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EXISTENCE OF THREE NON-NEGATIVE SOLUTIONS FOR A THREE-POINT BOUNDARY-VALUE PROBLEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This article concerns the existence of three non-negative solutions for two kinds of three-point boundary-value problems of nonlinear fractional differential equations, where the fractional derivative is taken in the Riemann-Liouville sense. Using Leggett-Williams fixed point theorem, we present some existence criteria and then illustrate our results with examples.

1. Introduction

With the development of fractional calculus and its applications [8, 9, 11] in mathematics, technology, biology, chemical process etc., increasing attention has been paid to the study of fractional differential equations including the existence of solutions to fractional differential equations [1, 2, 3, 7, 12, 13, 14, 15, 16, 17, 18], the stability analysis of fractional differential equations [4, 6, 10], and so on. As a fundamental issue of the theory of fractional differential equations, the existence of (positive) solutions for kinds of boundary-value problems (BVPs) of fractional differential equations has been studied recently by many scholars, and lots of excellent results have been obtained for both two-point BVPs and nonlocal BVPs by means of fixed point index theory [2], fixed point theorems [1, 15, 17], mixed monotone method [14], upper and lower solutions technique [12], and so on.

Xu et al [14, 15] investigated the fractional differential equation

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\ t\in(0,1), \tag{1.1}$$

subject to the following two kinds of three-point boundary conditions:

$$u(0)=0,\ D_{0+}^{\beta}u(1)=m_1D_{0+}^{\beta}u(\xi), \tag{1.2}$$

and

$$u(0)=0,\ u(1)=m_2u(\xi), \tag{1.3}$$

respectively, where $1<\alpha<2$, $0<\beta\le 1$, $\alpha-\beta-1\ge 0$, $0\le m_1\le 1$, $0<m_2$, $\xi<1$, and D_{0+}^{α} is the standard Riemann-Liouville derivative. They obtained some

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interesting positive properties of Green's functions for (1.1) with (1.2) and (1.1) with (1.3), respectively, and presented some existence criteria of positive solutions by using some fixed point theorems and the mixed monotone method.

It should be noted that there are fewer results on the existence of multiple (triple) solutions for nonlocal BVPs of fractional differential equations such as (1.1) with (1.2) and (1.1) with (1.3). As a result, the purpose of this paper is to establish the existence results for triple non-negative solutions of (1.1) with (1.2) and (1.1) with (1.3) by virtue of Leggett-Williams fixed point theorem and enrich this academic area.

Throughout this paper, we assume that the nonlinearity $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous in (1.1). Moreover, let $E=C[0,1]$ with the norm $\|x\|=\max_{t\in[0,1]}|x(t)|$. Then, E is a Banach space. We will consider the existence of non-negative solutions for (1.1) with (1.2) and (1.1) with (1.3) in E.

The rest of this work is organized as follows. Section 2 contains some preliminaries on the standard Riemann-Liouville derivative and the properties of Green's functions for (1.1) with (1.2) and (1.1) with (1.3). Section 3 investigates the existence of triple non-negative solutions for (1.1) with (1.2) and (1.1) with (1.3), respectively, and presents the main results of this paper. In Section 4, two illustrative examples are worked out to support our obtained results.

2. Preliminaries

In this section, we give some necessary preliminaries on the Riemann-Liouville derivative and the properties of Green's functions for (1.1) with (1.2) and (1.1) with (1.3), which will be used in the sequel.

We first recall some well known results about Riemann-Liouville derivative. For details, please refer to [9, 8, 11] and the references therein.

Definition 2.1 ([11]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\to\mathbb{R}$ is given by

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds, \tag{2.1}$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 ([11]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $y:(0,\infty)\to\mathbb{R}$ is given by

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t\frac{y(s)}{(t-s)^{\alpha-n+1}}ds, \tag{2.2}$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of α, provided that the right side is pointwise defined on $(0,\infty)$.

One can easily obtain the following properties from the definition of Riemann-Liouville derivative.

Proposition 2.3 ([11]). *Let $\alpha>0$, if we assume $u\in C(0,1)\cap L(0,1)$, then, the fractional differential equation $D_{0+}^{\alpha}u(t)=0$ has $u(t)=C_1t^{\alpha-1}+C_2t^{\alpha-2}+\dots+C_Nt^{\alpha-N}$, $C_i\in\mathbb{R}$, $i=1,2,\dots,N$ as unique solution, where N is the smallest integer greater than or equal to α.*

Proposition 2.4 ([11]). Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then,

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}, \tag{2.3}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to α.

In the following, we present some important properties of Green's functions for (1.1) with (1.2) and (1.1) with (1.3), which have been proved in [2, 7, 14, 15].

Lemma 2.5 ([15]). *$x(t) \in E$ is a solution to* (1.1) *with* (1.2), *if and only if $x(t) = T_1 x(t)$, where*

$$T_1 x(t) = \int_0^1 G_1(t,s) f(s, x(s)) ds, \tag{2.4}$$

$$G_1(t,s) = \begin{cases} \frac{Dt^{\alpha-1}(1-s)^{\alpha-\beta-1} - m_1 Dt^{\alpha-1}(\xi-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1,\ s \le \xi, \\ \frac{Dt^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 < \xi \le s \le t \le 1, \\ \frac{Dt^{\alpha-1}(1-s)^{\alpha-\beta-1} - m_1 Dt^{\alpha-1}(\xi-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le \xi < 1, \\ \frac{Dt^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1,\ \xi \le s, \end{cases} \tag{2.5}$$

and $D = (1 - m_1 \xi^{\alpha-\beta-1})^{-1}$.

Lemma 2.6 ([15]). *The Green's function $G_1(t,s)$ given in* (2.5) *satisfies*

$$\frac{\beta t^{\alpha-1} s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \le G_1(t,s) \le \frac{Dt^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, \quad \forall\ t, s \in [0,1]. \tag{2.6}$$

Lemma 2.7 ([14]). *$x(t) \in E$ is a solution to* (1.1) *with* (1.3), *if and only if $x(t) = T_2 x(t)$, where*

$$T_2 x(t) = \int_0^1 G_2(t,s) f(s, x(s)) ds, \tag{2.7}$$

and

$$G_2(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - m_2 t^{\alpha-1}(\xi-s)^{\alpha-1} - (t-s)^{\alpha-1}(1-m_2\xi^{\alpha-1})}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)}, & 0 \le s \le t \le 1,\ s \le \xi, \\ \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1-m_2\xi^{\alpha-1})}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)}, & 0 < \xi \le s \le t \le 1, \\ \frac{[t(1-s)]^{\alpha-1} - m_2 t^{\alpha-1}(\xi-s)^{\alpha-1}}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le \xi < 1, \\ \frac{[t(1-s)]^{\alpha-1}}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le 1,\ \xi \le s. \end{cases} \tag{2.8}$$

Lemma 2.8 ([14]). *The Green's function $G_2(t,s)$ given in* (2.8) *satisfies*

$$\frac{M_0 t^{\alpha-1} s(1-s)^{\alpha-1}}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} \le G_2(t,s) \le \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)}, \quad \forall\ t,\ s \in [0,1], \tag{2.9}$$

where $0 < M_0 = \min\{1 - m_2\xi^{\alpha-1}, m_2\xi^{\alpha-2}(1-\xi)(\alpha-1), m_2\xi^{\alpha-1}\} < 1$.

Finally, we recall the Leggett-Williams fixed point theorem. Let $E = (E, \|\cdot\|)$ be a Banach space and $P \subset E$ be a cone on E. A continuous mapping $\omega : P \to [0, +\infty)$ is said to be a concave non-negative continuous functional on P, if ω satisfies $\omega(\lambda x + (1-\lambda)y) \ge \lambda\omega(x) + (1-\lambda)\omega(y)$ for all $x, y \in P$ and $\lambda \in [0,1]$.

Let $a, b, d > 0$ be constants. Define $P_d = \{x \in P : \|x\| < d\}$, $\overline{P_d} = \{x \in P : \|x\| \le d\}$ and $P(\omega, a, b) = \{x \in P : \omega(x) \ge a, \|x\| \le b\}$.

Lemma 2.9 ([5])**.** *Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ be a cone of E and $c > 0$ be a constant. Suppose there exists a concave non-negative continuous functional ω on P with $\omega(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Let $T : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator. Assume that there are numbers a, b and d with $0 < d < a < b \leq c$, such that*

- (i) *$\{x \in P(\omega, a, b) : \omega(x) > a\} \neq \emptyset$ and $\omega(Tx) > a$ for all $x \in P(\omega, a, b)$;*
- (ii) *$\|Tx\| < d$ for all $x \in \overline{P_d}$;*
- (iii) *$\omega(Tx) > a$ for all $x \in P(\omega, a, c)$ with $\|Tx\| > b$.*

Then, T has at least three fixed points x_1, x_2 and x_3 in $\overline{P_c}$. Furthermore, $x_1 \in P_a$; $x_2 \in \{x \in P(\omega, a, c) : \omega(x) > a\}$; $x_3 \in \overline{P_c} \setminus (P(\omega, b, c) \cup \overline{P_a})$.

3. Main Results

In this section, we investigate the existence of triple non-negative solutions for (1.1) with (1.2) and (1.1) with (1.3), and present some existence criteria. Denote

$$\Phi_1 = \frac{\Gamma(\alpha)(\alpha - \beta)}{D}, \quad \Phi_2 = \alpha(1 - m_2\xi^{\alpha-1})\Gamma(\alpha),$$
$$\Psi_1 = \frac{\Gamma(\alpha)(\alpha-\beta)(\alpha-\beta+1)}{\beta\xi^{\alpha-1}(1-\xi)^{\alpha-\beta}(\alpha\xi - \beta\xi + 1)}, \quad \Psi_2 = \frac{\alpha(\alpha+1)(1-m_2\xi^{\alpha-1})\Gamma(\alpha)}{M_0\xi^{\alpha-1}(1-\xi)^{\alpha}(\alpha\xi+1)},$$
$$\Pi_1 = \frac{\Gamma(\alpha)(\alpha-\beta)(\alpha-\beta+1)}{\beta\xi^{\alpha-1}}, \quad \Pi_2 = \frac{\alpha(\alpha+1)(1-m_2\xi^{\alpha-1})\Gamma(\alpha)}{M_0\xi^{\alpha-1}},$$
$$\overline{f_0} = \limsup_{x\to 0^+} \sup_{t\in[0,1]} \frac{f(t,x)}{x},$$

where D and M_0 are given in Lemmas 2.5 and 2.8, respectively.

To use Lemma 2.9, we define a cone $P = \{x \in E : x(t) \geq 0, \forall t \in [0,1]\}$ and a functional $\omega : P \to [0, +\infty)$ by

$$\omega(x) = \min_{t\in[\xi,1]} x(t), \tag{3.1}$$

then, one can easily see that ω is a concave non-negative continuous functional on P, and satisfies $\omega(x) \leq \|x\|$ for all $x \in P$.

We first consider (1.1) with (1.2) and obtain the following result.

Theorem 3.1. *Consider (1.1) with (1.2). Assume that there exist two constants a and c with $0 < a < \min\{c, \frac{\Phi_1}{\Psi_1}c\}$, such that*

- (H1) *$f(t,x) \leq \Phi_1 c$, for all $(t,x) \in [0,1] \times [0,c]$;*
- (H2) *there exists a constant η with $0 \leq \eta < \Phi_1$, such that $\overline{f_0} = \eta$;*
- (H3) *$f(t,x) > \Psi_1 a$, for all $(t,x) \in [\xi,1] \times [a,c]$.*

Then (1.1) with (1.2) has at least three non-negative solutions.

Proof. Let us divide the proof into 4 steps.

Step1. From (H1) and Lemma 2.6, for all $x \in \overline{P_c}$, we have

$$\begin{aligned}\|T_1x\| &= \max_{t\in[0,1]} \int_0^1 G_1(t,s) f(s, x(s))ds \\ &\leq \Phi_1 c \max_{t\in[0,1]} \int_0^1 G_1(t,s)ds\end{aligned}$$

$$
\begin{aligned}
&\le \Phi_1 c \max_{t\in[0,1]} \int_0^1 \frac{Dt^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} ds \\
&\le c(\alpha-\beta)\int_0^1 (1-s)^{\alpha-\beta-1} ds = c.
\end{aligned}
$$

Thus, $T_1 : \overline{P_c} \to \overline{P_c}$.

Now, let us show that $T_1 : \overline{P_c} \to \overline{P_c}$ is completely continuous. Let $x_n, x_0 \in \overline{P_c}$ with $\|x_n - x_0\| \to 0$ as $n \to +\infty$. Then,

$$
\begin{aligned}
\|T_1x_n - T_1x_0\| &= \max_{t\in[0,1]} \Big|\int_0^1 G_1(t,s)f(s,x_n(s))ds - \int_0^1 G_1(t,s)f(s,x_0(s))ds\Big| \\
&\le \max_{t\in[0,1]} \int_0^1 G_1(t,s)|f(s,x_n(s)) - f(s,x_0(s))|ds \\
&\le \frac{D}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1}|f(s,x_n(s)) - f(s,x_0(s))|ds \to 0,
\end{aligned}
$$

as $n \to +\infty$. Hence, $T_1 : \overline{P_c} \to \overline{P_c}$ is continuous.

In addition, for any $t_1,\ t_2 \in [0,1]$ and $x \in \overline{P_c}$, we have

$$
|(T_1x)(t_1) - (T_1x)(t_2)| \le \Phi_1 c \int_0^1 |G_1(t_1,s) - G_1(t_2,s)|ds. \tag{3.2}
$$

Since $G_1(t,s)$ is uniformly continuous on $(t,s) \in [0,1]\times[0,1]$, it is easy to see that $|(T_1x)(t_1) - (T_1x)(t_2)| \to 0$ as $|t_1 - t_2| \to 0$. Moreover, $T_1(\overline{P_c})$ is bounded. Thus, the Arzela-Ascoli theorem guarantees that $T_1 : \overline{P_c} \to \overline{P_c}$ is compact. Therefore, $T_1 : \overline{P_c} \to \overline{P_c}$ is completely continuous.

Step2. Choose a constant $b \in (a,c]$. Let $x_0(t) = \frac{a+b}{2},\ \forall\ t \in [0,1]$. Then, $\omega(x_0) = \frac{a+b}{2} > a$ and $\|x_0\| = \frac{a+b}{2} < b$. Thus, $x_0 \in \{x \in P(\omega,a,b) : \omega(x) > a\} \neq \emptyset$.

Now, let us prove that $\omega(T_1x) > a$ holds for all $x \in P(\omega,a,b)$. In fact, $x \in P(\omega,a,b)$ implies that $a \le x(t) \le b$, for all $t \in [\xi,1]$. One can obtain from (H3) and Lemma 2.6 that

$$
\begin{aligned}
\omega(T_1x) &= \min_{t\in[\xi,1]}(T_1x)(t) = \min_{t\in[\xi,1]} \int_0^1 G_1(t,s)f(s,x(s))ds \\
&> \Psi_1 a \min_{t\in[\xi,1]} \int_\xi^1 G_1(t,s)ds \\
&\ge \Psi_1 a \min_{t\in[\xi,1]} \int_\xi^1 \frac{\beta t^{\alpha-1}s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} ds \\
&\ge \Psi_1 a \frac{\beta\xi^{\alpha-1}}{\Gamma(\alpha)} \int_\xi^1 s(1-s)^{\alpha-\beta-1} ds = a.
\end{aligned}
$$

Hence, the condition (i) of Lemma 2.9 holds.

Step3. It is easy to see from (H2) that for all $t \in [0,1]$, $\forall\ 0 < \varepsilon \le \frac{\alpha-\beta}{D}\Gamma(\alpha) - \eta$, there exists $\delta > 0$, such that for $0 \le x < \delta$, we have

$$
f(t,x) < (\eta+\varepsilon)x. \tag{3.3}
$$

Let $0 < d < \min\{\delta, a\}$. Now, we prove that $\|T_1x\| < d$ for all $x \in \overline{P_d} = \{x \in P : \|x\| \le d\}$. As a matter of fact, for all $x \in \overline{P_d}$, one can see that

$$\begin{aligned}
\|T_1x\| &= \max_{t\in[0,1]} \int_0^1 G_1(t,s)f(s,x(s))ds \\
&< (\eta+\varepsilon)\|x\| \max_{t\in[0,1]} \int_0^1 G_1(t,s)ds \\
&\le (\eta+\varepsilon)\|x\| \max_{t\in[0,1]} \int_0^1 \frac{Dt^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}ds \\
&\le (\eta+\varepsilon)\|x\| \frac{D}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1}ds \le \|x\| \le d
\end{aligned}$$

Thus, $\|T_1x\| < d$, for all $x \in \overline{P_d}$.

Step4. Let us prove that $\omega(T_1x) > a$ holds for all $x \in P(\omega, a, c)$ with $\|T_1x\| > b$. For $x \in P(\omega, a, c)$ with $\|T_1x\| > b$, we have $a \le x(t) \le c$, for all $t \in [\xi, 1]$. From (H3) and Lemma 2.6, one can see that

$$\begin{aligned}
\omega(T_1x) &= \min_{t\in[\xi,1]} (T_1x)(t) \\
&= \min_{t\in[\xi,1]} \int_0^1 G_1(t,s)f(s,x(s))ds \\
&\ge \min_{t\in[\xi,1]} \int_0^1 \frac{\beta t^{\alpha-1}s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&\ge \frac{\beta\xi^{\alpha-1}}{\Gamma(\alpha)} \int_\xi^1 s(1-s)^{\alpha-\beta-1} f(s,x(s))ds \\
&> \frac{\beta\xi^{\alpha-1}}{\Gamma(\alpha)} \Psi_1 a \int_\xi^1 s(1-s)^{\alpha-\beta-1}ds = a.
\end{aligned}$$

Therefore, the condition (iii) of Lemma 2.9 is satisfied.

To summing up, all conditions of Lemma 2.9 hold; therefore, BVP (1.1) with (1.2) has at least three non-negative solutions. □

Next, we study (1.1) with (1.3) and establish a sufficient condition for the existence of triple non-negative solutions of (1.1) with (1.3).

Theorem 3.2. *Consider* (1.1) *with* (1.3). *Assume that there exist two constants* a *and* c *with* $0 < a < \min\{c, \frac{\Phi_2}{\Psi_2}c\}$, *such that*

- (H4) $f(t,x) \le \Phi_2 c$ *for all* $(t,x) \in [0,1] \times [0,c]$;
- (H5) *there exists a constant* μ *with* $0 \le \mu < \Phi_2$, *such that* $\overline{f_0} = \mu$;
- (H6) $f(t,x) > \Psi_2 a$ *for all* $(t,x) \in [\xi,1] \times [a,c]$.

Then (1.1) *with* (1.3) *has at least three non-negative solutions.*

Proof. By (H4) and Lemma 2.8, for all $x \in \overline{P_c}$, we have

$$\begin{aligned}
\|T_2x\| &= \max_{t\in[0,1]} \int_0^1 G_2(t,s)f(s,x(s))ds \\
&\le \Phi_2 c \max_{t\in[0,1]} \int_0^1 G_2(t,s)ds
\end{aligned}$$

$$\begin{aligned}
&\le \Phi_2 c \max_{t\in[0,1]} \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} ds \\
&\le \Phi_2 c \frac{1}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds = c.
\end{aligned}$$

Thus, $T_2 : \overline{P_c} \to \overline{P_c}$. Similar to the proof of Theorem 3.1, it is easy to see that $T_2 : \overline{P_c} \to \overline{P_c}$ is completely continuous.

Choose a constant $b \in (a, c]$. Denote by $x_0(t) = \frac{a+b}{2}$ for all$t \in [0, 1]$. Then, $\omega(x_0) = \frac{a+b}{2} > a$ and $\|x_0\| = \frac{a+b}{2} < b$. Thus, $x_0 \in \{x \in P(\omega, a, b) : \omega(x) > a\} \neq \emptyset$.

Next, let us prove that $\omega(T_2x) > a$ holds for all $x \in P(\omega, a, b)$. In fact, $x \in P(\omega, a, b)$ implies that $a \le x(t) \le b$ for all $t \in [\xi, 1]$. One can obtain from (H6) and Lemma 2.8 that

$$\begin{aligned}
\omega(T_2x) &= \min_{t\in[\xi,1]} (T_2x)(t) = \min_{t\in[\xi,1]} \int_0^1 G_2(t,s) f(s, x(s)) ds \\
&> \Psi_2 a \min_{t\in[\xi,1]} \int_\xi^1 G_2(t,s) ds \\
&\ge \Psi_2 a \min_{t\in[\xi,1]} \int_\xi^1 \frac{M_0}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1} ds \\
&\ge \Psi_2 a \frac{M_0 \xi^{\alpha-1}}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} \int_\xi^1 s(1-s)^{\alpha-1} ds = a.
\end{aligned}$$

Hence, the condition (i) of Lemma 2.9 holds.

Next, we prove that the condition (ii) of Lemma 2.9 holds. (H5) implies that for all $t \in [0, 1]$, and all $0 < \varepsilon \le \Phi_2 - \mu$, there exists $\delta > 0$, such that for $0 \le x < \delta$, we have

$$f(t, x) < (\mu + \varepsilon)x. \tag{3.4}$$

Let $0 < d < \min\{\delta, a\}$. Now, we prove that $\|T_2x\| < d$ for all $x \in \overline{P_d} = \{x \in P : \|x\| \le d\}$.

For all $x \in \overline{P_d}$, one can see that

$$\begin{aligned}
\|T_2x\| &= \max_{t\in[0,1]} \int_0^1 G_2(t,s) f(s, x(s)) ds \\
&< (\mu+\varepsilon)\|x\| \max_{t\in[0,1]} \int_0^1 G_2(t,s) ds \\
&\le (\mu+\varepsilon)\|x\| \max_{t\in[0,1]} \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} ds \\
&\le (\mu+\varepsilon)\|x\| \frac{1}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\
&\le \|x\| \le d.
\end{aligned}$$

Thus, $\|T_2x\| < d$ for all $x \in \overline{P_d}$.

Finally, let us prove that $\omega(T_2x) > a$ holds for all $x \in P(\omega, a, c)$ with $\|T_2x\| > b$. For $x \in P(\omega, a, c)$ with $\|T_2x\| > b$, we have $a \le x(t) \le c$, $\forall\ t \in [\xi, 1]$. From (H6) and Lemma 2.8, it is easy to see

$$\omega(T_2x) = \min_{t\in[\xi,1]} (T_2x)(t)$$

$$
\begin{aligned}
&= \min_{t\in[\xi,1]} \int_0^1 G_2(t,s) f(s,x(s))ds \\
&\geq \min_{t\in[\xi,1]} \int_0^1 \frac{M_0}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1} f(s,x(s))ds \\
&\geq \frac{M_0}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} \xi^{\alpha-1} \int_\xi^1 s(1-s)^{\alpha-1} f(s,x(s))ds \\
&> \frac{M_0}{(1-m_2\xi^{\alpha-1})\Gamma(\alpha)} \xi^{\alpha-1} \Psi_2 a \int_\xi^1 s(1-s)^{\alpha-1} ds = a.
\end{aligned}
$$

Therefore, condition (iii) of Lemma 2.9 is satisfied.

Hence, all conditions of Lemma 2.9 hold; therefore, (1.1) with (1.3) has at least three non-negative solutions. □

Remark 3.3. It is noted that in the proof of Theorems 3.1 and 3.2, the condition $\|T_i x\| > b$, $i = 1, 2$ is not applied in Step 4. This is because (H3) or (H6) is sufficient for the proof, which makes (H3) or (H6) strong.

In the following, to apply the condition $\|T_i x\| > b$, $i = 1, 2$ in the proof and relax (H3) or (H6), we study (1.1) with (1.2) and (1.1) with (1.3) by constructing the following two cones:

$$P_1 = \{x \in E : x(t) \geq 0,\ \forall t \in [0,1];\ x(t) \geq \gamma_1\|x\|,\ \forall t \in [\xi,1]\}, \tag{3.5}$$

$$P_2 = \{x \in E : x(t) \geq 0,\ \forall t \in [0,1];\ x(t) \geq \gamma_2\|x\|,\ \forall t \in [\xi,1]\}, \tag{3.6}$$

where $0 < \gamma_i < \min\{1, \frac{\Phi_i}{\Pi_i}, \frac{\Psi_i}{\Pi_i}\}$, $i = 1, 2$. In this case, $\omega : P_i \to [0, +\infty)$, $i = 1, 2$.

We first consider (1.1) with (1.2) by using the cone P_1, and obtain the following result.

Theorem 3.4. *Consider* (1.1) *with* (1.2). *Assume that there exist constants a, b, c and d with $0 < \frac{\gamma_1\Pi_1}{\Phi_1}c < d < a$ and $0 < \frac{\gamma_1\Pi_1}{\Psi_1}c \leq a < \gamma_1 b < b \leq c$, such that*

- (H1') $\gamma_1\Pi_1 c \leq f(t,x) \leq \Phi_1 c$ *for all* $(t,x) \in [0,1]\times[0,c]$;
- (H2') $f(t,x) < \Phi_1 d$ *for all* $(t,x) \in [0,1]\times[0,d]$;
- (H3') $f(t,x) > \Psi_1 a$ *for all* $(t,x) \in [\xi,1]\times[a,b]$.

Then, (1.1) *with* (1.2) *has at least three non-negative solutions.*

Proof. Let us divide the proof into 4 steps.

Step 1. By (H1') and Lemma 2.6, for any $x \in \overline{P_c} = \{x \in P_1 : \|x\| \leq c\}$ we have

$$
\begin{aligned}
\|T_1 x\| &= \max_{t\in[0,1]} \int_0^1 G_1(t,s) f(s,x(s))ds \\
&\leq \Phi_1 c \max_{t\in[0,1]} \int_0^1 \frac{Dt^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} ds \\
&\leq \frac{D}{(\alpha-\beta)\Gamma(\alpha)} \Phi_1 c = c,
\end{aligned}
$$

and

$$
\begin{aligned}
\min_{t\in[\xi,1]} (T_1 x)(t) &= \min_{t\in[\xi,1]} \int_0^1 G_1(t,s) f(s,x(s))ds \\
&\geq \gamma_1\Pi_1 c \min_{t\in[\xi,1]} \int_0^1 G_1(t,s)ds
\end{aligned}
$$

$$
\begin{aligned}
&\geq \gamma_1 \Pi_1 c \min_{t\in[\xi,1]} \int_0^1 \frac{\beta t^{\alpha-1} s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} ds \\
&\geq \gamma_1 \Pi_1 c \frac{1}{\Pi_1} = \gamma_1 c \geq \gamma_1 \|T_1 x\|.
\end{aligned}
$$

Thus, $T_1 : \overline{P_c} \to \overline{P_c}$.

Next, let us show that $T_1 : \overline{P_c} \to \overline{P_c}$ is completely continuous. Let $x_n,\ x_0 \in \overline{P_c}$ with $\|x_n - x_0\| \to 0$ as $n \to +\infty$. Then

$$
\begin{aligned}
\|T_1 x_n - T_1 x_0\| &= \max_{t\in[0,1]} \Big| \int_0^1 G_1(t,s) f(s, x_n(s)) ds - \int_0^1 G_1(t,s) f(s, x_0(s)) ds \Big| \\
&\leq \max_{t\in[0,1]} \int_0^1 G_1(t,s) |f(s, x_n(s)) - f(s, x_0(s))| ds \\
&\leq \frac{D}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} |f(s, x_n(s)) - f(s, x_0(s))| ds \to 0,
\end{aligned}
$$

as $n \to +\infty$. Hence, $T_1 : \overline{P_c} \to \overline{P_c}$ is continuous.

In addition, for any $t_1, t_2 \in [0,1]$ and $x \in \overline{P_c}$, we have

$$
|(T_1 x)(t_1) - (T_1 x)(t_2)| \leq \Phi_1 c \int_0^1 |G_1(t_1, s) - G_1(t_2, s)| ds. \tag{3.7}
$$

Since $G_1(t,s)$ is uniformly continuous on $(t,s) \in [0,1] \times [0,1]$, it is easy to see that $|(T_1 x)(t_1) - (T_1 x)(t_2)| \to 0$ as $|t_1 - t_2| \to 0$. Moreover, $T_1(\overline{P_c})$ is bounded. Thus, the Arzela-Ascoli theorem guarantees that $T_1 : \overline{P_c} \to \overline{P_c}$ is compact. Therefore, $T_1 : \overline{P_c} \to \overline{P_c}$ is completely continuous.

Step 2. Let $x_0(t) = \frac{a+b}{2}$ for all $t \in [0,1]$. Then $\omega(x_0) = \frac{a+b}{2} > a$ and $\|x_0\| = \frac{a+b}{2} < b$. Thus, $x_0 \in \{x \in P(\omega, a, b) : \omega(x) > a\} \neq \emptyset$.

Now, let us prove that $\omega(T_1 x) > a$ holds for all $x \in P(\omega, a, b)$. In fact, $x \in P(\omega, a, b)$ implies that $a \leq x(t) \leq b$ for all $t \in [\xi, 1]$. One can obtain from (H3') and Lemma 2.6 that

$$
\begin{aligned}
\omega(T_1 x) = \min_{t\in[\xi,1]} (T_1 x)(t) &= \min_{t\in[\xi,1]} \int_0^1 G_1(t,s) f(s, x(s)) ds \\
&> \Psi_1 a \min_{t\in[\xi,1]} \int_\xi^1 G_1(t,s) ds \\
&\geq \Psi_1 a \min_{t\in[\xi,1]} \int_\xi^1 \frac{\beta t^{\alpha-1} s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} ds \\
&\geq \Psi_1 a \frac{\beta \xi^{\alpha-1}}{\Gamma(\alpha)} \int_\xi^1 s(1-s)^{\alpha-\beta-1} ds = a.
\end{aligned}
$$

Hence, condition (i) of Lemma 2.9 holds.

Step 3. It is easy to see from (H2') that for all $x \in \overline{P_d}$, we have

$$
\begin{aligned}
\|T_1 x\| &= \max_{t\in[0,1]} \int_0^1 G_1(t,s) f(s, x(s)) ds \\
&< \Phi_1 d \max_{t\in[0,1]} \int_0^1 G_1(t,s) ds \\
&\leq \Phi_1 d \max_{t\in[0,1]} \int_0^1 \frac{D t^{\alpha-1} (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} ds
\end{aligned}
$$

$$\leq \Phi_1 d \frac{D}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} ds = d.$$

Thus, $\|T_1 x\| < d$, for all $x \in \overline{P_d}$.

Step 4. Let us prove that $\omega(T_1 x) > a$ holds for all $x \in P(\omega, a, c)$ with $\|T_1 x\| > b$. For $x \in P(\omega, a, c)$ with $\|T_1 x\| > b$, we have $a \leq x(t) \leq c$ for all $t \in [\xi, 1]$. Then, one can see that

$$\omega(T_1 x) = \min_{t \in [\xi, 1]} (T_1 x)(t) \geq \gamma_1 \|T_1 x\| > \gamma_1 b \geq a.$$

Therefore, condition (iii) of Lemma 2.9 is satisfied. By Lemma 2.9, (1.1) with (1.2) has at least three non-negative solutions. □

Next, we study (1.1) with (1.3) in the cone P_2 and establish the following result.

Theorem 3.5. *Consider* (1.1) *with* (1.3). *Assume that there exist constants a, b, c and d with $0 < \frac{\gamma_2 \Pi_2}{\Phi_2} c < d < a$ and $0 < \frac{\gamma_2 \Pi_2}{\Psi_2} c \leq a < \gamma_2 b < b \leq c$, such that*

- (H4') *$\gamma_2 \Pi_2 c \leq f(t, x) \leq \Phi_2 c$ for all $(t, x) \in [0, 1] \times [0, c]$;*
- (H5') *$f(t, x) < \Phi_2 d$ for all $(t, x) \in [0, 1] \times [0, d]$;*
- (H6') *$f(t, x) > \Psi_2 a$ for all $(t, x) \in [\xi, 1] \times [a, b]$.*

Then (1.1) *with* (1.3) *has at least three non-negative solutions.*

The proof of the above theorem is similar to that of Theorem 3.4; thus we omit it.

Remark 3.6. From the proof of Theorems 3.1-3.5, one can see that at least two of the three non-negative solutions are positive.

Remark 3.7. Comparing Theorem 3.1 and Theorem 3.4, one can see that (H1) and (H2) are weaker than (H1') and (H2'), while (H3) is stronger than (H3'). Similarly, for Theorem 3.2 and Theorem 3.5: (H4) and (H5) are weaker than (H4') and (H5'), while (H6) is stronger than (H6').

4. Examples

In this section, we give two illustrative examples to support our new results.

Example 4.1. Consider the fractional order three-point BVP

$$\begin{gathered} D_{0+}^{3/2} u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \\ u(0) = 0, \quad D_{0+}^{1/2} u(1) = \frac{1}{2} D_{0+}^{1/2} u(\frac{1}{2}), \end{gathered} \tag{4.1}$$

where

$$f(t, x) = \begin{cases} \frac{1}{5}(1+t)x, & 0 \leq t \leq 1,\ 0 \leq x \leq 0.5, \\ \frac{99}{5}(1+t)x - \frac{49}{5}(1+t), & 0 \leq t \leq 1,\ 0.5 < x < 1, \\ 10(1+t), & 0 \leq t \leq 1,\ x \geq 1. \end{cases} \tag{4.2}$$

A simple calculation shows that $\Phi_1 \approx 0.4431$, $\Psi_1 \approx 6.6843$. Set $a = 1$ and $c = 50$, then one can see that

$$f(t, x) \geq 15 > \Psi_1 a, \quad \forall (t, x) \in [\frac{1}{2}, 1] \times [1, 50],$$

and

$$f(t, x) \leq 20 \leq \Phi_1 c, \quad \forall (t, x) \in [0, 1] \times [0, 50].$$

Thus, (H1) and (H3) hold. Since $\overline{f_0} = \overline{\lim}_{x\to 0^+} \sup_{t\in[0,1]} \frac{f(t,x)}{x} = 0.4 < \Phi_1$, we conclude that (H2) holds. By Theorem 3.1, (4.1) has at least three non-negative solutions.

Example 4.2. Consider the fractional order three-point BVP

$$\begin{gathered} D_{0+}^{3/2} u(t) + f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \frac{1}{2} u(\frac{1}{2}), \end{gathered} \tag{4.3}$$

where

$$f(t,x) = \begin{cases} \frac{1}{4}(1+t)x, & 0 \le t \le 1,\ 0 \le x \le 0.05, \\ \frac{239}{4}(1+t)x - \frac{119}{40}(1+t), & 0 \le t \le 1,\ 0.05 < x < 0.1, \\ 3(1+t), & 0 \le t \le 1,\ x \ge 0.1. \end{cases} \tag{4.4}$$

One can calculate that $\Phi_2 \approx 0.8593$, $\Psi_2 \approx 27.7778$. Set $a = 0.1$ and $c = 10$, then it is easy to see that

$$f(t,x) \ge \frac{9}{2} > \Psi_2 a, \quad \forall (t,x) \in [\frac{1}{2}, 1] \times [0.1, 10],$$

and

$$f(t,x) \le 6 \le \Phi_2 c, \quad \forall (t,x) \in [0,1] \times [0,10].$$

Thus, (H4) and (H6) are satisfied. A straightforward computation implies that $\overline{f_0} = \limsup_{x\to 0^+} \sup_{t\in[0,1]} \frac{f(t,x)}{x} = 0.5 < \Phi_2$, and thus (H5) holds. By Theorem 3.2, (4.3) has at least three non-negative solutions.

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