Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 103, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EQUI-ASYMPTOTICALLY ALMOST PERIODIC FUNCTIONS AND APPLICATIONS TO FUNCTIONAL INTEGRAL EQUATIONS

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Abstract. In this article, we introduce the notion of equi-asymptotically almost periodicity, and investigate some properties of equi-asymptotically almost periodic functions. In addition, by applying the results on equi-asymptotically almost periodic functions, we obtain an existence theorem for asymptotically almost periodic solutions to a class of functional integral equation.

1. Introduction and preliminaries

In 1940s, M. Fréchet introduced the notion of asymptotically almost periodicity, which turns out to be one of the most interesting and important generalizations of almost periodicity. In fact, asymptotically almost periodic functions now have been of great interest for many mathematicians and have been applied to various branches of pure and applied mathematics, especially to differential equations and dynamical systems. For example, we refer the reader to [9, 10, 11, 2] and references therein for some recent progress on asymptotically almost periodic functions and their applications to differential equations.

In a recent work, when studying the existence of asymptotically almost periodic solutions to a class of Volterra-type difference equations, Long et al [12] introduced the notion of equi-asymptotically almost periodic sequences. To the best of our knowledge, there are only a few publications about equi-asymptotically almost periodic functions. This motivates the publication of this work.

For the rest of this paper, if there is no special statement, we denote by $\mathbb{R}$ the set of real numbers, by X a Banach space, and by $C(\mathbb{R}, X)$ the set of all continuous functions from $\mathbb{R}$ to X. In addition, we denote

$$C_0(\mathbb{R}, X) = \{f \in C(\mathbb{R}, X) : \lim_{|t|\to\infty} f(t) = 0\}.$$

Next, let us recall some notation and basic results of almost periodic functions and asymptotically almost periodic functions (for more details, see [4, 8, 15, 14]).

2000 *Mathematics Subject Classification.* 34K14, 45G10.
Key words and phrases. Asymptotically almost periodic; equi-asymptotically almost periodic; functional integral equation.

Submitted February 6, 2013. Published April 24, 2013.

Definition 1.1. A set $E \subset \mathbb{R}$ is said to be relatively dense if there exists a number $l > 0$ such that

$$(a, a + l) \cap E \neq \emptyset$$

for every $a \in \mathbb{R}$.

Definition 1.2. A function $f \in C(\mathbb{R}, X)$ is said to be almost periodic if for every $\varepsilon > 0$ there exists a relatively dense set $T(f, \varepsilon) \subset \mathbb{R}$ such that

$$\|f(t + \tau) - f(t)\| < \varepsilon,$$

for all $t \in \mathbb{R}$ and $\tau \in T(f, \varepsilon)$. We denote the set of all such functions by $AP(\mathbb{R}, X)$.

Definition 1.3. A set $F \subset C(\mathbb{R}, X)$ is said to be equi-almost periodic if for every $\varepsilon > 0$, there exists a relatively dense set $T(F, \varepsilon) \subset \mathbb{R}$ such that

$$\|f(t + \tau) - f(t)\| < \varepsilon,$$

for all $f \in F$, $t \in \mathbb{R}$, and $\tau \in T(F, \varepsilon)$.

Definition 1.4. A function $f \in C(\mathbb{R}, X)$ is said to be asymptotically almost periodic if for every ε, there exist a constant $M(\varepsilon) > 0$ and a relatively dense set $T(f, \varepsilon) \subset \mathbb{R}$ such that

$$\|f(t + \tau) - f(t)\| < \varepsilon,$$

for all $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$ and $\tau \in T(F, \varepsilon)$ with $|t + \tau| > M(\varepsilon)$. We denote the set of all such functions by $AAP(\mathbb{R}, X)$.

The following result can be deduced from [15, Theorem 2.5].

Lemma 1.5. *$f \in AAP(\mathbb{R}, X)$ if and only if there exists a unique $g \in AP(\mathbb{R}, X)$ such that $f - g \in C_0(\mathbb{R}, X)$.*

Remark 1.6. By Lemma 1.5, we know that every asymptotically almost periodic function f admits a unique decomposition; i.e., there exist unique $g \in AP(\mathbb{R}, X)$ and $h \in C_0(\mathbb{R}, X)$ such that $f = g + h$. Thus, throughout the rest of this paper, for every $f \in AAP(\mathbb{R}, X)$, we denote by f_{AP} the almost periodic (or principal) component of f and f_{C_0} be the $C_0(\mathbb{R}, X)$ (or corrective) term of f.

Next, we list some basic properties of asymptotically almost periodic functions. For the proof, we refer the reader to [14, 15].

Lemma 1.7. *Let $f, g \in AAP(\mathbb{R}, X)$. Then*

(a) *f is uniformly continuous on $\mathbb{R}$;*
(b) *$\{f_{AP}(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$;*
(c) *$f + g \in AAP(\mathbb{R}, X)$, and $f \cdot g \in AAP(\mathbb{R}, \mathbb{R})$;*
(d) *$f(\cdot + a) \in AAP(\mathbb{R}, X)$ for any $a \in \mathbb{R}$.*

2. Equi-asymptotically almost periodic functions

In this section, we introduce the notion of equi-asymptotically almost periodicity, and present some basic and interesting properties for equi-asymptotically almost periodic functions.

Definition 2.1. A set $F \subset C(\mathbb{R}, X)$ is said to be equi-asymptotically almost periodic if for every $\varepsilon > 0$, there exist a constant $M(\varepsilon) > 0$ and a relatively dense set $T(F, \varepsilon) \subset \mathbb{R}$ such that

$$\|f(t + \tau) - f(t)\| < \varepsilon,$$

for all $f \in F$, $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$ and $\tau \in T(F, \varepsilon)$ with $|t + \tau| > M(\varepsilon)$.

Theorem 2.2. *Let $F \subset AAP(\mathbb{R}, X)$. Then the following assertions are equivalent:*

(i) *F is precompact in $AAP(\mathbb{R}, X)$.*

(ii) *F satisfies the following three conditions:*

 (a) *for every $t \in \mathbb{R}$, $\{f(t) : f \in F\}$ is precompact in X;*

 (b) *F is equi-uniformly continuous;*

 (c) *F is equi-asymptotically almost periodic.*

(iii) *G is precompact in $AP(\mathbb{R}, X)$ and H is precompact in $C_0(\mathbb{R}, X)$, where*

$$G = \{f_{AP} : f \in F\} \text{ and } H = \{f_{C_0} : f \in F\}.$$

Proof. (i) $\Rightarrow$ (ii): Let F be precompact in $AAP(\mathbb{R}, X)$. Then, obviously, for every $t \in \mathbb{R}$, $\{f(t) : f \in F\}$ is precompact in X. In addition, for every $\varepsilon > 0$, there exist $f_1, f_2, \dots, f_k \in F$ such that for every $f \in F$,

$$\min_{1 \le i \le k} \|f - f_i\| < \varepsilon,$$

where k is a positive integer dependent on ε. Combining this with the fact that $\{f_i\}_{i=1}^k$ is equi-uniformly continuous and equi-asymptotically almost periodic, we know that (b) and (c) hold.

(ii) $\Rightarrow$ (iii): Let $\{g_n\}_{n=1}^\infty \subset G$. For every n, there exist $f_n \in F$ and $h_n \in H$ such that $f_n = g_n + h_n$. By (a) and (b), applying Arzela-Ascoli theorem and choosing diagonal sequence, we can get a subsequence of $\{f_n\}_{n=1}^\infty$, which we still denote by $\{f_n\}_{n=1}^\infty$ for convenience, such that $\{f_n(t)\}_{n=1}^\infty$ is uniformly convergent on every compact subsets of $\mathbb{R}$.

Since $\{f_n\}_{n=1}^\infty$ is equi-asymptotically almost periodic, for every $\varepsilon > 0$, there exists $l(\varepsilon), M(\varepsilon) > 0$ such that for every $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$, there is a

$$\tau_t \in [M(\varepsilon) + 1 - t, M(\varepsilon) + 1 - t + l(\varepsilon)]$$

satisfying

$$\|f_n(t + \tau_t) - f_n(t)\| < \frac{\varepsilon}{3} \tag{2.1}$$

for all $n \in \mathbb{N}$. Noting that $\{f_n(t)\}_{n=1}^\infty$ is uniformly convergent on

$$[-M(\varepsilon) - l(\varepsilon) - 1, M(\varepsilon) + l(\varepsilon) + 1],$$

for the above $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \ge n \ge N$ and $t \in [-M(\varepsilon) - l(\varepsilon) - 1, M(\varepsilon) + l(\varepsilon) + 1]$,

$$\|f_m(t) - f_n(t)\| < \frac{\varepsilon}{3} \tag{2.2}$$

Combining (2.1) and (2.2), for all $m \ge n \ge N$ and $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$, we have

$$\begin{aligned} &\|f_m(t) - f_n(t)\| \\ &\le \|f_m(t) - f_m(t + \tau_t)\| + \|f_m(t + \tau_t) - f_n(t + \tau_t)\| + \|f_n(t + \tau_t) - f_n(t)\| \\ &\le \varepsilon, \end{aligned}$$

this and (2.2) yield $\{f_n(t)\}_{n=1}^\infty$ is uniformly convergent on $\mathbb{R}$. In view of

$$\{g_m(t) - g_n(t) : t \in \mathbb{R}\} \subset \overline{\{f_m(t) - f_n(t) : t \in \mathbb{R}\}}$$

for all $m, n \in \mathbb{N}$, we conclude that $\{g_n(t)\}_{n=1}^\infty$ is also uniformly convergent on $\mathbb{R}$; i.e., $\{g_n\}_{n=1}^\infty$ is convergent in $AP(\mathbb{R}, X)$. So G is precompact in $AP(\mathbb{R}, X)$. In addition, it follows from the above proof that F is precompact, and thus H is also precompact.

(iii) $\Rightarrow$ (i): The proof is straightforward; we omit it. $\square$

Remark 2.3. Theorem 2.2 can be seen as an extension of the corresponding compactness criteria for the subsets of $AP(\mathbb{R}, X)$ (see, e.g., [4, 14]).

Definition 2.4. $F \subset C_0(\mathbb{R}, X)$ is called equi-C_0 if

$$\lim_{|t|\to\infty} \sup_{f\in F} \|f(t)\| = 0.$$

Theorem 2.5. *The following two assertions are equivalent:*

(I) *F is equi-asymptotically almost periodic;*

(II) *G is equi-almost periodic and H is equi-C_0, where*

$$G = \{f_{AP} : f \in F\} \text{ and } H = \{f_{C_0} : f \in F\}.$$

Proof. The proof from (II) to (I) is straightforward. We will only give the proof from (I) to (II) by using the idea in the proof of [15, Theorem 2.5].

Since F is equi-asymptotically almost periodic, for every $k \in \mathbb{N}$, there exist a constant $M_k > 0$ and a relatively dense set $T(F, k) \subset \mathbb{R}$ such that

$$\|f(t+\tau) - f(t)\| < \frac{1}{k}, \tag{2.3}$$

for all $f \in F$, $t \in \mathbb{R}$ with $|t| > M_k$ and $\tau \in T(F, k)$ with $|t+\tau| > M_k$. Moreover, for every $f \in F \subset AAP(\mathbb{R}, X)$, noting that f is uniformly continuous, for the above $k \in \mathbb{N}$, there exists $\delta_k^f > 0$ such that

$$\|f(t_1) - f(t_2)\| < \frac{1}{k} \tag{2.4}$$

for all $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_k^f$.

Now, for every $t \in \mathbb{R}$ and $k \in \mathbb{N}$, we choose $\tau_k^t \in T(F, k)$ with $t + \tau_k^t > M_k$. Also, we denote

$$g_k^f(t) = f(t + \tau_k^t), \quad t \in \mathbb{R},\ k \in \mathbb{N},\ f \in F.$$

Next, we divide the remaining proof into eight steps.

Step 1. For every $f \in F$, there holds

$$\|g_k^f(t_1) - g_k^f(t_2)\| < \frac{5}{k} \tag{2.5}$$

for all $k \in \mathbb{N}$, and $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_k^f$. In fact, by (2.3) and (2.4), we have

$$\begin{aligned}
&\|g_k^f(t_1) - g_k^f(t_2)\| \\
&= \|f(t_1 + \tau_k^{t_1}) - f(t_2 + \tau_k^{t_2})\| \\
&\le \|f(t_1 + \tau_k^{t_1}) - f(t_1 + \tau_k^{t_1} + \tau)\| + \|f(t_1 + \tau_k^{t_1} + \tau) - f(t_2 + \tau_k^{t_1} + \tau)\| \\
&\quad + \|f(t_2 + \tau_k^{t_1} + \tau) - f(t_2 + \tau)\| + \|f(t_2 + \tau) - f(t_2 + \tau + \tau_k^{t_2})\| \\
&\quad + \|f(t_2 + \tau + \tau_k^{t_2}) - f(t_2 + \tau_k^{t_2})\| < \frac{5}{k},
\end{aligned}$$

where $\tau \in T(F, k)$ satisfies

$$\min\{t_1 + \tau_k^{t_1} + \tau, t_2 + \tau_k^{t_1} + \tau, t_2 + \tau, t_2 + \tau + \tau_k^{t_2}\} > M_k.$$

Step 2. For every $k \in \mathbb{N}$, there holds

$$\|g_k^f(t+\tau) - g_k^f(t)\| < \frac{5}{k} \tag{2.6}$$

for all $f \in F$, $\tau \in T(F,k)$, and $t \in \mathbb{R}$. In fact, by using (2.3), we have

$$\begin{aligned}
&\|g_k^f(t+\tau) - g_k^f(t)\| \\
&= \|f(t+\tau+\tau_k^{t+\tau}) - f(t+\tau_k^t)\| \\
&\le \|f(t+\tau+\tau_k^{t+\tau}) - f(t+\tau+\tau_k^{t+\tau}+\tau')\| \\
&\quad + \|f(t+\tau+\tau_k^{t+\tau}+\tau') - f(t+\tau_k^{t+\tau}+\tau')\| \\
&\quad + \|f(t+\tau_k^{t+\tau}+\tau') - f(t+\tau')\| + \|f(t+\tau') - f(t+\tau'+\tau_k^t)\| \\
&\quad + \|f(t+\tau'+\tau_k^t) - f(t+\tau_k^t)\| < \frac{5}{k},
\end{aligned}$$

where $\tau' \in T(F,k)$ satisfies

$$\min\{t+\tau+\tau_k^{t+\tau}+\tau', t+\tau_k^{t+\tau}+\tau', t+\tau', t+\tau'+\tau_k^t\} > M_k.$$

Step 3. For every $n \in \mathbb{N}$, there holds

$$\|g_m^f(t) - g_n^f(t)\| < \frac{4}{n} \tag{2.7}$$

for all $f \in F$, $t \in \mathbb{R}$, and $m, n \in \mathbb{N}$ with $m \ge n$. In fact, without loss of generality, we can assume that $M_{k+1} \ge M_k$ for all $k \in \mathbb{N}$. Then, by using (2.3), we have

$$\begin{aligned}
&\|g_m^f(t) - g_n^f(t)\| \\
&= \|f(t+\tau_m^t) - f(t+\tau_n^t)\| \\
&\le \|f(t+\tau_m^t) - f(t+\tau_m^t+\tau)\| + \|f(t+\tau_m^t+\tau) - f(t+\tau_m^t+\tau+\tau_n^t)\| \\
&\quad + \|f(t+\tau_m^t+\tau+\tau_n^t) - f(t+\tau+\tau_n^t)\| + \|f(t+\tau+\tau_n^t) - f(t+\tau_n^t)\| \\
&< \frac{1}{n} + \frac{1}{n} + \frac{1}{m} + \frac{1}{n} \le \frac{4}{n},
\end{aligned}$$

where $\tau \in T(F,n)$ satisfies

$$\min\{t+\tau_m^t+\tau, t+\tau_m^t+\tau+\tau_n^t, t+\tau+\tau_n^t\} > M_m.$$

Step 4. Let

$$g^f(t) = \lim_{n\to\infty} g_n^f(t), \quad t \in \mathbb{R},\ f \in F.$$

By Step 3, we know that for every $f \in F$, g^f is well-defined. Moreover, it follows from Step 3 that for every $n \in \mathbb{N}$, there holds

$$\|g^f(t) - g_n^f(t)\| \le \frac{4}{n} \tag{2.8}$$

for all $f \in F$, $t \in \mathbb{R}$, and $n \in \mathbb{N}$.

Step 5. For every $f \in F$, g^f is uniformly continuous on $\mathbb{R}$. In fact, by (2.5) and (2.8), we have

$$\begin{aligned}
\|g^f(t_1) - g^f(t_2)\| &\le \|g^f(t_1) - g_n^f(t_1)\| + \|g_n^f(t_1) - g_n^f(t_2)\| + \|g_n^f(t_2) - g^f(t_2)\| \\
&\le \frac{4}{n} + \frac{5}{n} + \frac{4}{n} = \frac{13}{n},
\end{aligned}$$

for all $n \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_n^f$.

Step 6. $\{g^f\}_{f\in F}$ is equi-almost periodic. By (2.6) and (2.8), for every $n \in \mathbb{N}$, we obtain

$$\|g^f(t+\tau) - g^f(t)\|$$

$$\begin{aligned}&\le \|g^f(t+\tau) - g_n^f(t+\tau)\| + \|g_n^f(t+\tau) - g_n^f(t)\| + \|g_n^f(t) - g^f(t)\| \\ &\le \frac{4}{n} + \frac{5}{n} + \frac{4}{n} = \frac{13}{n},\end{aligned}$$

for all $f \in F$, $\tau \in T(F, n)$, and $t \in \mathbb{R}$. Then, it follows that $\{g^f\}_{f\in F}$ is equi-almost periodic.

Step 7. $\{h^f\}_{f\in F}$ is equi-C_0, where $h^f(t) = f(t) - g^f(t)$ for all $f \in F$ and $t \in \mathbb{R}$. In fact, firstly, by Step 5, $h^f \in C(\mathbb{R}, X)$ for every $f \in F$; secondly, for every $n \in \mathbb{N}$, by (2.8) and the definition of τ_n^t, we have

$$\begin{aligned}\|h^f(t)\| &= \|f(t) - g^f(t)\| \\ &\le \|f(t) - g_n^f(t)\| + \|g_n^f(t) - g^f(t)\| \\ &\le \|f(t) - f(t+\tau_n^t)\| + \frac{4}{n} \\ &\le \frac{1}{n} + \frac{4}{n} = \frac{5}{n},\end{aligned}$$

for all $f \in F$, $t \in \mathbb{R}$ with $|t| > M_n$. Thus, $\{h^f\}_{f\in F}$ is equi-C_0.

Step 8. It follows from the above steps that $G = \{g^f\}_{f\in F}$ and $H = \{h^f\}_{f\in F}$. This completes the proof. □

3. Application to functional integral equations

In this section, we apply some results in Section 2 to discuss the existence of asymptotically almost periodic solutions for the following functional integral equation:

$$x(t) = \sum_{i=1}^{n} f_i(t, x(t)) \cdot \int_{\mathbb{R}} k_i(t, s) g_i(s, x(s)) ds, \quad t \in \mathbb{R}, \tag{3.1}$$

where n is a fixed positive integer, and $f_i : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $k_i : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g_i : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ $(i = 1, 2, \dots, n)$ satisfy some conditions stated below.

The study on the existence of solutions to various kinds of functional integral equations has been one of the most attractive topics in the theory of integral equations. We refer the reader to [1, 3, 5, 7, 13] and references therein for some of recent developments on this topic. Especially, in [7, 13], the authors obtained some results concerning periodic solutions and almost periodic solutions for some variants of equation (3.1). In addition, we would like to note that by a recent work [16], in the sense of category, the "amount" of almost periodic functions (not periodic) is far more than the "amount" of continuous periodic functions. Thus, studying the existence of almost periodic solutions for differential equations is necessary.

Next, we will extend the results in [7, 13] to asymptotically almost periodic case for equation (3.1). Before establishing our existence theorem, we first recall a fixed point theorem in a Banach algebra, which is due to [13, Theorem 2.1] (see also [6, Theorem 2.1]).

Theorem 3.1. *Let n be a positive integer, and C be a nonempty, closed, convex and bounded subset of a Banach algebra X. Assume that the operators $A_i : X \to X$ and $B_i : C \to X$, $i = 1, 2, \dots, n$, satisfy*

(a) *for each $i \in \{1, 2, \dots, n\}$, there exists $L_i > 0$ such that*

$$\|A_i x - A_i y\| \le L_i \|x - y\| \quad \text{for all } x, y \in X;$$

(b) *for each* $i \in \{1, 2, \dots, n\}$, B_i *is continuous and* $B_i(C)$ *is precompact;*
(c) *for each* $y \in C$, $x = \sum_{i=1}^{n} A_i x \cdot B_i y$ *implies that* $x \in C$;

Then, the operator equation $x = \sum_{i=1}^{n} A_i x \cdot B_i x$ *has a solution provided that* $\sum_{i=1}^{n}(L_i \cdot M_i) < 1$, *where* $M_i = \sup_{x \in C} \|B_i x\|$, $i = 1, 2, \dots, n$.

Now, we present our existence theorem.

Theorem 3.2. *Let* $p \ge 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that the following assumptions hold:*

(H1) *For each* $i \in \{1, 2, \dots, n\}$, $f_i(\cdot, x) \in AAP(\mathbb{R})$ *for any fixed* $x \in \mathbb{R}$ *and there exists a constant* $L_i > 0$ *such that*
$$|f_i(t, x) - f_i(t, y)| \le L_i |x - y|, \quad \forall t \in \mathbb{R},\ \forall x, y \in \mathbb{R}.$$
(H2) *For each* $i \in \{1, 2, \dots, n\}$, $g_i(\cdot, x)$ *is measurable for all* $x \in \mathbb{R}$, $g_i(t, \cdot)$ *is continuous for almost all* $t \in \mathbb{R}$, *and for each* $r > 0$, *there exists a function* $\mu_i^r \in L^p(\mathbb{R})$ *such that* $|g_i(t, x)| \le \mu_i^r(t)$ *for all* $|x| \le r$ *and almost all* $t \in \mathbb{R}$.
(H3) *For each* $i \in \{1, 2, \dots, n\}$, $\widetilde{k}_i \in AAP(\mathbb{R}, L^q(\mathbb{R}))$, *where* $[\widetilde{k}_i(t)](s) = k(t, s)$, $\forall t, s \in \mathbb{R}$.
(H4) *There exists a constant* $r_0 > 0$ *such that*
$$\sum_{i=1}^{n} L_i K_i \|\mu_i^{r_0}\|_p < 1, \tag{3.2}$$
where $K_i = \sup_{t \in \mathbb{R}} \|\widetilde{k}_i(t)\|_q$; *and*
$$\sum_{i=1}^{n} \big[\sup_{t \in \mathbb{R}, |x| \le r} |f_i(t, x)| \cdot K_i \cdot \|\mu_i^{r_0}\|_p \big] < r, \quad \forall r > r_0. \tag{3.3}$$

Then (3.1) *has an asymptotical almost periodic solution.*

Proof. For each $i \in \{1, 2, \dots, n\}$, we denote
$$(A_i x)(t) = f_i(t, x(t)), \quad x \in AAP(\mathbb{R}),\ t \in \mathbb{R},$$
and
$$(B_i x)(t) = \int_{\mathbb{R}} k_i(t, s) g_i(s, x(s)) ds, \quad x \in AAP(\mathbb{R}),\ t \in \mathbb{R}.$$

Now, we show that A_i, B_i map $AAP(\mathbb{R})$ into $AAP(\mathbb{R})$. Fix $x \in AAP(\mathbb{R})$. By using (H1), we can get $A_i x \in AAP(\mathbb{R})$. In addition, since (H2) holds, for all $t_1, t_2 \in \mathbb{R}$, we have
$$|(B_i x)(t_1) - (B_i x)(t_2)| \le \|\widetilde{k}_i(t_1) - \widetilde{k}_i(t_2)\|_q \cdot \|\mu_i^{\|x\|_{AAP(\mathbb{R})}}\|_p. \tag{3.4}$$
Combining (3.4) and the fact that $\widetilde{k}_i \in AAP(\mathbb{R}, L^q(\mathbb{R}))$, we conclude that $B_i x \in AAP(\mathbb{R})$.

Let $C = \{x \in AAP(\mathbb{R}) : \|x\| \le r_0\}$, where r_0 is the constant in (H4). Obviously, C is a nonempty, closed, convex and bounded subset in $AAP(\mathbb{R})$. In addition, it follows from (H1) that for each $i \in \{1, 2, \dots, n\}$, $\|A_i x - A_i y\| \le L_i \|x - y\|$ for all $x, y \in X$, i.e., the assumption (a) of Theorem 3.1 holds.

Next, we show that the assumption (b) of Theorem 3.1 holds. We first show that every B_i is continuous. Let $x_k \to x$ in $AAP(\mathbb{R})$. Since
$$|(B_i x_k)(t) - (B_i x)(t)| \le K_i \|g_i(\cdot, x_k(\cdot)) - g_i(\cdot, x(\cdot))\|_p,$$

and $g_i(t,\cdot)$ is continuous for almost all $t \in \mathbb{R}$, by using Lebesgue's dominated convergence theorem, we conclude that $B_i x_k \to B_i x$ in $AAP(\mathbb{R})$. So B_i is continuous. In addition, by a direct calculation, we have

$$|(B_i x)(t)| \le K_i \|\mu_i^{r_0}\|, \quad t \in \mathbb{R},\ x \in C,$$

which means that every $B_i(C)$ is uniformly bounded. Also, it follows from (3.4) that for all $t_1, t_2 \in \mathbb{R}$,

$$|(B_i x)(t_1) - (B_i x)(t_2)| \le \|\widetilde{k}_i(t_1) - \widetilde{k}_i(t_2)\|_q \cdot \|\mu_i^{r_0}\|_p\,,$$

which yields that every $B_i(C)$ is equi-uniformly continuous and equi-asymptotically almost periodic. Then, by using Theorem 2.2, we know that every $B_i(C)$ is precompact in $AAP(\mathbb{R})$.

Now, let us verify the assumption (c) of Theorem 3.1. Let $y \in C$ and $x = \sum_{i=1}^n A_i x \cdot B_i y$. We claim that $x \in C$. Otherwise, $\|x\| > r_0$. Then, by using (3.3), we obtain

$$\|x\| = \|\sum_{i=1}^n A_i x \cdot B_i y\| \le \sum_{i=1}^n \big[\sup_{t\in\mathbb{R}} |f_i(t,x(t))| \cdot K_i \cdot \|\mu_i^{r_0}\|_p\big] < \|x\|,$$

which is a contradiction.

It follows from (3.2) that

$$\sum_{i=1}^n (L_i \cdot M_i) = \sum_{i=1}^n (L_i \cdot \sup_{x\in C} \|B_i x\|) \le \sum_{i=1}^n L_i K_i \|\mu_i^{r_0}\|_p < 1.$$

So all the assumptions of Theorem 3.1 hold. Thus, the operator equation $x = \sum_{i=1}^n A_i x \cdot B_i x$ has a solution in $AAP(\mathbb{R})$; i.e., Equation (3.1) has an asymptotical almost periodic solution. $\square$

Next, we give an example, which does not aim to generality but illustrate our existence theorem.

Example 3.3. Let $n = 1$, $p = 1$, $q = \infty$,

$$f_1(t,x) = \frac{\arctan x \cdot (\sin t + \sin(\pi t) + e^{-t^2})}{10}, \quad g_1(t,x) = \frac{x \sin(x e^{t^2})}{4(1+t^2)},$$

$$k_1(t,s) = (\sin t + \sin(\sqrt{2}t) + \frac{1}{1+t^2})e^{-s^2}.$$

Since for each $x \in \mathbb{R}$, $f_1(\cdot,x) \in AAP(\mathbb{R})$ and

$$|f_1(t,x) - f_1(t,y)| \le \frac{3}{10}|x-y|, \quad t \in \mathbb{R},\ x,y \in \mathbb{R},$$

we know that (H1) holds with $L_1 = 3/10$. It is easy to verify that (H2) holds with

$$\mu_1(r) = \frac{r}{4(1+t^2)},\ r > 0.$$

Obviously, (H3) holds and $K_1 = \sup_{t\in\mathbb{R}} \|\widetilde{k}_1(t)\|_\infty \le 3$. Moreover, by a direct calculation, one can show that (H4) holds with $r_0 = 1$. So, by Theorem 3.2, the following functional integral equation

$$x(t) = f_1(t,x(t)) \cdot \int_{\mathbb{R}} k_1(t,s) g_1(s,x(s))ds, \quad t \in \mathbb{R},$$

has an asymptotical almost periodic solution.

Acknowledgements. This work was partially supported by grants 11101192 from the NSF of China, 211090 from the Key Project of Chinese Ministry of Education, 20114BAB211002 from the NSF of Jiangxi Province, and GJJ12173 from the Jiangxi Provincial Education Department.

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