).$$

Then w solves

$$x^q h(x,t) \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + x^q \frac{\partial h}{\partial t}(x,t) w \geq g(x,t) w, \quad (x,t) \in (0, a/2) \times (0, T_*), \tag{4.4}$$

where

$$h(x,t) = \frac{1}{m} \int_0^1 (\sigma u^m(x,t) + (1-\sigma) v^m(x,t))^{1/m-1} d\sigma,$$

$$g(x,t) = \int_0^1 f'(\sigma u^m(x,t) + (1-\sigma) v^m(x,t)) d\sigma \geq f'(0) > 0,$$

for $(x,t) \in (0, a/2) \times (0, T_*)$. From (4.1) and (4.3), for $(x,t) \in (0, a/2) \times (0, T_*)$, follows that

$$\frac{\partial h}{\partial t}(x,t) = \frac{1}{m}\Big(\frac{1}{m} - 1\Big) \int_0^1 \Big(\sigma \frac{\partial u^m(x,t)}{\partial t} + (1-\sigma) \frac{\partial v^m(x,t)}{\partial t}\Big)^{1/m-1} d\sigma < 0,$$
$$w(x,t) > 0.$$

Therefore, w satisfies

$$x^q h(x,t) \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} \geq 0, \quad (x,t) \in (0, a/2) \times (0, T_*). \tag{4.5}$$

For any $0 < \eta < a/4$, set

$$\delta = \min_{(\eta, a/2-\eta)} (u^m(\cdot, T_*/2) - v^m(\cdot, T_*/2)).$$

Let z be the solution to the problem

$$x^q h(x,t)\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = 0, \quad (x,t) \in (\eta, a/2-\eta) \times (T_*/2, T_*), \tag{4.6}$$

$$z(\eta, t) = z(a/2-\eta, t) = 0, \quad t \in (T_*/2, T_*), \tag{4.7}$$

$$z(x, T_*/2) = \delta \sin\Big(\frac{2\pi(x-\eta)}{a-4\eta}\Big), \quad x \in (\eta, a/2-\eta). \tag{4.8}$$

Since (4.6) is a uniformly parabolic equation in $(\eta, a/2-\eta) \times (T_*/2, T_*)$, from the classical strong maximum principle it follows that

$$z(x,t) > 0, \quad (x,t) \in (\eta, a/2-\eta) \times [T_*/2, T_*]. \tag{4.9}$$

By (4.5) and (4.3), w is a supersolution to (4.6)–(4.8). The classical comparison principle leads to

$$w(x,t) \ge z(x,t), \quad (x,t) \in (\eta, a/2-\eta) \times (T_*/2, T_*);$$

i.e.,

$$u^m(a-x,t) \le u^m(x,t) - z(x,t), \quad (x,t) \in (\eta, a/2-\eta) \times (T_*/2, T_*).$$

So, there is no quenching point in $(a/2+\eta, a-\eta)$ owing to (4.9). Then, (i) is proved due to the arbitrariness of $0 < \eta < a/4$. □

Theorem 4.3. *Assume that $a > a_*$ and $M = \int_0^c s^{m-1} f(s^m) ds < +\infty$. Then*

$$M \ge \frac{c^{2m}}{ma^2}$$

and

(i) *the quenching points belong to $[c^{2m}/(2Mma), a/2]$ if $q > 0$,*
(ii) *the quenching points belong to $[c^{2m}/(2Mma), a - c^{2m}/(2Mma)]$ if $q = 0$,*
(iii) *the quenching points belong to $[a/2, a - c^{2m}/(2Mma)]$ if $q < 0$.*

Proof. From Remark 2.4, one gets

$$\int_0^a \Big(\frac{\partial u^m}{\partial x}(x,t)\Big)^2 dx \le 2mMa, \quad t \in (0, T_*). \tag{4.10}$$

Then, it follows from (4.10) and the Schwarz inequality that

$$\begin{aligned} u^m(x,t) &= \int_0^x \frac{\partial u^m}{\partial x}(y,t)dy \le x^{1/2}\Big(\int_0^a \Big(\frac{\partial u^m}{\partial x}(y,t)\Big)^2 dy\Big)^{1/2} \\ &\le (2mMax)^{1/2}, \quad (x,t) \in [0, a/2] \times [0, T_*) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} u^m(x,t) &= -\int_x^a \frac{\partial u^m}{\partial x}(y,t)dy \le (a-x)^{1/2}\Big(\int_0^a \Big(\frac{\partial u^m}{\partial x}(y,t)\Big)^2 dy\Big)^{1/2} \\ &\le (2mMa(a-x))^{1/2}, \quad (x,t) \in [a/2, a] \times [0, T_*). \end{aligned} \tag{4.12}$$

Since

$$\lim_{t \to T_*^-} \sup_{(0,a)} u(\cdot, t) = c$$

by Theorem 3.5, from (4.11) and (4.12) it follows that

$$M \geq \frac{c^{2m}}{ma^2}.$$

Furthermore, (i)–(iii) follow from Theorem 4.2, (4.11) and (4.12) directly. $\square$

Theorem 4.4. *Assume that $a > a_*$ and $\int_0^c s^{m-1} f(s^m)ds < +\infty$. Then the solution u of (1.1)–(1.3) satisfies*

$$\lim_{t\to T_*^-} \sup_{(0,a)} \frac{\partial u}{\partial t}(\cdot,t) = +\infty.$$

Proof. From Theorem 4.3, there exist $0 < x_1 < x_2 < x_3 < x_4 < a$ such that

$$\lim_{t\to T_*^-} \sup_{(x_2,x_3)} u(\cdot,t) = c, \tag{4.13}$$

$$\sup_{(0,x_2)\times(0,T_*)} u < c, \qquad \sup_{(x_3,a)\times(0,T_*)} u < c. \tag{4.14}$$

Set

$$w(x,t) = u^m(x,t), \quad (x,t) \in [x_1,x_4]\times[0,T_*).$$

Then w and $\frac{\partial w}{\partial t}$ solve

$$\frac{x^q}{m} w^{1/m-1}\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = f(w), \quad (x,t) \in (x_1,x_4)\times(0,T_*) \tag{4.15}$$

and

$$\begin{aligned} &\frac{x^q}{m} w^{1/m-1}(x,t)\frac{\partial}{\partial t}\Big(\frac{\partial w}{\partial t}\Big) + \Big(\frac{1}{m}-1\Big)\frac{x^q}{m} w^{1/m-2}(x,t)\Big(\frac{\partial w}{\partial t}\Big)^2 \\ &- \frac{\partial^2}{\partial x^2}\Big(\frac{\partial w}{\partial t}\Big) = f'(w(x,t))\frac{\partial w}{\partial t}, \quad (x,t) \in (x_1,x_4)\times(0,T_*), \end{aligned} \tag{4.16}$$

respectively. By Remark 2.4,

$$\frac{\partial w}{\partial t}(x,t) > 0, \quad (x,t) \in (0,a)\times(0,T_*). \tag{4.17}$$

Let z be the solution to the problem

$$\frac{x^q}{m} w^{1/m-1}(x,t)\frac{\partial z}{\partial t} - \frac{\partial z^2}{\partial x^2} = 0, \quad (x,t) \in (x_1,x_4)\times(T_*/2,T_*), \tag{4.18}$$

$$z(x_1,t) = z(x_4,t) = 0, \quad t \in (T_*/2,T_*), \tag{4.19}$$

$$z(x,T_*/2) = \delta \sin\Big(\frac{\pi(x-x_1)}{x_4-x_1}\Big), \quad x \in (x_1,x_4), \tag{4.20}$$

where

$$\delta = \min_{(x_1,x_4)} \frac{\partial w}{\partial t}(\cdot,T_*/2).$$

Since (4.18) is a uniformly parabolic equation, one gets from the classical maximum principle that

$$z(x,t) > 0, \quad (x,t) \in (x_1,x_4)\times[T_*/2,T_*]. \tag{4.21}$$

By (4.16) and (4.17), the function $\frac{\partial w}{\partial t}$ is a supersolution to (4.18)–(4.21). The classical comparison principle leads to

$$\frac{\partial w}{\partial t}(x,t) \geq z(x,t), \quad (x,t) \in (x_1,x_4)\times(T_*/2,T_*). \tag{4.22}$$

Set

$$v(x,t) = \frac{\partial w}{\partial t}(x,t) - \kappa f(w(x,t)), \quad (x,t) \in [x_2,x_3]\times[T_*/2,T_*).$$

By (4.14), (4.17), (4.21) and (4.22), there exists $\kappa > 0$ such that

$$v(x,t) \geq 0, \quad (x,t) \in \{x_2, x_3\} \times [T_*/2, T_*) \cup [x_2, x_3] \times \{T_*/2\}. \tag{4.23}$$

From (4.15) and (4.16), v solves

$$\begin{aligned}
&\frac{x^q}{m} w^{1/m-1}(x,t)\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - f'(w(x,t))v \\
&= \frac{x^q}{m} w^{1/m-1}\Big(\frac{\partial^2 w}{\partial t^2} - \kappa f'(w)\frac{\partial w}{\partial t}\Big) - \frac{\partial^3 w}{\partial t \partial x^2} + \kappa f''(w)\Big(\frac{\partial w}{\partial x}\Big)^2 \\
&\quad + \kappa f'(w)\frac{\partial^2 w}{\partial x^2} - f'(w)\Big(\frac{\partial w}{\partial t} - \kappa f(w)\Big) \\
&= \Big(1 - \frac{1}{m}\Big)\frac{x^q}{m} w^{1/m-2}\Big(\frac{\partial w}{\partial t}\Big)^2 + \kappa f''(w)\Big(\frac{\partial w}{\partial x}\Big)^2 \\
&\geq 0, \quad (x,t) \in (x_2, x_3) \times (T_*/2, T_*).
\end{aligned}$$

Then, from the classical comparison principle with (4.23) it follows that

$$v(x,t) \geq 0, \quad (x,t) \in [x_2, x_3] \times [T_*/2, T_*),$$

which, together with (4.13), yields

$$\lim_{t \to T_*^-} \sup_{(x_2,x_3)} \frac{\partial u}{\partial t}(\cdot, t) = +\infty.$$

$\square$

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