Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 08, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND UNIQUENESS OF ANTI-PERIODIC SOLUTIONS FOR NONLINEAR THIRD-ORDER DIFFERENTIAL INCLUSIONS

TOUMA HADDAD, TAHAR HADDAD

Abstract. In this article, we study the existence of anti-periodic solutions for the third-order differential inclusion

$$\begin{gathered} u'''(t) \in \partial\varphi(u'(t)) + F(t, u(t)) \quad \text{a.e. on } [0, T] \\ u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T), \end{gathered}$$

where φ is a proper convex, lower semicontinuous and even function, and F is an upper semicontinuous convex compact set-valued mapping. Also uniqueness of anti-periodic solution is studied.

1. Introduction

Existence and uniqueness of anti-periodic solutions for differential inclusions generated by the subdifferential of a convex lower semicontinuous even function appear in several articles; see [2, 3, 4, 5, 6, 11, 12]. Okochi [13] initiated the study of anti-periodic solutions of the differential inclusion

$$\begin{gathered} f(t) \in u'(t) + \partial\varphi(u(t)) \quad \text{a.e. } t \in [0, T] \\ u(0) = -u(T) \end{gathered} \tag{1.1}$$

in Hilbert spaces, where $\partial\varphi$ is the subdifferential of an even function φ on a real Hilbert space H and $f \in L^2([0, T], H)$. It was shown in [14], by applying a fixed point theorem for nonexpansive mapping, that (1.1) has a unique solution. Later Aftabizadeh and al [1] studied the anti-periodic solution of third-order differential inclusion

$$\begin{gathered} u'''(t) \in \partial\varphi(u'(t)) + f(t) \quad \text{a.e. } t \in [0, T] \\ u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T), \end{gathered} \tag{1.2}$$

by using maximal monotone operator theory.

The aim of this article is to study the existence of anti-periodic solutions for the third-order differential inclusion

$$\begin{gathered} u'''(t) \in \partial\varphi(u'(t)) + F(t, u(t)) \quad \text{a.e. } t \in [0, T] \\ u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T), \end{gathered} \tag{1.3}$$

2000 *Mathematics Subject Classification.* 34C25, 34G20, 49J52.
Key words and phrases. Anti-periodic solution; differential inclusions; subdifferential.

Submitted September 28, 2012. Published January 9, 2013.

where $\varphi : H \to]-\infty, +\infty]$ is a convex lower semicontinuous even function and $F : [0,T]\times H \to 2^H$ is an upper semicontinuous convex compact set-valued mapping bounded above by L^2 function. Furthermore, an existence and uniqueness result when F is single-valued is also studied.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. The open ball centered at x with radius r is defined by $\mathbb{B}_r(x) = \{y \in H : \|y - x\| < r\}$, where $\overline{\mathbb{B}_r}(x)$ denotes its closure. For a proper lower semicontinuous convex function $\varphi : H \to]-\infty, +\infty]$, the set-valued mapping $\partial\varphi : H \to 2^H$ defined by

$$\partial\varphi(x) = \{\xi \in H : \varphi(y) - \varphi(x) \geq \langle \xi, y - x\rangle, \forall y \in H\}$$

which is the subdifferential of φ. Let us recall a classical closure type lemma from [7].

Lemma 2.1. *Let H be a separable Hilbert space. Let φ be a convex lower semicontinuous function defined on H with values in $]-\infty, +\infty]$. Let $(u_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence of measurable mappings from $[0,T]$ into H such that $u_n \to u_\infty$ pointwise with respect to the norm topology. Assume that $(\xi_n)_{n\in\mathbb{N}}$ is a sequence in $L^2([0,T],H)$ satisfying*

$$\xi_n(t) \in \partial\varphi(u_n(t)) \quad a.e.\ t \in [0,T]$$

for each $n \in \mathbb{N}$ and converging weakly to $\xi_\infty \in L^2([0,T],H)$. Then we have

$$\xi_\infty(t) \in \partial\varphi(u_\infty(t)) \quad a.e.\ t \in [0,T].$$

Let us recall a useful result.

Lemma 2.2 ([15])**.** *Let H be a real Hilbert space. Let $u \in W^{1,2}_{\rm loc}(\mathbb{R},H)$ be $2T$-periodic and satisfying $\int_0^{2T} u(t)dt = 0$, then*

$$\|u\|_{L^2([0,2T],H)} \leq \frac{T}{\pi}\|u'\|_{L^2([0,2T],H)}.$$

3. Main results

We state and summarize some useful results for anti-periodic mappings that are crucial for our purpose.

Proposition 3.1. *Let H be a real Hilbert space. Let $u \in W^{3,2}([0,T],H)$ satisfying $u(0) = -u(T)$, $u'(0) = -u'(T)$, $u''(0) = -u''(T)$, then the following inequalities hold*

- (A1) $\|u\|_{\mathcal{C}([0,T],H)} \leq \frac{\sqrt{T}}{2}\|u'\|_{L^2([0,T],H)}$;
- (A2) $\|u\|_{L^2([0,T],H)} \leq \frac{T}{\pi}\|u'\|_{L^2([0,T],H)}$;
- (B1) $\|u'\|_{\mathcal{C}([0,T],H)} \leq \frac{\sqrt{T}}{2}\|u''\|_{L^2([0,T],H)}$;
- (B2) $\|u'\|_{L^2([0,T],H)} \leq \frac{T}{\pi}\|u''\|_{L^2([0,T],H)}$;
- (C1) $\|u''\|_{\mathcal{C}([0,T],H)} \leq \frac{\sqrt{T}}{2}\|u'''\|_{L^2([0,T],H)}$.

Proof. (A1) Since $u(t) = u(0) + \int_0^t u'(s)ds$ and $u(t) = u(T) - \int_t^T u'(s)ds$, for all $t \in [0,T]$, by adding these equalities, by anti-periodicity, we obtain

$$2u(t) = \int_0^t u'(s)ds - \int_t^T u'(s)ds, \quad \forall t \in [0,T].$$

Hence we have

$$2\|u(t)\| \leq \int_0^t \|u'(s)\|ds + \int_t^T \|u'(s)\|ds = \int_0^T \|u'(s)\|ds, \quad \forall t \in [0,T],$$

and so, by Holder inequality

$$\|u\|_{\mathcal{C}([0,T],H)} = \sup_{t\in[0,T]} \|u(t)\| \leq \frac{\sqrt{T}}{2}\|u'\|_{L^2([0,T],H)}.$$

(A2) For the sake of simplicity, we use the same notation $u(t), t \in \mathbb{R}$, to denote the anti-periodic extension of $u(t)$, $t \in [0,T]$, such that

$$u(t+T) = -u(t) \quad \text{and} \quad u'(t+T) = -u'(t), \quad \text{for } t \in \mathbb{R}.$$

Then u is $2T$-periodic function since

$$u(t+2T) = u(t+T+T) = -u(t+T) = u(t).$$

Also, since

$$\int_0^{2T} u(t)dt = \int_0^T u(t)dt + \int_T^{2T} u(t)dt = \int_0^T u(t)dt - \int_0^T u(t)dt = 0,$$

so, by Lemma 2.2,

$$\int_0^{2T} \|u(t)\|^2 dt \leq \frac{T^2}{\pi^2}\int_0^{2T} \|u'(t)\|^2 dt.$$

Let us observe that $\|u(t)\|^2$ and $\|u'(t)\|^2$ are T-periodic because $\|u(t+T)\|^2 = \|-u(t)\|^2 = \|u(t)\|^2$ and similarly $\|u'(t+T)\|^2 = \|-u'(t)\|^2 = \|u'(t)\|^2$.
Hence we deduce that

$$\int_0^{2T} \|u(t)\|^2 dt = 2\int_0^T \|u(t)\|^2 dt \quad \text{and} \quad \int_0^{2T} \|u'(t)\|^2 dt = 2\int_0^T \|u'(t)\|^2 dt.$$

Finally we get the required inequality

$$\int_0^T \|u(t)\|^2 dt \leq \frac{T^2}{\pi^2}\int_0^T \|u'(t)\|^2 dt.$$

Similarly, we prove (B1), (B2) and (C1), using $u'(0) = -u'(T)$, and $u''(0) = -u''(T)$ respectively. □

The following result deal with convex compact valued perturbations of a third-order differential inclusion governed by subdifferential operators of convex lower semicontinuous functions with anti-periodic boundary conditions. First, to simplify we will assume that $H = \mathbb{R}^d$.

Theorem 3.2. *Let $H = \mathbb{R}^d$, $\varphi : H \to]-\infty,+\infty]$ be a proper, convex, lower semicontinuous and even function. Let $F : [0,T]\times H \to 2^H$ be a convex compact set-valued mapping, measurable on $[0,T]$ and upper semicontinuous on H satisfying: there is a function $\alpha(\cdot) \in L^2([0,T],\mathbb{R}_+)$ such that*

$$F(t,x) \subset \Gamma(t) := \overline{\mathbb{B}}_{\alpha(t)}(0) \quad \textit{for all } (t,x) \in [0,T]\times H.$$

Then the problem

$$u'''(t) \in \partial\varphi(u'(t)) + F(t,u(t)) \quad \textit{a.e. } t \in [0,T],$$
$$u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T),$$

has at least an anti-periodic $W^{3,2}([0,T],H)$ solution.

Proof. Recall that a $W^{3,2}([0,T],H)$ function $u:[0,T]\to H$ is a solution of the problem under consideration if there exists a function $h\in L^2([0,T];H)$ such that

$$\begin{gathered} u'''(t)\in\partial\varphi(u'(t))+h(t)\quad \text{a.e. } t\in[0,T],\\ h(t)\in F(t,u(t))\quad \text{a.e. } t\in[0,T],\\ u(0)=-u(T),\quad u'(0)=-u'(T),\quad u''(0)=-u''(T). \end{gathered}$$

Let us denote by S^2_Γ the set of all $L^2([0,T];H)$-selection of Γ

$$S^2_\Gamma:=\{f\in L^2([0,T],H): f(t)\in\Gamma(t) \text{ a.e. } t\in[0,T]\}.$$

By [2, Theorem 2.1], for all $f\in S^2_\Gamma$, there is a unique $W^{3,2}([0,T],H)$ solution u_f of

$$\begin{gathered} u_f'''(t)\in\partial\varphi(u_f'(t))+f(t)\quad \text{a.e. } t\in[0,T],\\ u_f(0)=-u_f(T),\quad u_f'(0)=-u_f'(T),\quad u_f''(0)=-u_f''(T), \end{gathered}$$

such that

$$\|u_f'''\|_{L^2([0,T],H)}\le\|f\|_{L^2([0,T],H)}. \tag{3.1}$$

For each $f\in S^2_\Gamma$, let us define the set-valued mapping

$$\Psi(f):=\{g\in L^2([0,T],H); g(t)\in F(t,u_f(t))\quad \text{a.e. } t\in[0,T]\}.$$

Then it is clear that $\Psi(f)$ is a nonempty convex weakly compact subset of S^2_Γ, here the nonemptiness follows from [10, theorem VI.4]. From the above consideration, we need to prove that the convex weakly compact set-valued mapping $\Psi: S^2_\Gamma\to 2^{S^2_\Gamma}$ admits a fixed point. By Kakutani-Ky Fan fixed point theorem, it is sufficient to prove that Ψ is upper semicontinuous when S^2_Γ is endowed with the weak topology of $L^2([0,T],H)$. As $L^2([0,T],H)$ is separable, S^2_Γ is compact metrizable with respect to the weak topology of $L^2([0,T],H)$. So it turns out to check that the graph $gph(\Psi)$ is sequentially weakly closed in $S^2_\Gamma\times S^2_\Gamma$. Let $(f_n,g_n)_n\in gph(\Psi)$ weakly converging to $(f,g)\in S^2_\Gamma\times S^2_\Gamma$. From the definition of Ψ, that means u_{f_n} is the unique $W^{3,2}([0,T],H)$ solution of

$$\begin{gathered} u_{f_n}'''(t)\in\partial\varphi(u_{f_n}'(t))+f_n(t)\quad \text{a.e. } t\in[0,T],\\ u_{f_n}(0)=-u_{f_n}(T),\quad u_{f_n}'(0)=-u_{f_n}'(T),\quad u_{f_n}''(0)=-u_{f_n}''(T), \end{gathered}$$

with $f_n\in S^2_\Gamma$ and $g_n(t)\in F(t,u_{f_n}(t))$ a.e. $t\in[0,T]$. Taking into account the antiperiodicity of u_{f_n}'', u_{f_n}' and u_{f_n}, proposition 3.1 gives

$$\begin{gathered} \|u_{f_n}''\|_{\mathcal{C}([0,T],H)}\le\frac{\sqrt{T}}{2}\|u_{f_n}'''\|_{L^2([0,T],H)},\\ \|u_{f_n}'\|_{\mathcal{C}([0,T],H)}\le\frac{T\sqrt{T}}{2\pi}\|u_{f_n}'''\|_{L^2([0,T],H)},\\ \|u_{f_n}\|_{\mathcal{C}([0,T],H)}\le\frac{T^2\sqrt{T}}{2\pi^2}\|u_{f_n}'''\|_{L^2([0,T],H)}. \end{gathered}$$

for all $n\ge1$. Using the estimate (3.1), we have

$$\|u_{f_n}'''\|_{L^2([0,T],H)}\le\|f_n\|_{L^2([0,T],H)}\le\|\alpha\|_{L^2([0,T],R)}<+\infty.$$

We may conclude that

$$\begin{gathered} \sup_{n\ge1}\|u_{f_n}''\|_{\mathcal{C}([0,T],H)}<+\infty,\\ \sup_{n\ge1}\|u_{f_n}'\|_{\mathcal{C}([0,T],H)}<+\infty, \end{gathered}$$

$$\sup_{n\geq 1}\|u_{f_n}\|_{\mathcal{C}([0,T],H)} < +\infty.$$

By extracting suitable subsequences, we may assume that (u'''_{f_n}) converges weakly in $L^2([0,T],H)$ to a function $\gamma \in L^2([0,T],H)$ and (u''_{f_n}) converges pointwise to a function w, namely

$$\begin{aligned} w(t) &:= \lim_{n\to+\infty} u''_{f_n}(t) = \lim_{n\to+\infty}(u''_{f_n}(0) + \int_0^t u'''_{f_n}(s)ds) \\ &= \lim_{n\to+\infty} u''_{f_n}(0) + \int_0^t \gamma(s)ds, \quad \forall t\in[0,T]. \end{aligned}$$

Then

$$\begin{aligned} v(t) &:= \lim_{n\to+\infty} u'_{f_n}(t) = \lim_{n\to+\infty}(u'_{f_n}(0) + \int_0^t u''_{f_n}(s)ds) \\ &= \lim_{n\to+\infty} u'_{f_n}(0) + \int_0^t w(s)ds, \quad \forall t\in[0,T]. \end{aligned}$$

So we have

$$\begin{aligned} u(t) &:= \lim_{n\to+\infty} u_{f_n}(t) = \lim_{n\to+\infty}(u_{f_n}(0) + \int_0^t u'_{f_n}(s)ds) \\ &= \lim_{n\to+\infty} u_{f_n}(0) + \int_0^t v(s)ds, \quad \forall t\in[0,T]. \end{aligned}$$

We conclude that $u \in W^{3,2}([0,T],H)$ with $u' = v$, $u'' = w$ and $u''' = \gamma$ and satisfying the anti-periodic conditions $u(0) = -u(T)$, $u'(0) = -u'(T)$ and $u''(0) = -u''(T)$. Furthermore, we see that u'_{f_n} converges pointwise to u' and u'''_{f_n} converges to u''' with respect to the weak topology of $L^2([0,T],H)$. Combining these facts and applying Lemma 2.1 to the inclusion

$$u'''_{f_n}(t) - f_n(t) \in \partial\varphi(u'_{f_n}(t)) \quad \text{a.e. } t\in[0,T]$$

it yields

$$u'''(t) - f(t) \in \partial\varphi(u'(t)) \quad \text{a.e. } t\in[0,T].$$

By uniqueness [1, Theorem 2.1], we have $u = u_f$. Further using the inclusion

$$g_n(t) \in F(t, u_{f_n}(t)) \quad \text{a.e. } t\in[0,T]$$

and invoking the closure type Lemma in [10, theorem VI.4], we have

$$g(t) \in F(t, u_f(t)) \quad \text{a.e. } t\in[0,T].$$

We may then applying the Kakutani-Ky Fan fixed point theorem to the set-valued mapping Ψ to obtain some $f \in S^2_\Gamma$ such that $f \in \Psi(f)$ or

$$f(t) \in F(t, u(t)) \quad \text{a.e. } t\in[0,T].$$

This means that

$$\begin{gathered} u'''(t) \in \partial\varphi(u'(t)) + f(t) \quad \text{a.e. } t\in[0,T], \\ f(t) \in F(t,u(t)) \quad \text{a.e. } t\in[0,T], \\ u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T). \end{gathered}$$

The proof is complete. $\square$

A more general version of the preceding result is available by introducing some inf-compactness assumption [2] on the function φ.

Theorem 3.3. *Let H be a separable Hilbert space, $\varphi : H \to [0, +\infty]$ be a proper, convex, lower semicontinuous and even function satisfying: $\varphi(0) = 0$ and for each $\beta_1, \beta_2 > 0$, the set $\{x \in D(\varphi) : \|x\| \le \beta_1, \varphi(x) \le \beta_2\}$ is compact. Let $F : [0,T] \times H \to 2^H$ be a convex compact set-valued mapping, measurable on $[0,T]$ and upper semicontinuous on H satisfying: there is $\alpha(\cdot) \in L^2([0,T], \mathbb{R}_+)$ such that*

$$F(t,x) \subset \Gamma(t) := \overline{\mathbb{B}}_{\alpha(t)}(0) \quad \forall (t,x) \in [0,T] \times H$$

Then the problem

$$\begin{gathered} u'''(t) \in \partial\varphi(u'(t)) + F(t,u(t)) \quad a.e.\ t \in [0,T], \\ u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T), \end{gathered}$$

has at least an anti-periodic $W^{3,2}([0,T], H)$ solution.

Proof. Using the notation of the proof of Theorem 3.2, we have

$$u'''_{f_n}(t) - f_n(t) \in \partial\varphi(u'_{f_n}(t)) \quad \text{a.e. } t \in [0,T],$$

for every $f_n \in S^2_\Gamma$. The absolute continuity of $\varphi(u'_{f_n}(\cdot))$ and the chain rule theorem [9], yield

$$\langle u'''_{f_n}(t), u''_{f_n}(t)\rangle - \langle f_n(t), u''_{f_n}(t)\rangle = \frac{d}{dt}\varphi(u'_{f_n}(t)),$$

for every $f_n \in S^2_\Gamma$, so that

$$+\infty > \sup_{n\ge1} \int_0^T |\langle u'''_{f_n}(t), u''_{f_n}(t)\rangle - \langle f_n(t), u''_n(t)\rangle| dt = \sup_{n\ge1} \int_0^T \big|\frac{d}{dt}\varphi(u'_{f_n}(t))\big| dt.$$

Further applying the classical definition of the subdifferential to convex function φ yields

$$0 = \varphi(0) \ge \varphi(u'_{f_n}(t)) + \langle 0 - u'_{f_n}(t), u'''_{f_n}(t) - f_n(t)\rangle$$

or

$$0 \le \varphi(u'_{f_n}(t)) \le \langle u'_{f_n}(t), u'''_{f_n}(t) - f_n(t)\rangle.$$

Hence

$$\sup_{n\ge1} |\varphi(u'_{f_n})|_{L^1_{\mathbb{R}}([0,T])} < +\infty.$$

For all $t \in [0,T]$, we have

$$\varphi(u'_{f_n}(t)) = \varphi(u'_{f_n}(0)) + \int_0^t \frac{d}{dt}\varphi(u'_{f_n}(s)) ds \le \varphi(u'_{f_n}(0)) + \sup_{n\ge1} |\varphi(u'_{f_n})|_{L^1_{\mathbb{R}}([0,T])}.$$

Now we assert that $\varphi(u'_{f_n}(t)) \le \beta_2$ for every $t \in [0,T]$, here β_2 is a positive constant. Indeed for all $t \in [0,T]$, we have

$$\begin{aligned} \varphi(u'_{f_n}(0)) &\le |\varphi(u'_{f_n}(t)) - \varphi(u'_{f_n}(0))| + \varphi(u'_{f_n}(t)) \\ &\le \int_0^T |\frac{d}{dt}\varphi(u'_{f_n}(t))| dt + \varphi(u'_{f_n}(t)). \end{aligned}$$

Hence

$$\varphi(u'_{f_n}(0)) \le \sup_{n\ge1} \int_0^T |\frac{d}{dt}\varphi(u'_{f_n}(t))| dt + \frac{1}{T} \sup_{n\ge1} \int_0^T \varphi(u'_{f_n}(t)) dt < +\infty.$$

Whence we have

$$\beta_1 := \sup_{n\geq 1}\sup_{t\in[0,T]} \|u'_{f_n}(t)\| < +\infty, \quad \beta_2 := \sup_{n\geq 1}\sup_{t\in[0,T]} \varphi(u'_{f_n}(t)) < +\infty.$$

So that $(u'_{f_n}(t))$ is relatively compact with respect to the norm topology of H using the inf-compactness assumption on φ. The proof can be therefore achieved as Theorem 3.2 by invoking Lemma2.1 and a closure type lemma in [10, Theorem VI-4]. □

Here is an existence and uniqueness result related to Theorem 3.3 when the perturbation is single-valued.

Theorem 3.4. *Let H be a separable Hilbert space, $\varphi : H \to [0, +\infty]$ is a proper, convex, lower semicontinuous and even function satisfying: $\varphi(0) = 0$ and for each $\alpha, \beta > 0$, the set $\{x \in D(\varphi) : \|x\| \leq \alpha, \varphi(x) \leq \beta\}$ is compact and $f : [0, T] \times H \to H$ is a Carathéodory mapping satisfying :*
$(\mathcal{H}_1)$ $\|f(t, u) - f(t, v)\| \leq L\|u - v\|$ for all $(t, u, v) \in [0, T] \times H \times H$, for some positive constant $L > 0$.
$(\mathcal{H}_2)$ There is a $L^2([0, T]; \mathbb{R}^+)$ integrable function $\alpha : \mathbb{R} \to \mathbb{R}^+$ such that $\|f(t, u)\| \leq \alpha(t)$ for all $(t, u) \in [0, T] \times H$. If $0 < T < \frac{\pi}{\sqrt[3]{L}}$, then the inclusion

$$\begin{gathered} u'''(t) \in \partial\varphi(u'(t)) + f(t, u(t)) \quad a.e.\ t \in [0, T], \\ u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T), \end{gathered}$$

admits a unique $W^{3,2}([0, T]; H)$-anti-periodic solution.

Proof. Existence of at least one $W^{3,2}([0, T]; H)$-anti-periodic solution is ensured by Theorem 3.3. Indeed, we put $F(t, u) := \{f(t, u)\}$ for all $(t, u) \in [0, T] \times H$, As f is a Carathéodory function, then: $u \longmapsto f(t, u)$ is continuous, for almost all $t \in [0, T]$ and $t \longmapsto f(t, u)$ is Lebesgue measurable, for all $u \in H$. More, by assumption; there is a $L^2([0, T]; \mathbb{R}^+)$ integrable function $\alpha(\cdot)$ such that

$$\|f(t, u)\| \leq \alpha(t) \quad \text{for all } (t, u) \in [0, T] \times H.$$

Therefore, F satisfies hypotheses of Theorem 3.3.

To prove uniqueness, we assume that (u_1) and (u_2) are two solutions of the inclusion under consideration.

$$\begin{gathered} u_1'''(t) \in \partial\varphi(u_1'(t)) + f(t, u_1(t)) \quad \text{a.e. } t \in [0, T], \\ u_1(0) = -u_1(T), \quad u_1'(0) = -u_1'(T), \quad u_1''(0) = -u_1''(T), \end{gathered}$$

and

$$\begin{gathered} u_2'''(t) \in \partial\varphi(u_2'(t)) + f(t, u_2(t)) \quad \text{a.e. } t \in [0, T], \\ u_2(0) = -u_2(T), \quad u_2'(0) = -u_2'(T), \quad u_2''(0) = -u_2''(T). \end{gathered}$$

For simplicity, let us set

$$\begin{gathered} v_1(t) = u_1'''(t) - f(t, u_1(t)), \quad \forall t \in [0, T], \\ v_2(t) = u_2'''(t) - f(t, u_2(t)), \quad \forall t \in [0, T]. \end{gathered}$$

Then we have

$$v_1(t) - v_2(t) = u_1'''(t) - u_2'''(t) - f(t, u_1(t)) + f(t, u_2(t)), \quad \text{a.e. } t \in [0, T]. \tag{3.2}$$

Multiplying scalarly (3.2) by $(u_1' - u_2')$ and integrating on $[0, T]$ yields

$$\begin{aligned}&\int_0^T \langle v_1(t) - v_2(t), u_1'(t) - u_2'(t)\rangle dt \\ &= \int_0^T \langle u_1'''(t) - u_2'''(t), u_1'(t) - u_2'(t)\rangle dt \\ &\quad - \int_0^T \langle f(t, u_1(t)) - f(t, u_2(t)), u_1'(t) - u_2'(t)\rangle dt\end{aligned} \tag{3.3}$$

As $v_1 \in \partial\varphi(u_1')$ and $v_2 \in \partial\varphi(u_2')$ by monotonicity of $(\partial\varphi)$, (3.3) implies

$$\begin{aligned}&\int_0^T \langle u_1'''(t) - u_2'''(t), u_1'(t) - u_2'(t)\rangle dt \\ &\geq \int_0^T \langle f(t, u_1(t)) - f(t, u_2(t)), u_1'(t) - u_2'(t)\rangle dt.\end{aligned} \tag{3.4}$$

By antiperiodicity, we have

$$\begin{aligned}&\int_0^T \langle u_1'''(t) - u_2'''(t), u_1'(t) - u_2'(t)\rangle dt \\ &= \langle u_1''(T) - u_2''(T), u_1'(T) - u_2'(T)\rangle - \langle u_1''(0) - u_2''(0), u_1'(0) - u_2'(0)\rangle \\ &\quad - \int_0^T \langle u_1''(t) - u_2''(t), u_1''(t) - u_2''(t)\rangle dt \\ &= -\int_0^T \|u_1''(t) - u_2''(t)\|^2 dt.\end{aligned}$$

The inequality (3.4) gives

$$\begin{aligned}\int_0^T \|u_1''(t) - u_2''(t)\|^2 dt &\leq \int_0^T \langle f(t, u_2(t)) - f(t, u_1(t)), u_1'(t) - u_2'(t)\rangle dt \\ &\leq L \int_0^T \|u_1(t) - u_2(t)\| \, \|u_1'(t) - u_2'(t)\| dt.\end{aligned}$$

By Holder's inequality, we obtain

$$\|u_1'' - u_2''\|^2_{L^2([0,T],H)} \leq L\|u_1 - u_2\|_{L^2([0,T],H)} \|u_1' - u_2'\|_{L^2([0,T],H)}.$$

Using the estimates (A1) and (A2) in proposition 3.1, we obtain

$$\frac{\pi^2}{T^2}\|u_1' - u_2'\|^2_{L^2([0,T],H)} \leq L\frac{T}{\pi}\|u_1' - u_2'\|^2_{L^2([0,T],H)}$$

or

$$\|u_1' - u_2'\|^2_{L^2([0,T],H)} \leq L\frac{T^3}{\pi^3}\|u_1' - u_2'\|^2_{L^2([0,T],H)}.$$

It follows from the choice of T that $\|u_1' - u_2'\|^2_{L^2([0,T],H)} = 0$. By inequality (A2) in lemma 2.2, we conclude that $u_1(t) - u_2(t) = 0$ for all $t \in [0, T]$. This completes the proof. □

4. Applications

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let γ be a maximal monotone operator on $\mathbb{R}$ such that $\gamma = \partial j$, where $j : \mathbb{R} \to [0, +\infty]$ is proper, convex, lower semicontinuous and even with $j(0) = 0$. We are concerned with the third-order boundary-value problem

$$\begin{gathered} -u_{ttt}(t,x) - \Delta_x u_t(t,x) + f(t, u(t,x)) = 0 \quad \text{in } [0,T]\times\Omega, \\ \frac{\partial u_t}{\partial \nu}(t,x) \in \gamma(u_t(t,x)) \quad \text{on } [0,T]\times\partial\Omega, \\ u(0,x) = -u(T,x), \quad u_t(0,x) = -u_t(T,x) \quad u_{tt}(0,x) = -u_{tt}(T,x) \quad \text{in } \Omega, \end{gathered} \tag{4.1}$$

where $\frac{\partial}{\partial\nu}$ denotes outward normal derivative, $\Delta_x = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, and $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying:

- (i) $|f(t,u) - f(t,v)| \le L|u - v|$ for all $(t,u,v) \in [0,T]\times\mathbb{R}^2$, for some positive constant $L > 0$,
- (ii) There is an $L^2([0,T];\mathbb{R}^+)$ integrable function $\alpha : [0,T] \to \mathbb{R}^+$ such that $|f(t,u)| \le \alpha(t)$ for all $(t,u) \in [0,T]\times\mathbb{R}$.

Let $H = L^2(\Omega)$, and define $\varphi : H \to [0, +\infty]$ by

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\operatorname{grad} u|^2 dx + \int_{\partial\Omega} j(u)d\sigma, & \text{if } u \in H^1(\Omega) \text{ and } j(u) \in L^1(\partial\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

According to Brézis [8, Theorem 12], φ is proper, convex and lower semicontinuous on H, with $\partial\varphi(u) = -\Delta_x u$, and $D(\varphi) = \{u \in W^{1,2}(\Omega) : -\frac{\partial u}{\partial\nu} \in \gamma(u), \text{ a.e. on } \partial\Omega\}$. We consider $u = u(t,x) = u(t)(x)$ and we rewriter the problem (4.1) in the abstract form

$$\begin{gathered} -u'''(t) + \partial\varphi(u'(t)) + f(t,u(t)) \ni 0 \quad \text{a.e. } t \in [0,T], \\ u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T), \end{gathered}$$

or

$$\begin{gathered} u'''(t) \in \partial\varphi(u'(t)) + f(t,u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T). \end{gathered}$$

We remark that $\varphi(0) = 0$, φ is even and that the inf-compactness condition on φ holds because $W^{1,2}(\Omega)$ is compactly imbedded in $L^2(\Omega)$. Then, we can applying Theorem 3.4 to derive the existence of a solution to (4.1). If $0 < T < \frac{\pi}{\sqrt[3]{L}}$, then the solution is unique.

Acknowledgments. The authors would like to thank the anonymous referees for their careful and thorough reading of the original manuscript.

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Touma Haddad
Département de Mathématiques, Faculté des Sciences, Université de Jijel, B.P. 98, Algérie
E-mail address: touma.haddad@yahoo.com

Tahar Haddad
Laboratoire LMPA, Faculté des Sciences, Université de Jijel, B.P. 98, Algérie
E-mail address: haddadtr2000@yahoo.fr