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GENERALIZED HEAVISIDE FUNCTIONS IN THE COLOMBEAU THEORY CONTEXT

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Abstract. We defined generalized Heaviside functions for a variable x in $\mathbb{R}^n$, and for variables (x,t) in $\mathbb{R}^n \times \mathbb{R}^m$. Then study properties such as: composition, invertibility, and association relation (the weak equality). This work is developed in the Colombeau generalized functions context.

Introduction

The study of Heaviside generalized functions considered in the work is developed in the Colombeau's theory context [1, 3, 4]. These generalized functions are introduced using basically products of the classical Heaviside's step functions. In particular, we present some basic properties of generalized functions of the form $\nu \circ (a_1 H_1 + b_1, \dots, a_\ell H_\ell + b_\ell)$, where ν is a real generalized function on $\mathbb{R}^\ell$ satisfying some conditions to be introduced in §4 and $H_1, \dots, H_\ell$ are real Heaviside generalized functions, in variables x in $\mathbb{R}^n$ or in variables (x,t) in $\mathbb{R}^n \times \mathbb{R}^m$, obtained by regularization way. One of the motivations for introducing this subject is its use in the study of shock wave solutions of partial differential equations that modeling some physics phenomena, for example (see [2, 7, 9, 10] and Remark 4.10).

In this work, unless otherwise stated, $E, F_1, \dots, F_\ell$ and G denote $\mathbb{K}$-Banach spaces (where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$), F denotes a $\mathbb{K}$-Banach algebra (as a $\mathbb{K}$-Banach space) and Ω (resp. Ω') denotes an open subset of E (resp. F). We will briefly describe the content of this paper. Generally speaking we can affirm that in the first three sections we collect the results to be used in the last one. In §1 we fix some basic notation about the generalized functions theory that will be used in the development of the work. In §2 and §3 we introduce the notion of Heaviside generalized functions in $\mathbb{R}^n$ and in $\mathbb{R}^n \times \mathbb{R}^m$, respectively. In §4 we present some basic properties about these generalized functions involving composition and invertibility. The main results of this work are proposition 2.7 and theorems 3.9, 3.11, 4.6, 4.8, 4.9.

Problems were studied in [2, 7] (resp. [9, 10]) involving real Heaviside generalized functions in the variables x in $\mathbb{R}$ (resp. in the variables x in $\mathbb{R}^n$, in the variables x in $\mathbb{R}^n$ and (x,t) in $\mathbb{R}^n \times \mathbb{R}$). In these works we use only the necessary results

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involving real Heaviside generalized functions. It is not made a deepened study on these generalized functions such as here is presented.

This work constitutes a considerable advancement of the results contained in [7]. In the present paper an additional complication arises from the expression $\nu \circ (a_1 H_1 + b_1, \dots, a_\ell H_\ell + b_\ell)$, which requires the hard study of composition and inverse multiplicative of generalized functions in the sense of Colombeau's theory. The basic references for Colombeau's theory are [1, 3, 4]. The general notation not mentioned in this work are those like [1].

1. The algebra $\mathcal{G}_s(\Omega; F)$ and association relation

We denote by (the same symbol) $|\cdot|$ the norms in the considered spaces. The symbol $\mathcal{L}(F_1, \dots, F_\ell; G)$ denotes the space of continuous ℓ-linear mappings from product space ($\mathbb{K}$-Banach space) $F_1 \times \cdots \times F_\ell$ into G endowed with the norm

$$|\cdot|_\ell \colon A \in \mathcal{L}(F_1, \dots, F_\ell; G) \mapsto \sup_{|y_i|=1,\ 1 \le i \le \ell} |A(y_1, \dots, y_\ell)| \in \mathbb{R}_+ \,.$$

If $F_1 = \cdots = F_\ell = F$ this space is denoted by $\mathcal{L}({}^\ell F; G)$ and $\mathcal{L}({}^0 F; G) =: G$.

Let $\mathcal{E}_s[\Omega; F] := \{u \in F^{]0,1] \times \Omega} : u(\varepsilon, \cdot) \in C^\infty(\Omega; F) \text{ for all } \varepsilon \in]0,1]\}$.

If $p \in \mathbb{N}$ we consider the linear map $u \in \mathcal{E}_s[\Omega; F] \mapsto u^{(p)} \in \mathcal{E}_s[\Omega; \mathcal{L}({}^p E; F)]$ where $u^{(p)}(\varepsilon, x) := [u(\varepsilon, \cdot)]^{(p)}(x)$ and $x \in \Omega$. The notation $K \Subset \Omega$ means that K is a compact subset of Ω and $|u^{(p)}(\varepsilon, \cdot)|_{p,K} := \sup_{x \in K} |u^{(p)}(\varepsilon, x)|_p$.

Let $\mathcal{E}_{s,M}[\Omega; F]$ denote the algebra of all $u \in \mathcal{E}_s[\Omega; F]$ such that for each $K \Subset \Omega$ and each $p \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that $|u^{(p)}(\varepsilon, \cdot)|_{p,K} = O(\varepsilon^{-N})$ as $\varepsilon \downarrow 0$. This algebra is called the *algebra of moderate functions on Ω with values in F*.

By $\mathcal{N}_s[\Omega; F]$ we denote the ideal of all $u \in \mathcal{E}_s[\Omega; F]$ such that for each $K \Subset \Omega$ and each $(p, q) \in \mathbb{N} \times \mathbb{N}$ we have $|u^{(p)}(\varepsilon, \cdot)|_{p,K} = O(\varepsilon^q)$ as $\varepsilon \downarrow 0$. This ideal is called the *ideal of null functions on Ω with values in F*.

The *Colombeau algebra of generalized mappings on Ω with values in F* is defined as the quotient algebra ([1, 3, 4, 6, 7, 8])

$$\mathcal{G}_s(\Omega; F) := \frac{\mathcal{E}_{s,M}[\Omega; F]}{\mathcal{N}_s[\Omega; F]} \,.$$

If $F = \mathbb{K}$ we write $\mathcal{G}_s(\Omega)$ instead of $\mathcal{G}_s(\Omega; \mathbb{K})$ and a similar notation is used for sets that generate (as well as for subsets of) $\mathcal{G}_s(\Omega; \mathbb{K})$.

We indicate by $\mathcal{G}_{s,\ell b}(\Omega; F)$ the set of maps $f \in \mathcal{G}_s(\Omega; F)$ so that f has a representative $\widehat{f}$ such that for each $K \Subset \Omega$ there are $C > 0$ and $\eta \in]0,1]$ such that $\sup_{x \in K} |\widehat{f}(\varepsilon, x)| \le C$, $(0 < \varepsilon < \eta)$.

If $f_i \in \mathcal{G}_s(\Omega; F_i)$, $1 \le i \le \ell$, we denote by $(f_1, \dots, f_\ell)$ the class of

$$(\widehat{f}_1, \dots, \widehat{f}_\ell) \colon (\varepsilon, x) \in]0,1] \times \Omega \mapsto \Big(\widehat{f}_1(\varepsilon, x), \dots, \widehat{f}_\ell(\varepsilon, x)\Big) \in F_1 \times \cdots \times F_\ell$$

in $\mathcal{G}_s(\Omega; F_1 \times \cdots \times F_\ell)$ where $\widehat{f}_i$ is an arbitrary representative of f_i. Here $(f_1, \dots, f_\ell)$ is the generalized mapping on Ω with values in the $\mathbb{K}$-Banach space $F_1 \times \cdots \times F_\ell$. The generalized mappings $f_1, \dots, f_\ell$ are called the *components* of $(f_1, \dots, f_\ell)$. Remark that $(f_1, \dots, f_\ell) \in \mathcal{G}_{s,\ell b}(\Omega; F_1 \times \cdots \times F_\ell)$ if and only if $f_i \in \mathcal{G}_{s,\ell b}(\Omega; F_i)$, $1 \le i \le \ell$.

Let $\mathcal{E}_{s,M}[\Omega; \Omega'] := \{u \in \mathcal{E}_{s,M}[\Omega; F] : u(]0,1] \times \Omega) \subset \Omega'\}$ be.

We denote by $\mathcal{E}_{s,M,*}[\Omega; \Omega']$ the set of all $u \in \mathcal{E}_{s,M}[\Omega; \Omega']$ so that for each $K \Subset \Omega$ there are $K' \subset\subset \Omega'$ and $\eta \in]0,1]$ such that $u(]0, \eta[\times K) \subset K'$. We indicate by

$\mathcal{G}_{s,*}(\Omega;\Omega')$ the set of all elements of $\mathcal{G}_s(\Omega;F)$ so that each one has representative in $\mathcal{E}_{s,M,*}[\Omega;\Omega']$.

If $(u,w) \in \mathcal{E}_{s,M}[\Omega;\Omega'] \times \mathcal{E}_{s,M}[\Omega';G]$ let $w \circ u \in \mathcal{E}_{s,M}[\Omega;G]$ be defined by

$$(w \circ u)(\varepsilon,x) := w(\varepsilon,u(\varepsilon,x)), \quad ((\varepsilon,x) \in]0,1] \times \Omega).$$

If $\dim F < +\infty$ and $(f,g) \in \mathcal{G}_{s,*}(\Omega;\Omega') \times \mathcal{G}_s(\Omega';G)$ we define the *composite function*

$$g \circ f := \widehat{g} \circ \widehat{f} + \mathcal{N}_s[\Omega;G]$$

where $\widehat{f}$ and $\widehat{g}$ are arbitrary representatives of f and g, respectively [1, 6, 7, 8].

Let $\mathcal{E}_{s,M}(F)$ be the set of all $\mu \in F^{]0,1]}$ such that there is $N \in \mathbb{N}$ satisfying $|\mu(\varepsilon)| = O(\varepsilon^{-N})$ as $\varepsilon \downarrow 0$ and let $\mathcal{N}_s(F)$ be the set of all functions $\mu \in \mathcal{E}_{s,M}(F)$ such that for each $q \in \mathbb{N}$ we have $|\mu(\varepsilon)| = O(\varepsilon^q)$ as $\varepsilon \downarrow 0$. The algebra of the *Colombeau generalized vectors in* F is defined by

$$\bar{F}_s := \frac{\mathcal{E}_{s,M}(F)}{\mathcal{N}_s(F)}.$$

We can identify F with a subspace of $\bar{F}_s$ and $\bar{F}_s$ with a subspace of $\mathcal{G}_s(\Omega;F)$. The elements of the image of $\bar{F}_s$ in $\mathcal{G}_s(\Omega;F)$ are called *generalized constants* ([6, 7]).

In the cases $E = \mathbb{R}^n$ and $F = \mathbb{K}$, we say that an element f in $\mathcal{G}_s(\Omega)$ is *associated with the null function* 0 ($f \approx 0$) if for some representative $\widehat{f}$ of f we have $\widehat{f}(\varepsilon,\cdot) \to 0$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \downarrow 0$. We say that a map $f = (f_1,\dots,f_\ell)$ in $\mathcal{G}_s(\Omega;\mathbb{K}^\ell)$ is associated with 0 if $f_j \approx 0$ for each $j = 1,\dots,\ell$. We say that two elements f, $g \in \mathcal{G}_s(\Omega;\mathbb{K}^\ell)$ are associated with each other if $f - g \approx 0$.

Let Ω (resp. W) be an open subset of $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Let us suppose that $(x,t) = (x_1,\dots,x_n,t_1,\dots,t_m)$ denotes a generic element of $\mathbb{R}^n \times \mathbb{R}^m$. If $f \in \mathcal{G}_s(\Omega \times W;\mathbb{R})$, we introduce the following notation: $\nabla_t f := \big(\frac{\partial f}{\partial t_1},\dots,\frac{\partial f}{\partial t_m}\big)$;

$$\operatorname{div}_x f := \frac{\partial f}{\partial x_1} + \dots + \frac{\partial f}{\partial x_n}; \quad \partial_x^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$

for $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$; $\partial_x^n f := \frac{\partial^n f}{\partial x_1 \dots \partial x_n}$.

For a proof of the following result see [1, Thm. 6.3.1].

Proposition 1.1. *If $f \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}^m)$ and $\partial_x^n f \approx 0$, there is $\Phi \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}^m)$ such that $\partial_x^n \Phi = 0$ and $f \approx \Phi$. In particular, if $n = m = 1$, Φ is a generalized constant.*

The next result follows from the dominated convergence theorem.

Proposition 1.2. *Let $(\phi,f) \in \mathbb{K}^\Omega \times \mathcal{G}_{s,\ell b}(\Omega)$ be such that $\widehat{f}(\varepsilon,\cdot) \to \phi$ a.e. in Ω as $\varepsilon \downarrow 0$ for some representative $\widehat{f}$ of f. Then, $\widehat{f}(\varepsilon,\cdot) \to \phi$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \downarrow 0$. In particular, if $\phi \in C^\infty(\Omega)$ then $f \approx \phi$.*

Lemma 1.3. *There is a function $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\int_{(\mathbb{R}_+^*)^n} \operatorname{div}_\lambda \psi(\lambda) d\lambda \neq 0$ where $\mathbb{R}_+^* := \{\lambda \in \mathbb{R}^* : \lambda > 0\}$ and $\mathbb{R}^* := \{\lambda \in \mathbb{R} : \lambda \neq 0\}$.*

Proof. Consider $\psi \colon \mathbb{R}^n \to \mathbb{R}$ defined by $\psi(\lambda) = \exp\big(\frac{1}{|\lambda|^2 - 1}\big)$ if $|\lambda| < 1$ and $\psi(\lambda) = 0$ if $|\lambda| \geq 1$, for example. $\square$

2. Heaviside GFs in $\mathbb{R}^n$

In this section will be considered the cases $E = \mathbb{R}^n$ (or $E = \mathbb{R}^n \times \mathbb{R}^m$), $F = \mathbb{R}^n$ and $G = \mathbb{K}^\ell$. We indicate by δ_n the Dirac measure in $\mathbb{R}^n$ at $x = 0$; that is, the function $\delta_n \colon \varphi \in \mathcal{D}(\mathbb{R}^n) \mapsto \varphi(0) \in \mathbb{K}$. An element $f \in \mathcal{G}_s(\mathbb{R}^n)$ is said to be a *Dirac generalized function in* $\mathbb{R}^n$ if there is a representative $\widehat{f}$ of f such that $\widehat{f}(\varepsilon, \cdot) \to \delta_n$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$.

The *Heaviside function in* $\mathbb{R}^n$ is the function $\mathbb{Y}_n \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathbb{Y}_n(\lambda_1, \dots, \lambda_n) := \begin{cases} 0, & \text{if } \lambda_j < 0 \text{ for some } j = 1, \dots, n \\ 1, & \text{if } \lambda_j > 0 \text{ for each } j = 1, \dots, n. \end{cases}$$

Lemma 2.1. *The Heaviside and Dirac functions verifies $\partial_\lambda^n \mathbb{Y}_n = \delta_n$ in $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi) \subset [-a_1, a_1] \times \dots \times [-a_n, a_n]$ then

$$\langle \partial_\lambda^n \mathbb{Y}_n, \varphi \rangle = (-1)^n \int_0^{a_n} \dots \int_0^{a_1} \frac{\partial^n \varphi}{\partial \lambda_1 \dots \partial \lambda_n}(\lambda_1, \dots, \lambda_n) d\lambda_1 \dots d\lambda_n = \langle \delta_n, \varphi \rangle$$

since $\langle \partial_\lambda^n \mathbb{Y}_n, \varphi \rangle = (-1)^n \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \mathbb{Y}_n(\lambda_1, \dots, \lambda_n) \partial_\lambda^n \varphi(\lambda_1, \dots, \lambda_n) d\lambda_1 \dots d\lambda_n$. □

An element $H \in \mathcal{G}_s(\mathbb{R}^n)$ is said to be a *Heaviside generalized function in* $\mathbb{R}^n$ if there is a representative $\widehat{H}$ of H such that $\widehat{H}(\varepsilon, \cdot) \to \mathbb{Y}_n$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$. We indicate by $\mathcal{H}(\mathbb{R}^n)$ the set of all Heaviside generalized functions in $\mathbb{R}^n$.

We denote by $\mathcal{H}_p(\mathbb{R}^n)$ the set of all elements H of $\mathcal{G}_{s,\ell b}(\mathbb{R}^n)$ so that it has representative $\widehat{H}$ such that $\widehat{H}(\varepsilon, \cdot) \to \mathbb{Y}_n$ in $(\mathbb{R}^*)^n$ as $\varepsilon \downarrow 0$.

We denote by $(\mathbb{R}_-^+)^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j < 0 \text{ for some } j = 1, \dots, n\}$.

Proposition 2.2. (a) *If $H \in \mathcal{H}(\mathbb{R}^n)$ then $\partial_\lambda^n H$ is a Dirac generalized function.*

(b) *If $H \in \mathcal{H}(\mathbb{R}^n)$ then $H|_{(\mathbb{R}_-^+)^n} \approx 0$, $H|_{(\mathbb{R}_+^*)^n} \approx 1$ and $\partial_\lambda^\alpha H|_{(\mathbb{R}^*)^n} \approx 0$, ($\alpha$ in $\mathbb{N}^n$, $\alpha \neq 0$).*

(c) *If $(k, H) \in \mathbb{K} \times \mathcal{H}(\mathbb{R}^n)$, then $k \operatorname{div}_\lambda H \approx 0$ if and only if $k = 0$.*

(d) *$\mathcal{H}_p(\mathbb{R}^n) \subset \mathcal{H}(\mathbb{R}^n)$ and if $(H, K) \in \mathcal{H}(\mathbb{R}^n) \times \mathcal{H}(\mathbb{R}^n)$ then $H \approx K$.*

(e) *If $(\alpha_j, H_j) \in \mathbb{N}^* \times \mathcal{H}_p(\mathbb{R}^n), 1 \leq j \leq m$, then $H_1^{\alpha_1} \dots H_m^{\alpha_m} \in \mathcal{H}_p(\mathbb{R}^n)$.*

Proof. The statement (a) follows by using lemma 2.1. The statements (b) and (e) result from given definitions. The statements (c) and (d) follow by using lemma 1.3 and Lebesgue's dominated convergence theorem, respectively. □

We say that a function $\widehat{H} \in \mathcal{E}_{s,M}[\mathbb{R}^n]$ *verifies the property* $(\mathcal{H}_r)^n$ if there is $\mu = (\mu_1, \dots, \mu_n) \in \big((\mathbb{R}_+^*)^n\big)^{]0,1]}$ such that $\lim_{\varepsilon \downarrow 0} \mu(\varepsilon) = 0$ and

$$\widehat{H}(\varepsilon, \lambda_1, \dots, \lambda_n) = \begin{cases} 0, & \text{if } \lambda_j < -\mu_j(\varepsilon) \text{ for some } j = 1, \dots, n \\ 1, & \text{if } \lambda_j > \mu_j(\varepsilon) \text{ for each } j = 1, \dots, n. \end{cases} \tag{2.1}$$

We indicate by $\mathcal{H}_r(\mathbb{R}^n)$ the set of all elements of $\mathcal{G}_{s,\ell b}(\mathbb{R}^n)$ so that each one has representative verifying the property $(\mathcal{H}_r)^n$.

Lemma 2.3. *If $\widehat{H} \colon]0, 1 \times \mathbb{R}^n \to \mathbb{R}$ verifies the property $(\mathcal{H}_r)^n$, hold:*

(a) *$\widehat{H}(\varepsilon, \cdot) \to \mathbb{Y}_n$ in $(\mathbb{R}^*)^n$ as $\varepsilon \downarrow 0$ and hence $\mathcal{H}_r(\mathbb{R}^n) \subset \mathcal{H}_p(\mathbb{R}^n)$.*

(b) *For each $K \Subset (\mathbb{R}^*)^n$ there is $\eta \in]0, 1]$ such that $0 \leq \widehat{H} \leq 1$ in $]0, \eta[\times K$.*

(c) *If* $\widehat{H} \in \mathcal{E}_M[\mathbb{R}^n;\mathbb{R}]$ *and* H *is the class of* $\widehat{H}$ *then* $H|_{(\mathbb{R}_-^+)^n} = 0$, $H|_{(\mathbb{R}_+^*)^n} = 1$ *and* $\frac{\partial H}{\partial \lambda_j}|_{(\mathbb{R}^*)^n} = 0$, $(1 \le j \le n)$.

Proof. The statement (a) follows directly from conditions of property $(\mathcal{H}_r)^n$.

(b). From $\lim_{\varepsilon\downarrow 0}\mu(\varepsilon) = 0$ we have $(\forall K \Subset (\mathbb{R}^*)^n)\,(\exists \eta \in]0,1])$: $K \subset \Omega_\mu(\varepsilon)$, $(\varepsilon \in]0,\eta[)$, which together with (2.1) implies the statement (b), where

$$\Omega_\mu(\varepsilon) := \{\lambda \in (\mathbb{R}^*)^n : \lambda > \mu(\varepsilon)\} \,\dot\cup\, \{\lambda : \lambda_j < -\mu_j(\varepsilon) \text{ for some } j = 1,\dots,n\}.$$

(c). From (2.1) we also have $\frac{\partial \widehat{H}}{\partial \lambda_j}(\varepsilon,\lambda) = 0$ for each $\varepsilon \in]0,1]$ and $\lambda \in \Omega_\mu(\varepsilon)$, $(j = 1,\dots,n)$. From (b) for each open subset V of $\mathbb{R}^n$ with $\overline{V} \Subset (\mathbb{R}^*)^n$ there is $\eta \in]0,1]$ such that $\overline{V} \subset \Omega_\mu(\varepsilon)$ for each $\varepsilon \in]0,\eta[$. From this, together with the previus condition, it follows the third statement. □

We set $B_r[0] := \{\lambda \in \mathbb{R}^n : |\lambda| \le r\}$, $(r \in \mathbb{R}_+^*)$, and we denote by

$$\Lambda(n) := \{\varphi \in \mathcal{D}(\mathbb{R}^n) : \varphi \ge 0,\ \varphi(0) > 0,\ \mathrm{supp}(\varphi) \subset \mathrm{B}_1[0] \text{ and } \int \varphi(\lambda)\,d\lambda = 1\}.$$

The following proposition shows concrete examples of Heaviside generalized functions in $\mathbb{R}^n$.

Proposition 2.4. *If* $\varphi \in \Lambda(n)$ *and* $\widehat{H}_\varphi\colon]0,1] \times \mathbb{R}^n \to \mathbb{R}$ *is defined by*

$$\widehat{H}_\varphi(\varepsilon,\lambda_1,\dots,\lambda_n) := \int_{-\infty}^{\lambda_1} \dots \int_{-\infty}^{\lambda_n} \frac{1}{\varepsilon^n}\varphi\big(\frac{t_1}{\varepsilon},\dots,\frac{t_n}{\varepsilon}\big)dt_1\dots dt_n$$

we have $0 \le \widehat{H}_\varphi \le 1$ *in* $]0,1] \times \mathbb{R}^n$ *and* $\widehat{H}_\varphi(\varepsilon,\lambda) = 0$ *(resp.* $= 1$*) if* $\lambda_j \le -\varepsilon$ *for some* $j = 1,\dots,n$ *(resp. if* $\lambda_j \ge \varepsilon$ *for each* $j = 1,\dots,n$*),* $(\lambda = (\lambda_1,\dots,\lambda_n))$*, for all* $\varepsilon \in]0,1]$*. Furthermore, if* δ_φ *is the class of* $\widehat{\delta}_\varphi\colon (\varepsilon,x) \mapsto \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$ *and if* H_φ *is the class of* $\widehat{H}_\varphi$ *then* $H_\varphi \in \mathcal{H}_r(\mathbb{R}^n)$ *and* $\partial_\lambda^n H_\varphi = \delta_\varphi$.

Proof. Since $\widehat{\delta}_\varphi \ge 0$, for each $(\varepsilon,\lambda_1,\dots,\lambda_n) \in]0,1] \times \mathbb{R}^n$, we have

$$0 \le \widehat{H}_\varphi(\varepsilon,\lambda_1,\dots,\lambda_n) \le \int_{\mathbb{R}^n} \widehat{\delta}_\varphi(\varepsilon,t)dt = \int_{B_\varepsilon[0]} \frac{1}{\varepsilon^n}\varphi\big(\frac{t}{\varepsilon}\big)dt = 1\,.$$

Let $\varepsilon \in]0,1]$ and $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n$ be fixed. Assume that $\lambda_j \ge \varepsilon$ for each $j = 1,\dots,n$. As $\mathrm{supp}[\widehat{\delta}_\varphi(\varepsilon,\cdot)] \subset B_\varepsilon[0]$, for each $j = 1,\dots,n$ and $t_j \in [\varepsilon,\lambda_j]$, we have $\widehat{\delta}_\varphi(\varepsilon,t_1,\dots,t_j,\dots,t_n) = 0$, $(t_i \in]-\infty,\lambda_i]$, $i = 1,\dots,j-1,j+1,\dots,n)$. Thus it follows that

$$\widehat{H}_\varphi(\varepsilon,\lambda_1,\dots,\lambda_n) = \int_{-\varepsilon}^{\varepsilon} \dots \int_{-\varepsilon}^{\varepsilon} \frac{1}{\varepsilon^n}\varphi\big(\frac{t_1}{\varepsilon},\dots,\frac{t_n}{\varepsilon}\big)dt_1\dots dt_n = \int_{B_\varepsilon[0]} \widehat{\delta}_\varphi(\varepsilon,t)dt = 1.$$

If $\lambda_j \le -\varepsilon$ for some $j = 1,\dots,n$ it is clear that $\widehat{H}_\varphi(\varepsilon,\lambda_1,\dots,\lambda_n) = 0$ considering that, for each $i = 1,\dots,j-1,j+1,\dots,n$ and each $t_i \in]-\infty,\lambda_i]$, we have $\widehat{\delta}_\varphi(\varepsilon,t_1,\dots,t_j,\dots,t_n) = 0$. Thus $H_\varphi \in \mathcal{H}_r(\mathbb{R}^n)$. Finally, as $\partial_\lambda^n \widehat{H}_\varphi = \widehat{\delta}_\varphi$, we have $\partial_\lambda^n H_\varphi = \delta_\varphi$. □

Remark 2.5. If $n = 1$, given $\psi \in \Lambda(1)$ let $K_\psi \in \mathcal{H}(\mathbb{R})$ be the class of $\widehat{K}_\psi$ defined as in proposition 2.4. If $p_j\colon (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n \mapsto \lambda_j \in \mathbb{R}$, $(j = 1,\dots,n)$, we have $(K \circ p_1)\dots(K \circ p_n) = H_\varphi$ where $\varphi := (\psi \circ p_1)\dots(\psi \circ p_n)$ and H_φ is the class of $(\widehat{K}_\psi \circ p_1)\dots(\widehat{K}_\psi \circ p_n)$.

If $\mathcal{H}^n_\Lambda(\mathbb{R}^n) := \{(K \circ p_1)\dots(K \circ p_n) : K \in \mathcal{H}_\Lambda(\mathbb{R})\}$ (see [7, Remark 2.1]) and if $\mathcal{H}_\Lambda(\mathbb{R}^n) := \{H_\varphi \in \mathcal{G}_s(\mathbb{R}^n) : \varphi \in \Lambda(n)\}$, where H_φ is defined as in proposition 2.4, we have the following inclusions

$$\mathcal{H}^n_\Lambda(\mathbb{R}^n) \subset \mathcal{H}_\Lambda(\mathbb{R}^n) \subset \mathcal{H}_r(\mathbb{R}^n) \subset \mathcal{H}_p(\mathbb{R}^n) \subset \mathcal{H}(\mathbb{R}^n)\,.$$

A proof of the next result can be found in [9, Prop. 3] or in [2, Prop. 2.11].

Proposition 2.6. *If $g \in \mathcal{G}_s(\mathbb{R}^n;\mathbb{K}^\ell)$, Ω is an open subset of $\mathbb{R}^n$ and S is a C^∞ -diffeomorphism on $\mathbb{R}^n$ such that $J_S(x) > 0$ for all $x \in \mathbb{R}^n$ (J_S denotes the jacobian of S) the following statements hold:*

(a) *$(g\circ S)|_\Omega \approx 0$ (resp. $(g\circ S)|_\Omega = 0$) if and only if $g|_{S(\Omega)} \approx 0$ (resp. $g|_{S(\Omega)} = 0$).*
(b) *If $\pi_m : (\lambda, t) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto \lambda \in \mathbb{R}^n$ then $(g \circ \pi_m)|_{\Omega\times\mathbb{R}^m} \approx 0$ if and only if $g|_\Omega \approx 0$ and $(g \circ \pi_m)|_{\Omega\times\mathbb{R}^m} = 0$ if and only if $g|_\Omega = 0$.*

The proof of the next result is similar at proof of [7, Prop. 2.2]. We present it for the convenience of the reader.

Proposition 2.7. *If $\mu \in \mathcal{E}_{s,M}(\mathbb{R})$ and $\mu \geq 1$, then there exists $\widehat{V} \in \mathcal{E}_{s,M}[\mathbb{R}^n;\mathbb{R}]$ verifying the property $(\mathcal{H}_r)^n$ and the two conditions: $0 \leq \widehat{V}(\varepsilon,\cdot) \leq \mu(\varepsilon)$ in $\mathbb{R}^n$ and $\widehat{V}(\varepsilon,\cdot) \equiv \mu(\varepsilon)$ in*

$$A_\varepsilon := \cup_{i=1}^n \{x \in \mathbb{R}^n : |x_i| \leq \varepsilon \text{ and } x_j \geq -\varepsilon,\ (j = 1,\dots,n,\ j \neq i)\}$$

for all $\varepsilon \in]0,1]$. Furthermore, if V is the class of $\widehat{V}$ then $V \notin \mathcal{H}_\Lambda(\mathbb{R}^n)$.

Proof. If $\varphi \in \Lambda(n)$ we consider the function $u : (\varepsilon, x) \mapsto [\chi(\varepsilon,\cdot) * \widehat{\delta}_\varphi(\varepsilon/4,\cdot)](x)$, where $\widehat{\delta}_\varphi : (\varepsilon, x) \mapsto \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon^n})$ and $\chi(\varepsilon,\cdot)$ is the characteristic function of

$$B_\varepsilon := \cup_{i=1}^n \{x \in \mathbb{R}^n : |x_i| \leq 3\frac{\varepsilon}{2} \text{ and } x_j \geq -3\frac{\varepsilon}{2},\ \ (j = 1,\dots,n,\ j \neq i)\}\,.$$

For a fixed $\varepsilon \in]0,1]$, we have $\operatorname{supp}[u(\varepsilon,\cdot)] \subset \operatorname{supp}[\chi(\varepsilon,\cdot)] + \operatorname{supp}[\widehat{\delta}_\varphi(\frac{\varepsilon}{4},\cdot)] = D_\varepsilon$, where

$$D_\varepsilon := \cup_{i=1}^n \{x \in \mathbb{R}^n : |x_i| \leq 7\frac{\varepsilon}{4} \text{ and } x_j \geq -7\frac{\varepsilon}{4},\ \ (j = 1,\dots,n,\ j \neq i)\}$$

and $u(\varepsilon,\cdot) \in C^\infty(\mathbb{R}^n;\mathbb{R})$. Since $0 \leq \chi(\varepsilon,\cdot) \leq 1$ and $\varphi \in \Lambda(n)$ it follows that $0 \leq u(\varepsilon,\cdot) \leq 1$ in $\mathbb{R}^n$. Moreover, we have

$$|\partial^\alpha_x u(\varepsilon, x)| \leq \frac{4^{|\alpha|}}{\varepsilon^{|\alpha|}} \int |\partial^\alpha_y \varphi(y)\|dy\,, \quad ((\varepsilon, x) \in]0,1] \times \mathbb{R}^n\,,\ \ \alpha \in \mathbb{N}^n)$$

since $\partial^\alpha_x u(\varepsilon, x) = [\chi(\varepsilon,\cdot) * (\partial^\alpha_x\widehat{\delta}_\varphi)(\frac{\varepsilon}{4},\cdot)](x)$ and $(\partial^\alpha_x\widehat{\delta}_\varphi)(\frac{\varepsilon}{4},\cdot) = \frac{4^{|\alpha|}}{\varepsilon^{|\alpha|}}\widehat{\delta}_{\partial^\alpha_x\varphi}(\frac{\varepsilon}{4},\cdot)$. Thus $u \in \mathcal{E}_{s,M}[\mathbb{R}^n;\mathbb{R}]$. For fixed $\varepsilon \in]0,1]$, if

$$W_\varepsilon := \cup_{i=1}^n \{x \in \mathbb{R}^n : |x_i| < 5\frac{\varepsilon}{4} \text{ and } x_j > -5\frac{\varepsilon}{4},\ \ (j = 1,\dots,n,\ j \neq i)\}$$

then $W_\varepsilon - \operatorname{supp}[\widehat{\delta}_\varphi(\frac{\varepsilon}{4},\cdot)] \subset \operatorname{Int}(B_\varepsilon)$ (interior set of B_ε). So, for each $x \in W_\varepsilon$ and each $y \in \operatorname{supp}[\widehat{\delta}_\varphi(\frac{\varepsilon}{4},\cdot)]$, we have $\chi(\varepsilon, x-y) = 1$. Therefore

$$u(\varepsilon, x) = \int_{\operatorname{supp}[\widehat{\delta}_\varphi(\frac{\varepsilon}{4},\cdot)]} \widehat{\delta}_\varphi(\frac{\varepsilon}{4}, y)\chi(\varepsilon, x-y)dy = \int \widehat{\delta}_\varphi(\frac{\varepsilon}{4}, y)dy = 1,\ \ (x \in W_\varepsilon).$$

The function $\widehat{V} : (\varepsilon, x) \mapsto 1+v(\varepsilon, x)[\mu(\varepsilon)u(\varepsilon, x)-1]$ satisfies the required properties, where $v \in \mathcal{E}_M[\mathbb{R}^n;\mathbb{R}]$ is defined by $v(\varepsilon, x) = 1$ (resp. $= u(\varepsilon, x)$) if $x_i < 5\frac{\varepsilon}{4}$ for some $i = 1,\dots,n$ (resp. if $x_i \geq 5\frac{\varepsilon}{4}$ for each $i = 1,\dots,n$). $\square$

The auxiliary functions y, y^*, π_m and y_*. For a fixed $y \in C^\infty(\mathbb{R}^m;\mathbb{R}^n)$, if $y^*\colon (x,t) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto x - y(t) \in \mathbb{R}^n$ and $y_* : (x,t) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto (y^*(x,t),t) \in \mathbb{R}^n \times \mathbb{R}^m$, we have

$$y^* = \pi_m \circ y_* \text{ in } \mathbb{R}^n \times \mathbb{R}^m \quad (\text{and hence } y^*(\mathbb{R}^n \times \mathbb{R}^m) = \mathbb{R}^n) \tag{2.2}$$

where $\pi_m\colon (\lambda,t) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto \lambda \in \mathbb{R}^n$. Moreover,

$$y_*(\Omega) = y^*(\Omega) \times \mathbb{R}^n\,, \quad \text{for } \Omega = \Omega_-^+\,, \Omega_+\,, \Omega^*\,, \mathbb{R}^n \times \mathbb{R}^m, \tag{2.3}$$

where $\Omega^* := \Omega_-^+ \cup \Omega_+$, $\Omega_-^+ := \{(x,t) : y_j^*(x,t) < 0 \text{ for some } j = 1,\dots,n\}$, $\Omega_+ := \{(x,t) : y_j^*(x,t) > 0 \text{ for all } j = 1,\dots,n\}$ and $y_1^*,\dots,y_n^*$ are the component functions of y^* associated to the component functions of $y = (y_1,\dots y_n)$, respectively. From proposition 2.6 we have the following corollary.

Corollary 2.8. *If $f \in \mathcal{G}_s(\mathbb{R}^n;\mathbb{K}^\ell)$ and Ω is an open subset of $\mathbb{R}^n \times \mathbb{R}^m$ such that $y_*(\Omega) = y^*(\Omega) \times \mathbb{R}^m$, then $(f \circ y^*)|_\Omega \approx 0$ if and only if $f|_{y^*(\Omega)} \approx 0$ and $(f \circ y^*)|_\Omega = 0$ if and only if $f|_{y^*(\Omega)} = 0$.*

From the previous corollary and proposition 2.2 we have the next result.

Proposition 2.9. *For each $H \in \mathcal{H}(\mathbb{R}^n)$ the following statements hold.*

(a) *$(H \circ y^*)|_{\Omega_+} \approx 1$, $(H \circ y^*)|_{\Omega_-^+} \approx 0$ and $\partial_x^\alpha (H \circ y^*)|_{\Omega^*} \approx 0$, ($\alpha \in \mathbb{N}^n$, $\alpha \neq 0$).*
(b) *If $k \in \mathbb{K}$, then $k \operatorname{div}_x(H \circ y^*) \approx 0$ if and only if $k = 0$.*
(c) *If $(\alpha_j, H_j) \in \mathbb{N}^* \times \mathcal{H}_p(\mathbb{R}^n)$, $1 \le j \le m$, then $(H_1^{\alpha_1} \dots H_m^{\alpha_m}) \circ y^* \approx H \circ y^*$.*

3. Heaviside GFs in $\mathbb{R}^n \times \mathbb{R}^m$

In this section will be considered the cases $E = \mathbb{R}^n \times \mathbb{R}^m$, $F = \mathbb{K}^\ell$ and $G = \mathbb{R}$. The *Heaviside function in* $\mathbb{R}^n \times \mathbb{R}^m$ is the function $\mathbb{Y}_{nm}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\mathbb{Y}_{nm}(\lambda_1,\dots,\lambda_n,t) = \begin{cases} 0, & \text{if } \lambda_j < 0 \text{ for some } j = 1,\dots,n \\ 1, & \text{if } \lambda_j > 0 \text{ for each } j = 1,\dots,n, \end{cases} \quad (t \in \mathbb{R}^m).$$

Lemma 3.1. (a) *$\nabla_t \mathbb{Y}_{nm} = 0$ and $\operatorname{div}_t \mathbb{Y}_{nm} = 0$ in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$.*
(b) *Given a, b in $\mathbb{R}$, $a < b$, let $X\colon \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ be defined by*

$$X(\lambda_1,\dots,\lambda_n,t) := \begin{cases} b, & \text{if } \lambda_j < 0 \text{ for some } j = 1,\dots,n \\ a, & \text{if } \lambda_j > 0 \text{ for each } j = 1,\dots,n, \end{cases} \quad (t \in \mathbb{R}^m).$$

Then $\nabla_t X = 0$ and $div_t X = 0$ in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$.

Setting $\pi_m = (p_{1m},\dots,p_{nm})$, where $p_{jm}\colon (\lambda,t) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto \lambda_j \in \mathbb{R}$, we have $\mathbb{Y}_{nm} = \mathbb{Y}_n \circ \pi_m$. Then, since $\partial_\lambda^n \mathbb{Y}_n = \delta_n$ (lemma 2.1) and $(p_{im})_{\lambda_j} = \delta_{ij}$ (Kronecker's delta) for i, $j = 1,\dots,n$, by the chain rule we have

$$\partial_\lambda^n \mathbb{Y}_{nm} = \delta_n \circ \pi_m\,.$$

An element $H \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}^m)$ is said to be a *Heaviside generalized function in* $\mathbb{R}^n \times \mathbb{R}^m$ if there is a representative $\widehat{H}$ of H such that $\widehat{H}(\varepsilon,\cdot) \to \mathbb{Y}_{nm}$ in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$ as $\varepsilon \downarrow 0$. We indicate by $\mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$ the set of all Heaviside generalized functions in $\mathbb{R}^n \times \mathbb{R}^m$.

We denote by $\mathcal{H}_p(\mathbb{R}^n \times \mathbb{R}^m)$ the set of all $H \in \mathcal{G}_{s,\ell b}(\mathbb{R}^n \times \mathbb{R}^m)$ so that it has representative $\widehat{H}$ such that $\widehat{H}(\varepsilon,\cdot) \to \mathbb{Y}_{nm}$ in $(\mathbb{R}^*)^n \times \mathbb{R}^m$ as $\varepsilon \downarrow 0$. The next result follows from the above definitions.

Proposition 3.2. (a) *For each $\mathcal{J} = \mathcal{H}, \mathcal{H}_p$, if $\pi_m\colon (\lambda, t) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto \lambda \in \mathbb{R}^n$ and $K \in \mathcal{J}(\mathbb{R}^n)$ we have $K \circ \pi_m \in \mathcal{J}(\mathbb{R}^n \times \mathbb{R}^m)$.*

(b) *$\mathcal{H}_p(\mathbb{R}^n \times \mathbb{R}^m) \subset \mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$ and if H, $K \in \mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$, then $H \approx K$.*

(c) *If $(\alpha_j, H_j) \in \mathbb{N}^* \times \mathcal{H}_p(\mathbb{R}^n \times \mathbb{R}^m)$, $1 \le j \le k$, then $H_1^{\alpha_1} \dots H_k^{\alpha_k} \in \mathcal{H}_p(\mathbb{R}^n \times \mathbb{R}^m)$.*

(d) *If $H \in \mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$, then $H|_{(\mathbb{R}^+_-)^n \times \mathbb{R}^m} \approx 0$, $H|_{(\mathbb{R}^*_+)^n \times \mathbb{R}^m} \approx 1$, $\nabla_t H \approx 0$ and $div_t H \approx 0$.*

We say that a function $\widehat{H}\colon]0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *verifies the property* $(\mathcal{H}_r)^n_m$ if there is $\mu = (\mu_1, \dots, \mu_n)\colon]0,1] \to (\mathbb{R}^*_+)^n$ such that $\lim_{\varepsilon \downarrow 0} \mu(\varepsilon) = 0$ and

$$\widehat{H}(\varepsilon, \lambda, t) = \begin{cases} 1, & \text{if } \lambda_j > \mu_j(\varepsilon) \text{ for each } j = 1, \dots, n \\ 0, & \text{if } \lambda_j < -\mu_j(\varepsilon) \text{ for some } j = 1, \dots, n, \end{cases} \quad ((\varepsilon, t) \in]0,1] \times \mathbb{R}^m)$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$. We indicate by $\mathcal{H}_r(\mathbb{R}^n \times \mathbb{R}^m)$ the set of all elements of $\mathcal{G}_{s,\ell b}(\mathbb{R}^n \times \mathbb{R}^m)$ so that each one has representative verifying the property $(\mathcal{H}_r)^n_m$. The next result follows from this definition.

Proposition 3.3. *The following statements hold:*

(a) *If $(\alpha_j, H_j) \in \mathbb{N}^* \times \mathcal{H}_r(\mathbb{R}^n \times \mathbb{R}^m)$, $1 \le j \le k$, then $H_1^{\alpha_1} \dots H_k^{\alpha_k} \in \mathcal{H}_r(\mathbb{R}^n \times \mathbb{R}^m)$.*

(b) *If $\pi_m\colon (\lambda, t) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto \lambda \in \mathbb{R}^n$, then $K \circ \pi_m \in \mathcal{H}_r(\mathbb{R}^n \times \mathbb{R}^m)$ for all $K \in \mathcal{H}_r(\mathbb{R}^n)$.*

(c) *If $H \in \mathcal{H}_r(\mathbb{R}^n \times \mathbb{R}^m)$ and $t \in \mathbb{R}^m$, then $H(\cdot, t) \in \mathcal{H}_r(\mathbb{R}^n)$.*

The proof of the next result follows in an analogous way to the proof of lemma 2.3.

Proposition 3.4. *If $\widehat{H}\colon]0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ verifies the property $(\mathcal{H}_r)^n_m$, then the following statements hold:*

(a) *$\widehat{H}(\varepsilon, \cdot) \to \mathbb{Y}_{nm}$ in $(\mathbb{R}^*)^n \times \mathbb{R}^m$ as $\varepsilon \downarrow 0$ and $\mathcal{H}_r(\mathbb{R}^n \times \mathbb{R}^m) \subset \mathcal{H}_p(\mathbb{R}^n \times \mathbb{R}^m)$.*

(b) *For each $K \Subset (\mathbb{R}^*)^n$ there is $\eta \in]0,1]$ such that $0 \le \widehat{H} \le 1$ in $]0, \eta[\times K \times \mathbb{R}^m$.*

(c) *If $\widehat{H} \in \mathcal{E}_M[\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}]$ and H is the class of $\widehat{H}$, then $H|_{(\mathbb{R}^+_-)^n \times \mathbb{R}^m} = 0$, $H|_{(\mathbb{R}^*_+)^n \times \mathbb{R}^m} = 1$ and $\frac{\partial H}{\partial \lambda_j}|_{(\mathbb{R}^*)^n \times \mathbb{R}^m} = 0$, $(1 \le j \le n)$.*

Remark 3.5. Denoting by (see Remark 2.5)

$$\mathcal{H}_\Lambda(\mathbb{R}^n \times \mathbb{R}^m) := \{H \circ \pi_m : H \in \mathcal{H}_\Lambda(\mathbb{R}^n)\} = \{H_\varphi \circ \pi_m : \varphi \in \Lambda(n)\},$$
$$\mathcal{H}^{n,m}_\Lambda(\mathbb{R}^n \times \mathbb{R}^m) := \{H \circ \pi_m : H \in \mathcal{H}^n_\Lambda(\mathbb{R}^n)\}$$

where H_φ is defined as in proposition 2.4 and $\pi_m\colon (\lambda, t) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto \lambda \in \mathbb{R}^n$, we have the following inclusions:

$$\mathcal{H}^{n,m}_\Lambda(\mathbb{R}^n \times \mathbb{R}^m) \subset \mathcal{H}_\Lambda(\mathbb{R}^n \times \mathbb{R}^m) \subset \mathcal{H}_r(\mathbb{R}^n \times \mathbb{R}^m) \subset \mathcal{H}_p(\mathbb{R}^n \times \mathbb{R}^m) \subset \mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)\,.$$

From propositions 2.6, 3.2, 3.4 we have the following corollary.

Corollary 3.6. *If $y \in C^\infty(\mathbb{R}^m; \mathbb{R}^n)$ and $(\alpha_j, H_j) \in \mathbb{N}^* \times \mathcal{H}_p(\mathbb{R}^n \times \mathbb{R}^m)$, $1 \le j \le k$, then $(H_1^{\alpha_1} \dots H_k^{\alpha_k}) \circ y_* \approx H \circ y_*$, $(H \in \mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$.*

Using the chain rule we have the following result.

Proposition 3.7. *If $f = f(\lambda, t)$, $g = g(x, t) \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}^m)$ and $y \in C^\infty(\mathbb{R}^m; \mathbb{R}^n)$, $y = (y_1, \dots, y_n)$, hold: $\operatorname{div}_x(f \circ y_*) = (\operatorname{div}_\lambda f) \circ y_*$; $\partial_x^\alpha(f \circ y_*) = (\partial_\lambda^\alpha f) \circ y_*$; $\partial_\lambda^\alpha\left(g \circ y_*^{-1}\right) = (\partial_x^\alpha g) \circ y_*^{-1}$, $(\alpha \in \mathbb{N}^n)$;*

$$\nabla_t(f \circ y_*) = -\sum_{i=1}^n \big(\frac{\partial f}{\partial \lambda_i} \circ y_*\big)\nabla y_i + (\nabla_t f) \circ y_*,$$

$$\nabla_t(g \circ y_*^{-1}) = \sum_{i=1}^n \big(\frac{\partial g}{\partial x_i} \circ y_*^{-1}\big)\nabla y_i + (\nabla_t g) \circ y_*^{-1}$$

where $\nabla y_i = (\frac{\partial y_i}{\partial t_1}, \dots, \frac{\partial y_i}{\partial t_m})$.

The next proposition follows from the above result, using formula (2.2).

Proposition 3.8. *If $f = f(\lambda) \in \mathcal{G}_s(\mathbb{R}^n)$ and $y = (y_1, \dots, y_n) \in C^\infty(\mathbb{R}^m; \mathbb{R}^n)$, then $\operatorname{div}_x(f \circ y^*) = (\operatorname{div}_\lambda f) \circ y^*$; $\partial_x^\alpha(f \circ y^*) = (\partial_\lambda^\alpha f) \circ y^*$, $(\alpha \in \mathbb{N}^n)$, and*

$$\nabla_t(f \circ y^*) = -\sum_{i=1}^n \frac{\partial(f \circ y^*)}{\partial x_i}\nabla y_i\,.$$

From propositions 2.6, 3.2, 3.7, 3.8, using formulas given in (2.3), we have the following result.

Theorem 3.9. *If $H \in \mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$ and $y = (y_1, \dots, y_n) \in C^\infty(\mathbb{R}^m; \mathbb{R}^n)$, then $(H \circ y_*)|_{\Omega_+} \approx 1$; $(H \circ y_*)|_{\Omega_-^+} \approx 0$; $[\partial_x^\alpha(H \circ y_*)]\,|_{\Omega^*} \approx 0$, $(\alpha \in \mathbb{N}^n$, $\alpha \neq 0)$; $(\nabla_t H) \circ y_* \approx 0$ and*

$$\nabla_t(H \circ y_*) \approx -\sum_{i=1}^n \big(\frac{\partial H}{\partial \lambda_i} \circ y_*\big)\nabla y_i\,.$$

Proposition 3.10. *Given $y \in C^\infty(\mathbb{R}^m; \mathbb{R}^n)$ let $\Phi \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{K}^\ell)$ be such that $\partial_x^n \Phi = 0$ and $\Phi|_\Omega \approx 0$ for some open subset $\Omega \neq \emptyset$ of $\mathbb{R}^n \times \mathbb{R}^m$ such that $y_*(\Omega) = y^*(\Omega) \times \mathbb{R}^m$. Then $\Phi \approx 0$.*

The proof of the above Proposition follows by using proposition 2.6, with a minor modification in the proof of [7, Prop. 2.5] (also see [5, p. 336]). By using theorem 3.9 and propositions 1.1 and 3.10 we obtain the next result.

Theorem 3.11. *If $y \in C^\infty(\mathbb{R}^m; \mathbb{R}^n)$, $(\phi, \varphi) \in C^\infty(\mathbb{R}^m; \mathbb{K}^\ell) \times C^\infty(\mathbb{R}^m; \mathbb{K}^\ell)$ and $H \in \mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$, then the following statements hold:*

(a) *If $f \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{K}^\ell)$, $f|_{\Omega^*} \approx 0$ and $(H \circ y_*)\phi + [\partial_x^n(H \circ y_*)]\,\varphi \approx \partial_x^n f$, then $\varphi \equiv \phi \equiv 0$ and $f \approx 0$.*

(b) *If $\psi \in C^\infty(\mathbb{R}^m; \mathbb{K}^\ell)$ and $(H \circ y_*)\phi + [\partial_x^n(H \circ y_*)]\varphi \approx \psi$, then $\psi \equiv \varphi \equiv \phi \equiv 0$.*

Proof. (a) By restriction to $\Omega_+ \subset \Omega^*$, being $(H \circ y_*)|_{\Omega_+} \approx 1$, $[\partial_x^n(H \circ y_*)]|_{\Omega_+} \approx 0$ (theorem 3.9) and $(\partial_x^n f)|_{\Omega_+} \approx 0$, we obtain $\phi|_{\Omega_+} \approx 0$. From proposition 3.10 it follows that $\phi \approx 0$. Since $\phi \in C^\infty(\mathbb{R}^m; \mathbb{K}^\ell)$ we have $\phi \equiv 0$, which together with hypothesis, implies that $[\partial_x^n(H \circ y_*)]\varphi \approx \partial_x^n f$. Hence, by proposition 1.1, one can find a map $\Phi \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{K}^\ell)$ such that $\partial_x^n \Phi = 0$ and $(H \circ y_*)\varphi \approx f + \Phi$. By restriction to $\Omega_-^+ \subset \Omega^*$, as $(H \circ y_*)|_{\Omega_-^+} \approx 0$ and $f|_{\Omega_-^+} \approx 0$, we have $\Phi|_{\Omega_-^+} \approx 0$ and hence $\Phi \approx 0$, by proposition 3.10. So we get $(H \circ y_*)\varphi \approx f$. From this, by restriction to Ω_+ and by a similar argument it was made to obtain $\phi \equiv 0$, we conclude that $\varphi \equiv 0$. Thus, from former condition, it follows that $f \approx 0$. The proof of (b) it follows in an analogous way at proof of (a). $\square$

From theorem 3.11, and using proposition 3.2 and (2.2), we have the analogous result for the case of Heaviside generalized functions in $\mathbb{R}^n$.

Corollary 3.12. *If $y \in C^\infty(\mathbb{R}^m;\mathbb{R}^n)$, $(\phi,\varphi) \in C^\infty(\mathbb{R}^m;\mathbb{K}^\ell) \times C^\infty(\mathbb{R}^m;\mathbb{K}^\ell)$ and $H \in \mathcal{H}(\mathbb{R}^n)$ the following statements hold:*

(a) *If $f \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}^m;\mathbb{K}^\ell)$, $f|_{\Omega^*} \approx 0$ and $(H \circ y^*)\,\phi + [\partial_x^n\,(H \circ y^*)]\,\varphi \approx \partial_x^n f$, then $\varphi \equiv \phi \equiv 0$ and $f \approx 0$.*

(b) *If $\psi \in C^\infty(\mathbb{R}^m;\mathbb{K}^\ell)$ and $(H \circ y^*)\,\phi + [\partial_x^n\,(H \circ y^*)]\,\varphi \approx \psi$, then $\psi \equiv \varphi \equiv \phi \equiv 0$.*

4. Additional properties of Heaviside GFs

More information about the special cases of composition and invertibility considered in what follows can be found in [6, Section 3] or in [7, Section 3].

Special case of composition. Here we suppose that $F = \mathbb{R}^\ell$. We fix $\alpha = (\alpha_1,\dots,\alpha_\ell)$ and $\beta = (\beta_1,\dots,\beta_\ell)$ in $(\tilde{\mathbb{R}}_+)^\ell$ ($\tilde{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$) with $\alpha < \beta$. We will use the notation

$$I_\alpha^\beta := \prod_{i=1}^{\ell}]\alpha_i,\beta_i[\subset (\mathbb{R}_+^*)^\ell,\;\; [\alpha,\beta] := \prod_{i=1}^{\ell} [\alpha_i,\beta_i] \text{ and }]\alpha,\beta] := \prod_{i=1}^{\ell}]\alpha_i,\beta_i];$$

and we will consider $\Omega' = I_\alpha^\beta \subset (\mathbb{R}_+^*)^\ell$.

We remark that $I_\alpha^\beta = (\mathbb{R}_+^*)^\ell$ if $\alpha = 0$ and $\beta = (+\infty,\dots,+\infty)$. In the case $\ell = 1$: $I_\alpha^\beta =]\alpha,\beta[$ and $I_\alpha^\beta = \mathbb{R}_+^*$, if $\alpha = 0$ and $\beta = +\infty$.

We denote by $\mathcal{E}_{s,M,\oslash}[\Omega; I_\alpha^\beta]$ the set of all $u \in \mathcal{E}_{s,M}[\Omega;\mathbb{R}^\ell]$ such that for each $K \Subset \Omega$ there are $\eta \in]0,1]$; a, $b \in I_\alpha^\beta$, $a < b$; and a function $\mu = (\mu_1,\dots,\mu_\ell)$ from $]0,1]$ into $]\alpha,a]$ such that $(\varepsilon \mapsto \frac{1}{\mu_i(\varepsilon)-\alpha_i}) \in \mathcal{E}_{s,M}(\mathbb{R})$, $(1 \le i \le \ell)$, and $u(\varepsilon,x) \in [\mu(\varepsilon),b]$ for all $(\varepsilon,x) \in]0,\eta[\times K$. We denote by $\mathcal{G}_{s,\otimes}(\Omega; I_\alpha^\beta)$ the set of all elements of $\mathcal{G}_s(\Omega;\mathbb{R}^\ell)$ so that each one has representative in (see $\mathcal{E}_{s,M}[\Omega; I_\alpha^\beta]$ for $\Omega' = I_\alpha^\beta$ in §1)

$$\mathcal{E}_{s,M,\otimes}[\Omega; I_\alpha^\beta] := \mathcal{E}_{s,M}[\Omega; I_\alpha^\beta] \cap \mathcal{E}_{s,M,\oslash}[\Omega; I_\alpha^\beta].$$

We denote by $\mathcal{E}_{s,QM}[I_\alpha^\beta; G]$ the set of all functions $w \in \mathcal{E}_s[I_\alpha^\beta; G]$ such that for each $p \in \mathbb{N}$; each a, $b \in I_\alpha^\beta$, $a < b$; and each function $\mu = (\mu_1,\dots,\mu_\ell)$ from $]0,1]$ into $]\alpha,a]$ such that $(\varepsilon \mapsto \frac{1}{\mu_i(\varepsilon)-\alpha_i}) \in \mathcal{E}_{s,M}(\mathbb{R})$, $(1 \le i \le \ell)$, there are $N \in \mathbb{N}$, $C > 0$ and $\eta \in]0,1]$ satisfying

$$\sup_{y \in [\mu(\varepsilon),b]} |w^{(p)}(\varepsilon,y)|_p \le C\varepsilon^{-N}, \;\; (0 < \varepsilon < \eta).$$

Note that $\mathcal{E}_{s,QM}[I_\alpha^\beta; G] \subset \mathcal{E}_{s,M}[I_\alpha^\beta; G]$. We define

$$C_{s,QM}^\infty[I_\alpha^\beta; G] := C^\infty(I_\alpha^\beta; G) \cap \mathcal{E}_{s,QM}[I_\alpha^\beta; G].$$

Proving the next result follows by a similar argument as in [1, Prop. 2.1.5].

Proposition 4.1. *We have $w \in \mathcal{E}_{s,QM}[I_\alpha^\beta; G]$ if and only if for each (γ,a,b) in $\mathbb{N}^m \times I_\alpha^\beta \times I_\alpha^\beta$, $a < b$, and for each $\mu = (\mu_1,\dots,\mu_m) \in (]\alpha,a])^{]0,1]}$ such that $(\varepsilon \mapsto \frac{1}{\mu_i(\varepsilon)-\alpha_i}) \in \mathcal{E}_{s,M}(\mathbb{R})$, $(1 \le i \le m)$, there are $N \in \mathbb{N}$, $C > 0$ and $\eta \in]0,1]$ satisfying*

$$\sup_{y \in [\mu(\varepsilon),b]} |\partial_y^\gamma w(\varepsilon,y)| \le C\varepsilon^{-N}, \quad (0 < \varepsilon < \eta).$$

Given $f \in \mathcal{G}_{s,\otimes}(\Omega; I_\alpha^\beta)$ and $\varphi \in C^\infty_{s,QM}[I_\alpha^\beta; G]$ we define the *composite function*

$$\varphi \circ f := \varphi \circ \widehat{f} + \mathcal{N}_s[\Omega; G]$$

where $\widehat{f} \in \mathcal{E}_{s,M}[\Omega; I_\alpha^\beta]$ is any representative of f (see [6, Thm. 3.3]).

Special case of invertibility. In this part we consider $F = G = \mathbb{R}$ and $\Omega' = \mathbb{R}^*$. We denote by $\mathcal{E}_{s,M,\odot}[\Omega; \mathbb{R}^*]$ the set of all $u \in \mathcal{E}_{s,M}[\Omega; \mathbb{R}]$ such that for each $K \Subset \Omega$ there are $\eta \in]0,1]$ and a function $\mu \in (\mathbb{R}_+^*)^{]0,1]}$ such that $(\varepsilon \mapsto \frac{1}{\mu(\varepsilon)}) \in \mathcal{E}_{s,M}(\mathbb{R})$ and $\mu(\varepsilon) \leq |u(\varepsilon, x)|$ for all $(\varepsilon, x) \in]0, \eta[\times K$.

Note that $\mathcal{E}_{s,M,\oslash}[\Omega; I_\alpha^\beta] \subset \mathcal{E}_{s,M,\odot}[\Omega; \mathbb{R}^*]$ and hence, for all α and β in $\tilde{\mathbb{R}}_+$ with $\alpha < \beta$, we have $\mathcal{E}_{s,M,\otimes}[\Omega; I_\alpha^\beta] \subset \mathcal{E}_{s,M,\odot}[\Omega; \mathbb{R}^*]$.

We denote by $\mathcal{G}_{s,\circ}(\Omega; \mathbb{R}^*)$ the set of all elements of $\mathcal{G}_s(\Omega; \mathbb{R})$ so that each one has representative in (see $\mathcal{E}_{s,M}[\Omega; \mathbb{R}^*]$ for $\Omega' = \mathbb{R}^*$ in §1)

$$\mathcal{E}_{s,M,\circ}[\Omega; \mathbb{R}^*] := \mathcal{E}_{s,M}[\Omega; \mathbb{R}^*] \cap \mathcal{E}_{s,M,\odot}[\Omega; \mathbb{R}^*].$$

Note that $\mathcal{E}_{s,M,\otimes}[\Omega; I_\alpha^\beta] \subset \mathcal{E}_{s,M,\circ}[\Omega; \mathbb{R}^*]$, thus $\mathcal{G}_{s,\otimes}(\Omega; I_\alpha^\beta) \subset \mathcal{G}_{s,\circ}(\Omega; \mathbb{R}^*)$ for all α and β in $\tilde{\mathbb{R}}_+$ with $\alpha < \beta$.

The proof of the following result can be found in [6, Thm. 3.6].

Proposition 4.2. *If* $\dim E < +\infty$; α, $\beta \in \tilde{\mathbb{R}}_+$, $\alpha < \beta$; *and* $f \in \mathcal{G}_{s,\otimes}(\Omega; I_\alpha^\beta)$ *then* f *(resp.* $f - \alpha$*) has a multiplicative inverse and* $\frac{1}{\widehat{f}}$ *(resp.* $\frac{1}{\widehat{f}-\alpha}$*) is a representative of* $\frac{1}{f}$ *(resp.* $\frac{1}{f-\alpha}$*), for every* $\widehat{f} \in \mathcal{E}_{s,M}[\Omega; I_\alpha^\beta]$ *representative of* f.

Some properties of Heaviside GFs. In what follows, will consider the cases $E = \mathbb{R}^n$ (or $E = \mathbb{R}^n \times \mathbb{R}^m$), $F = \mathbb{K}^\ell$ and $G = \mathbb{R}$.

Hypothesis 4.1 (For proposition 4.3 and theorems 4.6, 4.9) We fix $a = (a_1, \dots, a_\ell)$ and $b = (b_1, \dots, b_\ell)$ in $C^\infty(\mathbb{R}^m; \mathbb{R}^\ell)$ such that $0 < a(t) < b(t)$, $(t \in \mathbb{R}^m)$. For $i = 1, \dots, \ell$ we define $\Delta_i(t) := a_i(t) - b_i(t)$, $(t \in \mathbb{R}^m)$. We also fix $\alpha_* = (\alpha_1, \dots, \alpha_\ell)$ and $\beta_* = (\beta_1, \dots, \beta_\ell)$ in $(\tilde{\mathbb{R}}_+)^\ell$ with $\alpha_* < \beta_*$ and $\nu_* = (\nu_1, \dots, \nu_\ell)$ in $C^\infty(\mathbb{R}_+^*; \mathbb{R}^\ell)$ such that each component of ν_* is an increasing function. Let us suppose that $\{1, \dots, \ell\}$ is a reunion of two disjoint subsets $\mathbb{I}$ and $\mathbb{J}$ such that for $i \in \mathbb{I}$ (resp. for $j \in \mathbb{J}$) ν_i is strictly increasing with $\mathrm{Im}(\nu_i) = I_{\alpha_i}^{\beta_i}$ and inverse function ν_i^{-1} (resp. ν_j not strictly increasing with $\mathrm{Im}(\nu_j) \subset [\alpha_j, \beta_j[$).

Let us consider the real numbers $\alpha_{(s)} := \alpha_1 + \dots + \alpha_\ell$, $\beta_{(s)} := \beta_1 + \dots + \beta_\ell$ and $\nu_{(s)}$ defined by $\nu_{(s)}(y_1, \dots, y_\ell) := \nu_1(y_1) + \dots + \nu_\ell(y_\ell)$, $((y_1, \dots, y_\ell) \in (\mathbb{R}_+^*)^\ell)$. Moroever, if $\alpha_j > 0$ for all $j \in \mathbb{J}$, we consider $\alpha_{(\pi)} := \alpha_1 \dots \alpha_\ell$, $\beta_{(\pi)} := \beta_1 \dots \beta_\ell$ and $\nu_{(\pi)}$ defined by $\nu_{(\pi)}(y_1, \dots, y_\ell) := \nu_1(y_1) \dots \nu_\ell(y_\ell)$.

With these notation, for each $\nu = \nu_{(s)}$, $\nu_{(\pi)}$, we consider the associated map $(\nu_R, \nu_L, \Delta\nu)$, where $\nu_R := \nu \circ a$, $\nu_L := \nu \circ b$ and $\Delta\nu := \nu_R - \nu_L$. In the sequel (ν, α, β) it indistinctly indicates $(\nu_{(s)}, \alpha_{(s)}, \beta_{(s)})$ or $(\nu_{(\pi)}, \alpha_{(\pi)}, \beta_{(\pi)})$.

Proposition 4.3. *Given* $H_i \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}^m)$, $1 \leq i \leq \ell$, *let* $f = (f_1, \dots, f_\ell)$ *be being that* $f_i := \Delta_i H_i + b_i$. *If each* H_i *has representative* $\widehat{H}_i$ *such that*

$$\sup_{\lambda \in \mathbb{R}^n} |\widehat{H}_i(\varepsilon, \lambda, t)| \leq \frac{a_i(t)\varepsilon - b_i(t)}{\Delta_i(t)}, \quad ((\varepsilon, t) \in]0,1] \times \mathbb{R}^m)$$

then $\widehat{f} = (\widehat{f}_1, \dots, \widehat{f}_\ell) := (\Delta_1 \widehat{H}_1 + b_1, \dots, \Delta_\ell \widehat{H}_\ell + b_\ell) \in \mathcal{E}_{s,M,\otimes}[\mathbb{R}^n \times \mathbb{R}^m; (\mathbb{R}_+^*)^\ell]$. *Furthermore, if* $\nu_i \in \mathcal{E}_{s,QM}[\mathbb{R}_+^*; \mathbb{R}]$, $1 \leq i \leq \ell$, *then* $\nu \in \mathcal{E}_{s,QM}[(\mathbb{R}_+^*)^\ell; \mathbb{R}]$ *and the following statements hold*

(1) *if* $\widehat{H}_i(\varepsilon,\cdot) \to \mathbb{Y}_{nm}$ *in* $(\mathbb{R}^*)^n \times \mathbb{R}^m$ *as* $\varepsilon \downarrow 0$*, for each* $i = 1,\dots,\ell$*, then* $\frac{\nu\circ f-\nu_L}{\Delta\nu} \in \mathcal{H}_p(\mathbb{R}^n \times \mathbb{R}^m)$.
(2) *if* $1 \in \mathbb{I}$ *and* $\big(\varepsilon \mapsto \frac{1}{\nu_1(r\varepsilon)-\alpha_1}\big) \in \mathcal{E}_{s,M}(\mathbb{R})$ *for each* $r > 0$*, then the map* $\nu \circ \widehat{f} \in \mathcal{E}_{s,M,\otimes}[\mathbb{R}^n \times \mathbb{R}^m; I_\alpha^\beta]$*,* $\frac{\nu_1\circ f_1-\alpha_1}{\nu\circ f-\alpha} \in \mathcal{G}_{s,\ell b}(\mathbb{R}^n \times \mathbb{R}^m)$ *and*
$$\Big(\frac{\nu_1 \circ \widehat{f}_1 - \alpha_1}{\nu \circ \widehat{f} - \alpha}\Big)(\varepsilon,\cdot) \to \begin{cases} \frac{\nu_1(b_1)-\alpha_1}{\nu_L-\alpha} & \text{in } (\mathbb{R}^+_-)^n \times \mathbb{R}^m \\ \frac{\nu_1(a_1)-\alpha_1}{\nu_R-\alpha} & \text{in } (\mathbb{R}^*_+)^n \times \mathbb{R}^m \end{cases} \quad \text{as } \varepsilon \downarrow 0$$
In the case of $(\nu,\alpha,\beta) = (\nu_{(\pi)},\alpha_{(\pi)},\beta_{(\pi)})$ *we set the additional assumption* $\alpha_k > 0$ *whenever* $2 \le k \le \ell$.

The above result follows by using proposition 4.1 (to check $\nu \in \mathcal{E}_{s,QM}[(\mathbb{R}^*_+)^\ell;\mathbb{R}]$), and by a minor modification in the proof of [7, Prop. 4.2].

Hypothesis 4.2 (For propositions 4.4, 4.5) We fix (α,β) in $\mathbb{R}_+ \times \tilde{\mathbb{R}}_+$ with $\alpha < \beta$, $\nu \in C^\infty(\mathbb{R}^*_+;\mathbb{R})$ strictly increasing such that $\mathrm{Im}(\nu) = I_\alpha^\beta$ (ν^{-1} denotes the inverse function of ν) and (a,b) in $\mathbb{R}^*_+ \times \mathbb{R}^*_+$ with $a < b$. Let $\Delta := a - b$ be, $\nu_R := \nu(a)$, $\nu_L := \nu(b)$, $\Delta\nu := \nu_R - \nu_L$ and $\theta := \frac{\alpha-\nu_L}{\Delta\nu}$.

Proposition 4.4. *If* $\nu^{-1} \in \mathcal{E}_{s,QM}[I_\alpha^\beta;\mathbb{R}]$*, then for each* $\mu\colon]0,1] \to [1,\theta[$ *such that* $\big(\varepsilon \in]0,1] \mapsto \frac{1}{\Delta\nu\mu(\varepsilon)+\nu_L-\alpha} \in \mathbb{R}\big) \in \mathcal{E}_{s,M}(\mathbb{R})$ *and* $\nu(a\varepsilon) \le \Delta\nu\mu(\varepsilon) + \nu_L$*, for all* $\varepsilon \in]0,1]$)*, there is* $\widehat{H} \in \mathcal{E}_{s,M}[\mathbb{R}^n;\mathbb{R}]$ *verifying the property* $(\mathcal{H}_r)^n$ *and the following conditions:*
$$\sup_{x\in\mathbb{R}^n} |\widehat{H}(\varepsilon,x)| \le \frac{a\varepsilon - b}{\Delta}, \quad (\varepsilon \in]0,1]), \tag{4.1}$$
$$(\forall K \Subset (\mathbb{R}^*)^n)(\exists \eta \in]0,1]) : \sup_{(\varepsilon,x)\in]0,\eta[\times K} |\widehat{H}(\varepsilon,x)| < -\frac{b}{\Delta} \tag{4.2}$$
and $\widehat{H}(\varepsilon,\cdot) \equiv \mu^*(\varepsilon)$ *in* A_ε*,* $(\varepsilon \in]0,1])$*, where*
$$A_\varepsilon := \cup_{i=1}^n \{x \in \mathbb{R}^n : |x_i| \le \varepsilon \text{ and } x_j \ge -\varepsilon,\ (j = 1,\dots,n,\ j \ne i)\}$$
and $\mu^*\colon \varepsilon \in]0,1] \mapsto \frac{\nu^{-1}(\Delta\nu\mu(\varepsilon)+\nu_L)-b}{\Delta} \in [1,-\frac{b}{\Delta}[$*. Furthermore, if* H *is the class of* $\widehat{H}$*, then* $H \in \mathcal{H}_r(\mathbb{R}^n) \setminus \mathcal{H}_\Lambda(\mathbb{R}^n)$.

The proof of the above result follows from proposition 2.7 and lemma 2.3, by a minor modification of the proof in [7, Prop. 4.1].

Proposition 4.5. *Let* $\widehat{H} \in \mathcal{E}_{s,M}[\mathbb{R}^n;\mathbb{R}]$ *be such that* $(\widehat{H},a,b,\Delta)$ *verifies* (4.1) *and let* f *be the class of* $\Delta\widehat{H} + b$*. If* $\nu \in \mathcal{E}_{s,QM}[\mathbb{R}^*_+;\mathbb{R}]$*, then for each* $n \in \mathbb{N}$*,* $n \ge 2$*, there is a strictly increasing function* $\varphi \in C^\infty_{s,QM}[\mathbb{R}^*_+;\mathbb{R}]$ *such that*
$$\frac{\partial(\varphi \circ f)}{\partial\lambda_j} = \frac{\nu \circ f - \alpha}{f^n}\frac{\partial f}{\partial\lambda_j}, \quad (1 \le j \le n)\,.$$
Furthermore, if $\widehat{H}(\varepsilon,\cdot) \to \mathbb{Y}_n$ *in* $(\mathbb{R}^*)^n$ *as* $\varepsilon \downarrow 0$ *and if* $(\widehat{H},b,\Delta)$ *verifies* (4.2) *then* $(\varphi \circ f)|_{(\mathbb{R}^+_-)^n} \approx \varphi(b)$ *and* $(\varphi \circ f)|_{(\mathbb{R}^*_+)^n} \approx \varphi(a)$.

The above result follows from propositions 4.1, 1.2, by a minor modification of the in the proof of [7, Prop. 4.4]. In what follows assume that the elements a and b in hypothesis 4.1 are constant functions. More precisely, we give the following additional hypothesis.

Hypothesis 4.3 (For theorems 4.6, 4.9) We fix $a=(a_1,\dots,a_\ell)$ and $b=(b_1,\dots,b_\ell)$ in $\mathbb{R}^\ell$ with $0<a<b$. For $i=1,\dots,\ell$, let $\Delta_i:=a_i-b_i$. For $\nu=\nu_{(s)}$, $\nu_{(\pi)}$ (see hypothesis 4.1), the components of $(\nu_R,\nu_L,\Delta\nu)$ are given by $\nu_R:=\nu(a)$, $\nu_L:=\nu(b)$ and $\Delta\nu:=\nu_R-\nu_L$.

Theorem 4.6. *We assume that $\nu_i\in\mathcal{E}_{s,QM}[\mathbb{R}_+^*;\mathbb{R}]$ and $\nu_i^{-1}\in\mathcal{E}_{s,QM}[I_{\alpha_i}^{\beta_i};\mathbb{R}]$ for $i\in\mathbb{I}$ and that $\nu_j\equiv\alpha_j$ in $\mathbb{R}_+^*$ for $j\in\mathbb{J}$. Then there are $H_1,\dots,H_\ell$ in $\mathcal{H}_r(\mathbb{R}^n)\setminus\mathcal{H}_\Lambda(\mathbb{R}^n)$ verifying the following properties.*

(a) *Each H_i, $i=1,\dots,\ell$, has representative $\widehat{H}_i$ such that*
$$\sup_{\lambda\in\mathbb{R}^n}|\widehat{H}_i(\varepsilon,\lambda)|\le\frac{a_i\varepsilon-b_i}{\Delta_i},\qquad(\varepsilon\in]0,1]),$$
$$(\forall K\Subset(\mathbb{R}^*)^n)\ (\exists\eta\in]0,1]):\sup_{(\varepsilon,\lambda)\in]0,\eta[\times K}|\widehat{H}_i(\varepsilon,\lambda)|<-\frac{b_i}{\Delta_i}\,.$$

(b) *For each $H\in\mathcal{H}_\Lambda^n(\mathbb{R}^n)$ we have $[\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_\ell H_\ell+b_\ell)-\alpha]\frac{\partial H}{\partial\lambda_j}\approx0$ and $[\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_\ell H_\ell+b_\ell)-\alpha]H\frac{\partial H}{\partial\lambda_j}\approx0$, for $j=1,\dots,n$.*

(c) *For each $H\in\mathcal{H}_\Lambda(\mathbb{R}^n)$ we have $[\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_\ell H_\ell+b_\ell)-\alpha]\partial_\lambda^nH\approx0$ and $[\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_\ell H_\ell+b_\ell)-\alpha]H\partial_\lambda^nH\approx0$.*

Proof. Fix $j\in\mathbb{J}$. Since $-\frac{b_j}{\Delta_j}>1$ and $\frac{a_j\varepsilon-b_j}{\Delta_j}\ge1$, $(\varepsilon\in]0,1])$, by proposition 2.7 and lemma 2.3, there is $H_j\in\mathcal{H}_r(\mathbb{R}^n)\setminus\mathcal{H}_\Lambda(\mathbb{R}^n)$ having a representative $\widehat{H}_j$ verifying the conditions of statement (a). On the other hand, for a fixed $i\in\mathbb{I}$, as ν_i is strictly increasing with $\mathrm{Im}(\nu_i)=I_{\alpha_i}^{\beta_i}$, we can choose $\eta\in]0,1]$ such that $\nu_i(a_i)\varepsilon+\nu_i(a_i\varepsilon)\le\nu_i(a_i)$, $(0<\varepsilon<\eta)$. If $\Delta\nu_i:=\nu_i(a_i)-\nu_i(b_i)$, $\theta_i:=\frac{\alpha_i-\nu_i(b)}{\Delta\nu_i}$ and $\mu_i\colon]0,1]\to[1,\theta_i[$ is given by $\mu_i(\varepsilon):=\frac{\nu_i(a_i)\varepsilon+\nu_i(a_i\varepsilon)-\nu_i(b_i)}{\Delta\nu_i}$ (resp. $:=1$) for $\varepsilon\in]0,\eta[$ (resp. $\varepsilon\in[\eta,1]$) we have that $\nu_i(a_i\varepsilon)\le\Delta\nu_i\mu_i(\varepsilon)+\nu_i(b)$, $(\varepsilon\in]0,1])$, $\lim_{\varepsilon\downarrow0}\mu_i(\varepsilon)=\theta_i$, and [since $\Delta\nu_i\mu_i(\varepsilon)+\nu_i(b)-\alpha_i=\nu_i(a_i)\varepsilon+\nu_i(a_i\varepsilon)-\alpha_i\ge\nu_i(a_i)\varepsilon$, $(\varepsilon\in]0,1])$]
$$\Big(\varepsilon\mapsto\frac{1}{\Delta\nu_i\mu_i(\varepsilon)+\nu_i(b)-\alpha_i}\Big)\in\mathcal{E}_{s,M}(\mathbb{R})\,.$$
Then, as $\nu_i^{-1}\in\mathcal{E}_{s,QM}[I_{\alpha_i}^{\beta_i};\mathbb{R}]$, by proposition 4.4, there is H_i in $\mathcal{H}_r(\mathbb{R}^n)\setminus\mathcal{H}_\Lambda(\mathbb{R}^n)$ having a representative $\widehat{H}_i$ satisfying the conditions of the statement (a) and $\widehat{H}_i(\varepsilon,\cdot)\equiv\mu_i^*(\varepsilon)$ in A_ε, where $\mu_i^*\colon\varepsilon\in]0,1]\mapsto\frac{\nu_i^{-1}(\Delta\nu_i\mu_i(\varepsilon)+\nu_i(b_i))-b_i}{\Delta_i}\in[1,-\frac{b_i}{\Delta_i}[$ and $A_\varepsilon:=\cup_{k=1}^n\{\lambda\in\mathbb{R}^n:|\lambda_k|\le\varepsilon$ and $\lambda_r\ge-\varepsilon$, $(r=1,\dots,n,\,r\ne k)\}$. Therefore, $\Big[\nu_i\circ\Big(\Delta_i\widehat{H}_i\Big)+b_i\Big](\varepsilon,\cdot)\equiv\Delta\nu_i\mu_i(\varepsilon)+\nu_i(b_i)$ in A_ε, $(\varepsilon\in]0,1]$, $i\in\mathbb{I})$. If $\widehat{f}:=\nu\circ(\Delta_1\widehat{H}_1+b_1,\dots,\Delta_\ell\widehat{H}_\ell+b_\ell)$, we have the formulas
$$\nu_{(s)}\circ\widehat{f}=\sum_{i\in\mathbb{I}}\nu_i\circ\big(\Delta_i\widehat{H}_i+b_i\big)+\sum_{j\in\mathbb{J}}\alpha_j,\quad\nu_{(\pi)}\circ\widehat{f}=\prod_{i\in\mathbb{I}}\big[\nu_i\circ\big(\Delta_i\widehat{H}_i+b_i\big)\big]\prod_{j\in\mathbb{J}}\alpha_j$$
If $\mu_{(s)}\colon]0,1]\to[\alpha_{(s)},\nu_{(s)}(a)[$ is defined by
$$\mu_{(s)}(\varepsilon):=\sum_{i\in\mathbb{I}}[\Delta\nu_i\mu_i(\varepsilon)+\nu_i(b_i)]+\sum_{j\in\mathbb{J}}\alpha_j\,,\quad(\varepsilon\in]0,1])$$
we have that $(\nu_{(s)}\circ\widehat{f})(\varepsilon,\cdot)\equiv\mu_{(s)}(\varepsilon)$ in A_ε, $(\varepsilon\in]0,1])$, and (since $\mu_i(\varepsilon)\to\theta_i$ as $\varepsilon\downarrow0$, for each $i\in\mathbb{I}$) we also have that $\mu_{(s)}(\varepsilon)\to\alpha_{(s)}$ as $\varepsilon\downarrow0$. In a similar way, if

$\mu_{(\pi)}\colon]0,1] \to [\alpha_{(\pi)}, \nu_{(\pi)}(a)[$ is defined by

$$\mu_{(\pi)}(\varepsilon) := \prod_{i\in\mathbb{I}}[\Delta\nu_i\mu_i(\varepsilon) + \nu_i(b_i)] \prod_{j\in\mathbb{J}} \alpha_j\,, \quad (\varepsilon \in]0,1])$$

we have that $\mu_{(\pi)}(\varepsilon) \to \alpha_{(\pi)}$ as $\varepsilon \downarrow 0$ and that $(\nu_{(\pi)} \circ \widehat{f})(\varepsilon,\cdot) \equiv \mu_{(\pi)}(\varepsilon)$ in A_ε, $(\varepsilon \in]0,1])$. Thus, if μ indistinctly denotes $\mu_{(s)}$ or $\mu_{(\pi)}$, we have that $\mu(\varepsilon) \to \alpha$ as $\varepsilon \downarrow 0$ and that $(\nu \circ \widehat{f})(\varepsilon,\cdot) \equiv \mu(\varepsilon)$ in A_ε, $(\varepsilon \in]0,1])$.

(b) For a fix $H \in \mathcal{H}^n_\Lambda(\mathbb{R}^n)$, let $\phi \in \Lambda(1)$ be such that H is the class of the function $\widehat{H}\colon]0,1] \times \mathbb{R}^n \to \mathbb{R}$ defined by $\widehat{H}(\varepsilon,\lambda) := \widehat{K}(\varepsilon,\lambda_1)\dots\widehat{K}(\varepsilon,\lambda_n)$ where $\widehat{K}\colon (\varepsilon,s) \in]0,1] \times \mathbb{R} \mapsto \int_{-\varepsilon}^{s} \widehat{\delta}(\varepsilon,t)dt \in \mathbb{R}$ and $\widehat{\delta}\colon (\varepsilon,t) \mapsto \frac{1}{\varepsilon}\phi(\frac{t}{\varepsilon})$. To check that H verifies the required relations, for the sake of simplicity, it will be worked with $\frac{\partial H}{\partial\lambda_1}$. To prove the first, fix $\varphi \in \mathcal{D}(\mathbb{R}^n)$, it is sufficient to see that $I(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ where

$$I(\varepsilon) := \int_{\mathbb{R}^n} [(\nu \circ \widehat{f})(\varepsilon,\lambda) - \alpha]\frac{\partial \widehat{H}}{\partial \lambda_1}(\varepsilon,\lambda)\varphi(\lambda)d\lambda\,, \quad (\varepsilon \in]0,1])\,.$$

Since $(\nu \circ \widehat{f})(\varepsilon,\cdot) \equiv \mu(\varepsilon)$ in A_ε, $\frac{\partial \widehat{H}}{\partial\lambda_1}(\varepsilon,\lambda) = \widehat{\delta}(\varepsilon,\lambda_1)\widehat{K}(\varepsilon,\lambda_2)\dots\widehat{K}(\varepsilon,\lambda_n)$ and

$$\operatorname{supp}[\frac{\partial \widehat{H}}{\partial \lambda_1}(\varepsilon,\cdot)] \subset \{\lambda \in \mathbb{R}^n : |\lambda_1| \le \varepsilon \text{ and } \lambda_k \ge -\varepsilon \text{ for some } k = 2,\dots,n\} \subset A_\varepsilon$$

we have $I(\varepsilon) = [\mu(\varepsilon) - \alpha]\int_{-\varepsilon}^{\varepsilon} \widehat{\delta}(\varepsilon,\lambda_1)u(\varepsilon,\lambda_1)d\lambda_1$ where

$$u(\varepsilon,\lambda_1) := \int_{-\varepsilon}^{+\infty}\dots\int_{-\varepsilon}^{+\infty} \widehat{K}(\varepsilon,\lambda_2)\dots\widehat{K}(\varepsilon,\lambda_n)\varphi(\lambda_1,\lambda_2,\dots,\lambda_n)d\lambda_2\dots d\lambda_n\,.$$

Choosing $C > 0$ such that $|u(\varepsilon,\lambda_1)| \le C$ for all $(\varepsilon,\lambda_1) \in]0,1] \times \mathbb{R}$ we have

$$|I(\varepsilon)| \le C|\mu(\varepsilon) - \alpha| \int_{-\varepsilon}^{\varepsilon} \widehat{\delta}(\varepsilon,\lambda_1)d\lambda_1 = C|\mu(\varepsilon) - \alpha|\,, \;\; (\varepsilon \in]0,1])$$

thus it follows that $I(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. Proving the second relation and the statement (c) follows in similar way. ☐

What follows we present an auxiliary lemma of theorem 4.8.

Lemma 4.7. *Given $\nu \in C^\infty(\mathbb{R}^*_+;\mathbb{R})$ with $\operatorname{Im}(\nu) = I^\beta_\alpha$, α, β in $\tilde{\mathbb{R}}^*_+$ $(\alpha < \beta)$ and $\theta > 0$ we consider $\varphi\colon y \in \mathbb{R}^*_+ \mapsto \int_\theta^y \frac{\nu(s)-\alpha}{s^2}ds \in \mathbb{R}$. If ν is strictly increasing and $\nu \in \mathcal{E}_{s,QM}[\mathbb{R}^*_+;\mathbb{R}]$, then φ is strictly increasing, $\varphi \in C^\infty_{s,QM}[\mathbb{R}^*_+;\mathbb{R}]$ and*

$$|\varphi(y)| \le \begin{cases} [\nu(\theta) - \alpha](\frac{1}{y} - \frac{1}{\theta}), & \text{if } y < \theta \\ [\nu(y) - \alpha](\frac{1}{\theta} - \frac{1}{y}), & \text{if } y \ge \theta\,. \end{cases} \tag{4.3}$$

Proof. Since $\varphi'(y) = \frac{\nu(y)-\alpha}{y^2}$ and $\nu(y) > \alpha$, $(y > 0)$, it follows that φ is strictly increasing, and hence

$$\text{If } A, B \in \mathbb{R}^*_+ \text{ and } y \in [A,B], \text{ then } |\varphi(y)| \le |\varphi(A)|, \text{ or } |\varphi(y)| \le |\varphi(B)|. \tag{4.4}$$

To check $\varphi \in \mathcal{E}_{s,QM}[\mathbb{R}^*_+;\mathbb{R}]$, fix $p \in \mathbb{N}$, $a, b \in \mathbb{R}^*_+$, $a < b$, and μ from $]0,1]$ in $]0,a]$ such that $(\varepsilon \mapsto \frac{1}{\mu(\varepsilon)}) \in \mathcal{E}_{s,M}(\mathbb{R})$, we must find $N \in \mathbb{N}$, $C > 0$ and $\eta \in]0,1]$ verifying

$$\sup_{y\in[\mu(\varepsilon),b]} |\varphi^{(p)}(y)| \le C\varepsilon^{-N}, \quad (0 < \varepsilon < \eta). \tag{4.5}$$

In fact, since $\nu \in \mathcal{E}_{s,QM}[\mathbb{R}_+^*;\mathbb{R}]$ and $\varphi^{(p)}(y) = \sum_{q=0}^{p-1} C_q^p(\nu-\alpha)^{(q)}(y)y^{q-p-n+1}$, $(y>0,\, p\geq 1)$, where

$$C_q^p := (-1)^{p-q-1}n(n+1)\dots(n+p-q-2)\binom{p-1}{q}$$

it is enough to check (4.5) in the case $p=0$. For $b' := \max\{\theta, b\}$, let $N \in \mathbb{N}$ be, $C>0$ and $\eta \in]0,1]$ such that

$$\sup_{s\in[\mu(\varepsilon),b']} |\nu(s)-\alpha| \leq C\varepsilon^{-N}, \quad (0<\varepsilon<\eta). \tag{4.6}$$

For a fixed $\varepsilon \in]0,1]$ and $y \in [\mu(\varepsilon), b]$, we have $|\varphi(y)| \leq |\varphi(b)|$ or $|\varphi(y)| \leq |\varphi(\mu(\varepsilon))|$ (by (4.4)). In the last case, by definition of φ and using (4.6), it follows that

$$|\varphi(y)| \leq C\varepsilon^{-N}|\frac{1}{\theta} - \frac{1}{\mu(\varepsilon)}|, \quad (0<\varepsilon<\eta,\ \ y>0)$$

which together with condition $(\varepsilon \mapsto \frac{1}{\mu(\varepsilon)}) \in \mathcal{E}_{s,M}(\mathbb{R})$ implies (4.5).

To prove the inequalities in (4.3), let $y \in \mathbb{R}_+^*$ be fixed. If $y<\theta$ we have

$$\varphi(y) = \int_\theta^y \frac{\nu(s)-\alpha}{s^2}ds = \int_y^\theta \frac{\alpha-\nu(s)}{s^2}ds$$

thus, as $\nu(s)>\alpha$ for $s>0$, it follows that

$$|\varphi(y)| \leq \int_y^\theta \frac{|\alpha-\nu(s)|}{s^2}ds = \int_y^\theta \frac{\nu(s)-\alpha}{s^2}ds$$

and hence, since $\nu(s) \leq \nu(\theta)$ for $s \in [y,\theta]$, we have

$$|\varphi(y)| \leq [\nu(\theta)-\alpha]\int_y^\theta \frac{1}{s^2}ds = [\nu(\theta)-\alpha]\left(\frac{1}{y}-\frac{1}{\theta}\right).$$

On the other hand, if $y \geq \theta$ we have $|\varphi(y)| = \varphi(y) = \int_\theta^y \frac{\nu(s)-\alpha}{s^2}ds$ and thus, being $\alpha(s) \leq \nu(y)$ for $s \in [\theta, y]$, it follows that

$$|\varphi(y)| \leq [\nu(y)-\alpha]\int_\theta^y \frac{1}{s^2}ds = [\nu(y)-\alpha]\left(\frac{1}{\theta}-\frac{1}{y}\right).$$

□

Hypothesis 4.4 (For theorem 4.8) We fix $(\alpha,\beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\alpha<\beta$, and ν in $C^\infty(\mathbb{R}_+^*;\mathbb{R})$ with $\mathrm{Im}(\nu) = I_\alpha^\beta$. For each $\tau = \rho$, u also will be considered

- $\tau_r, \tau_\ell \in C^\infty(\mathbb{R}^m;\mathbb{R})$ such hat $0<\tau_r(t)<\tau_\ell(t)$, $(t\in\mathbb{R}^m)$
- $\Delta\tau \in C^\infty(\mathbb{R}^m;\mathbb{R})$ defined by $\Delta\tau := \tau_r - \tau_\ell$
- $\widehat{H}_\tau \in \mathcal{E}_{s,M}[\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}]$ such that $\widehat{H}_\tau(\varepsilon,\cdot) \to \mathbb{Y}_{nm}$ in $(\mathbb{R}^*)^n\times\mathbb{R}^m$ as $\varepsilon\downarrow 0$ and

$$\sup_{\lambda\in\mathbb{R}^n} |\widehat{H}_\tau(\varepsilon,\lambda,t)| \leq \frac{\tau_r(t)\varepsilon - \tau_\ell(t)}{\Delta\tau(t)}, \quad ((\varepsilon,t)\in]0,1]\times\mathbb{R}^m) \tag{4.7}$$

- the generalized function τ_* as being the class of $\widehat{\tau}_* := \Delta\tau\widehat{H}_\tau + \tau_\ell$.

Theorem 4.8. *If ν is strictly increasing and $\nu \in \mathcal{E}_{s,QM}[\mathbb{R}_+^*;\mathbb{R}]$, then there is a strictly increasing function $\varphi \in C_{s,QM}^\infty[\mathbb{R}_+^*;\mathbb{R}]$ satisfying the following conditions*

$$\begin{gathered}\frac{\partial(\varphi\circ\rho_*)}{\partial\lambda_j} = \frac{\nu\circ\rho_* - \alpha}{\rho_*^2}\frac{\partial\rho_*}{\partial\lambda_j}, \quad (1\le j\le n),\\ \nabla_t(\varphi\circ\rho_*) = \frac{\nu\circ\rho_*-\alpha}{\rho_*^2}\nabla_t\rho_*;\end{gathered} \tag{4.8}$$

$$\lim_{\varepsilon\downarrow 0}[\widehat{\rho}_*(\varphi\circ\widehat{\rho}_*)](\varepsilon,\cdot) = (\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)[\varphi\circ(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)] \quad \text{in } \mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^m); \tag{4.9}$$

$$\begin{gathered}|_{(\mathbb{R}_-^+)^n\times\mathbb{R}^m} \approx \rho_\ell(u_\ell - c)(\varphi\circ\rho_\ell),\\ [(u_*-c)\rho_*(\varphi\circ\rho_*)]|_{(\mathbb{R}_+^*)^n\times\mathbb{R}^m} \approx \rho_r(u_r-c)(\varphi\circ\rho_r);\end{gathered} \tag{4.10}$$

for all $c\in C^\infty(\mathbb{R}^m;\mathbb{R})$. In particular, if c, (ρ_r,ρ_ℓ) and (u_r,u_ℓ) are constant functions, we have $\nabla_t[\rho_*(\varphi\circ\rho_*)]\approx 0$, $[(u_*-c)\rho_*(\varphi\circ\rho_*)]|_{(\mathbb{R}_-^+)^n\times\mathbb{R}^m}\approx \rho_\ell(u_\ell-c)\varphi(\rho_\ell)$ and $[(u_*-c)\rho_*(\varphi\circ\rho_*)]|_{(\mathbb{R}_+^*)^n\times\mathbb{R}^m}\approx\rho_r(u_r-c)\varphi(\rho_r)$.

Proof. Fix $\theta>0$. By lemma 4.7, $\varphi\colon y\mapsto\int_\theta^y\frac{\nu(s)-\alpha}{s^2}ds$ is strictly increasing, φ in $C_{QM}^\infty[\mathbb{R}_+^*;\mathbb{R}]$ and satisfies the inequalities (4.3). From (4.7) and definition of $\widehat{\tau}_*$, $\tau=\rho$, u, it follows that

$$\widehat{\tau}_*(\varepsilon,\lambda,t)\in[\varepsilon\tau_r(t),2\tau_\ell(t)] \quad \text{for all } (\varepsilon,\lambda,t)\in]0,1]\times\mathbb{R}^n\times\mathbb{R}^m\,.$$

Fix $K\Subset\mathbb{R}^m$. For each $\tau=\rho$, u, let $A_\tau>0$ and $B_\tau>0$ be such that $A_\tau\le\tau_r(t)$ and $2\tau_\ell(t)\le B_\tau$, $(t\in K)$. From above conditions it follows that

$$\widehat{\tau}_*(\varepsilon,\lambda,t)\in[A_\tau\varepsilon,B_\tau], \quad ((\varepsilon,\lambda,t)\in]0,1]\times\mathbb{R}^n\times K,\,\tau=\rho,\,u)\,. \tag{4.11}$$

This condition shows that $\widehat{\rho}_*\in\mathcal{E}_{s,M,\otimes}[\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}_+^*]$ and, by proposition 4.2, ρ_* has multiplicative inverse in $\mathcal{G}_s(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R})$ and $\frac{1}{\widehat{\rho}_*}$ is a representative of $\frac{1}{\rho_*}$. By the chain rule we have

$$\begin{gathered}\frac{\partial(\varphi\circ\rho_*)}{\partial\lambda_j} = (\varphi'\circ\rho_*)\frac{\partial\rho_*}{\partial\lambda_j}, \quad (j=1,\dots,n)\\ \frac{\partial(\varphi\circ\rho_*)}{\partial t_i} = (\varphi'\circ\rho_*)\frac{\partial\rho_*}{\partial t_i}, \quad (i=1,\dots,m).\end{gathered}$$

From $\varphi'(y)=\frac{\nu(y)-\alpha}{y^2}$ it follows that $\varphi'\circ\widehat{\rho}_*=\frac{\nu\circ\widehat{\rho}_*-\alpha}{\widehat{\rho}_*^2}$ and hence $\varphi'\circ\rho_*=\frac{\nu\circ\rho_*-\alpha}{\rho_*^2}$ which together with previous formulas implies the required equalities in (4.8).

On the other hand, from (4.3) we have

$$\begin{gathered}|y\varphi(y)|\le[\nu(\theta)-\alpha](1-\frac{y}{\theta}), \quad \text{if } y<\theta\\ |y\varphi(y)|\le[\nu(y)-\alpha](\frac{y}{\theta}-1), \quad \text{if } y\ge\theta\,.\end{gathered}$$

Thus, noting that $0<1-\frac{y}{\theta}<1$ and by using (4.11) two times in the second inequality, we have

$$\begin{gathered}|[\widehat{\rho}_*(\varphi\circ\widehat{\rho}_*)](\varepsilon,\lambda,t)|\le[\nu(\theta)-\alpha], \quad \text{if } \widehat{\rho}_*(\varepsilon,\lambda,t)<\theta\\ |[\widehat{\rho}_*(\varphi\circ\widehat{\rho}_*)](\varepsilon,\lambda,t)|\le[\nu(B_\rho)-\alpha](\frac{B_\rho}{\theta}-1), \quad \text{if } \widehat{\rho}_*(\varepsilon,\lambda,t)\ge\theta\end{gathered}$$

for each $(\varepsilon,\lambda,t)\in]0,1]\times\mathbb{R}^n\times K$. Thus, if

$$M:=\max\{[\nu(\theta)-\alpha],[\nu(B_\rho)-\alpha](\frac{B_\rho}{\theta}-1)\}$$

we have $\sup_{(\lambda,t)\in\mathbb{R}^n\times K}|[\widehat{\rho}_*(\varphi\circ\widehat{\rho}_*)](\varepsilon,\lambda,t)|\le M$, $(0<\varepsilon\le 1)$, which together with (4.11) implies

$$\sup_{(\lambda,t)\in\mathbb{R}^n\times K}|[(\widehat{u}_*-c)\widehat{\rho}_*(\varphi\circ\widehat{\rho}_*)]\,(\varepsilon,\lambda,t)|\le M(B_u+B_c),\ \ (0<\varepsilon\le 1),$$

where $B_c>0$ is such that $\sup_{t\in K}|c(t)|\le B_c$. From the last two conditions it follows that $\rho_*(\varphi\circ\rho_*)$, $(u_*-c)\rho_*(\varphi\circ\rho_*)\in\mathcal{G}_{s,\ell b}(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R})$.

From hypothesis 4.4, remarking that $|\Delta\tau(t)|>0$, $(t\in\mathbb{R}^m)$, we have

$$\widehat{\tau}_*(\varepsilon,\cdot)\to\Delta\tau\mathbb{Y}_{nm}+\tau_\ell\quad\text{in }(\mathbb{R}*)^n\times\mathbb{R}^m\text{ as }\varepsilon\downarrow 0,\quad(\tau=\rho,\,u)\tag{4.12}$$

and thus $(\varphi\circ\widehat{\rho}_*)(\varepsilon,\cdot)\to\varphi\circ(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)$ in$(\mathbb{R}^*)^n\times\mathbb{R}^m$ as $\varepsilon\downarrow 0$. From these two conditions it follows that

$$\lim_{\varepsilon\downarrow 0}[\widehat{\rho}_*(\varphi\circ\widehat{\rho}_*)](\varepsilon,\cdot)=(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)\,[\varphi\circ(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)]\quad\text{in }(\mathbb{R}^*)^n\times\mathbb{R}^m.\tag{4.13}$$

Like this, as $\rho_*(\varphi\circ\rho_*)\in\mathcal{G}_{s,\ell b}(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R})$, by proposition 1.2 the statement (4.9) is verified.

Using (4.12) and (4.13) we have

$$\begin{gathered}[(\widehat{u}_*-c)\widehat{\rho}_*(\varphi\circ\widehat{\rho}_*)]\,(\varepsilon,\cdot)\to\rho_\ell(u_\ell-c)(\varphi\circ\rho_\ell)\quad\text{in }(\mathbb{R}^+_-)^n\times\mathbb{R}^m\text{ as }\varepsilon\downarrow 0\\ [(\widehat{u}_*-c)\widehat{\rho}_*(\varphi\circ\widehat{\rho}_*)]\,(\varepsilon,\cdot)\to\rho_r(u_r-c)(\varphi\circ\rho_r)\quad\text{in }(\mathbb{R}^*_+)^n\times\mathbb{R}^m\text{ as }\varepsilon\downarrow 0.\end{gathered}$$

Therefore, as $(u_*-c)\rho_*(\varphi\circ\rho_*)\in\mathcal{G}_{s,\ell b}(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R})$ and $\rho_\ell(u_\ell-c)(\varphi\circ\rho_\ell)$, $\rho_r(u_r-c)(\varphi\circ\rho_r)\in C^\infty(\mathbb{R}^m;\mathbb{R})$, by proposition 1.2 the statements in (4.10) hold.

Finally, if ρ_r and ρ_ℓ are constants we have

$$(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)[\varphi\circ(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)]=\begin{cases}\rho_\ell\varphi(\rho_\ell), & \text{in }(\mathbb{R}^+_-)^n\times\mathbb{R}^m\\ \rho_r\varphi(\rho_r), & \text{in }(\mathbb{R}^*_+)^n\times\mathbb{R}^m\end{cases}$$

and hence, by lemma 3.1, we have

$$\frac{\partial}{\partial t_i}\big\{(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)\,[\varphi\circ(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)]\,\big\}=0\quad\text{in }\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^m),\ (i=1,\dots,n)$$

and, from hypothesis 4.4, for each $i=1,\dots,n$, we have

$$\lim_{\varepsilon\downarrow 0}\left\{\frac{\partial}{\partial t_i}\,[\widehat{\rho}_*(\varphi\circ\widehat{\rho}_*)]\right\}(\varepsilon,\cdot)=\frac{\partial}{\partial t_i}\,\{(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)\,[\varphi\circ(\Delta\rho\mathbb{Y}_{nm}+\rho_\ell)]\}$$

in $\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^m)$. From these two conditions it follows that $\nabla_t[\rho_*(\varphi\circ\rho_*)]\approx 0$. $\square$

Theorem 4.9. *We assume that $\nu_i\in\mathcal{E}_{s,QM}[\mathbb{R}^*_+;\mathbb{R}]$ and $\nu_i^{-1}\in\mathcal{E}_{s,QM}[I^{\beta_i}_{\alpha_i};\mathbb{R}]$ for each $i\in\mathbb{I}$ and that $\nu_j\equiv\alpha_j$ in $\mathbb{R}^*_+$ for each $j\in\mathbb{J}$. Then there are generalized functions $H_1,\dots,H_\ell$ in $\mathcal{H}_r(\mathbb{R}^n\times\mathbb{R}^m)\setminus\mathcal{H}_\Lambda(\mathbb{R}^n\times\mathbb{R}^m)$ verifying the following properties.*

(1) *Each H_i, $i=1,\dots,\ell$, has representative $\widehat{H}_i$ such that*

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}^m}|\widehat{H}_i(\varepsilon,x,t)|\le\frac{a_i\varepsilon-b_i}{\Delta_i},\quad(\varepsilon\in]0,1]),$$

$$(\forall K\Subset(\mathbb{R}^*)^n)\,(\exists\eta\in]0,1]):\sup_{(\varepsilon,x,t)\in]0,\eta[\times K\times\mathbb{R}^m}|\widehat{H}_i(\varepsilon,x,t)|<-\frac{b_i}{\Delta_i}.$$

(2) *If $H\in\mathcal{H}^{n,m}_\Lambda(\mathbb{R}^n\times\mathbb{R}^m)$ we have $[\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_\ell H_\ell+b_\ell)-\alpha]\frac{\partial H}{\partial x_j}\approx 0$ and $[\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_\ell H_\ell+b_\ell)-\alpha]H\frac{\partial H}{\partial x_j}\approx 0$, for $j=1,\dots,n$.*

(3) *If $H\in\mathcal{H}_\Lambda(\mathbb{R}^n\times\mathbb{R}^m)$ we have $[\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_\ell H_\ell+b_\ell)-\alpha]\partial^n_x H\approx 0$ and $[\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_\ell H_\ell+b_\ell)-\alpha]H\partial^n_x H\approx 0$.*

The proof of the above theorem follows from theorem 4.6 and propositions 3.3, 2.6.

Remark 4.10. (1) The existence of shock wave solutions for the following system of hydrodynamic equations, with viscosity ν, was studied in [7]:

$$\begin{gathered}
\rho_t + (\rho u)_x \approx 0 \\
(\rho u)_t + (p + \rho u^2)_x \approx \{[\nu \circ (\rho, p, e) - \alpha] u_x\}_x \\
e_t + [(e+p)u]_x \approx \{[\nu \circ (\rho, p, e) - \alpha] u u_x\}_x \\
e \approx \lambda p + \frac{1}{2}\rho u^2, \quad \lambda \in \mathbb{R}^*,
\end{gathered} \tag{4.14}$$

where ρ is the density, u the velocity, p the pressure, e the total energy, ν in $C^\infty\big((\mathbb{R}_+^*)^3; I_\alpha^\beta\big)$ (α, $\beta \in \tilde{\mathbb{R}}_+$, $\alpha < \beta$) satisfying some adequate conditions and $\approx$ denotes the association relation in $\mathcal{G}_s(\mathbb{R}^2;\mathbb{R})$. In [7] was also studied the nonexistence of shock wave solutions for the system obtained by replacing in the two first equations of (4.14) the association relation by the equality of generalized functions. In [2] was studied the existence and nonexistence of shock wave solutions for the systems: (4.14) and the system obtained by replacing in the first equation of (4.14) the association by the equality, in the cases $\alpha = 0$ and $\nu \circ (\rho, p, e) = \nu \circ (\rho)$. These studies were realized using Heaviside generalized functions in variables x in $\mathbb{R}$.

(2) The existence of shock wave solutions for the following system, using Heaviside generalized functions in variables x in $\mathbb{R}^n$, was studied in [9]:

$$\begin{gathered}
\rho_t + \operatorname{div}_x(\rho u) \approx 0 \\
(\rho u)_t + \operatorname{div}_x(p + \rho u^2) \approx 0 \\
e_t + \operatorname{div}_x[(e+p)u] \approx 0 \\
e \approx \lambda p + \frac{1}{2}(\rho u^2), \quad \lambda \in \mathbb{R}^*
\end{gathered}$$

where ρ, u, p and e are real generalized functions in $\mathbb{R}^n \times \mathbb{R}$. The introduction of this system was suggested by the system considered in (1), in the case $\nu = 0$.

(3) The existence and nonexistence of shock wave solutions for Burger's equations $u_t + u \operatorname{div}_x u \approx 0$ and $u_t + u \operatorname{div}_x u = 0$, where $u \in \mathcal{G}_s(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$, was studied in [10], using Heaviside generalized functions in variables x in $\mathbb{R}^n$ and in variables (x,t) in $\mathbb{R}^n \times \mathbb{R}$.

(4) The existence and nonexistence of shock wave solutions for systems suggested by the systems in (1), in the case $\nu \neq 0$, are being studied by the author of this article. These studies are realized using Heaviside generalized functions in variables x in $\mathbb{R}^n$ and in variables (x,t) in $\mathbb{R}^n \times \mathbb{R}$ and the tools considered in this work. The results will be stated in a work which is in preparation and it will appear in a forthcoming publication.

5. Appendix: notation and definitions

- $A := B$ means that A is defined as being equal to B
- $\mathbb{R}^* := \{x \in \mathbb{R} \colon x \neq 0\}$, $\mathbb{R}_+^* := \{x \in \mathbb{R}^* \colon x > 0\}$, $\tilde{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$
- $(\mathbb{R}_-^+)^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_j < 0 \text{ for some } j = 1, \dots, n\}$
- $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$
- $B_r[0] := \{x \in \mathbb{R}^n \colon |x| \leq r\}$, $(r > 0)$
- $K \Subset \Omega$ means that K is a compact subset of Ω

- $\mathcal{L}(F_1, \dots, F_\ell; G)$ is the space of continuous ℓ-linear mappings from $F_1 \times \dots \times F_\ell$ into G endowed with the norm

$$|A|_\ell = \sup_{|y_i|=1,\, 1\le i\le \ell} |A(y_1, \dots, y_\ell)|$$

  for $A \in \mathcal{L}(F_1, \dots, F_\ell; G)$. When $F_1 = \dots = F_\ell = F$ this space is denoted by $\mathcal{L}({}^\ell F; G)$, and $\mathcal{L}({}^0 F; G) =: G$
- $\mathcal{E}_s[\Omega; F] := \{u \in F^{]0,1]\times\Omega} \colon u(\varepsilon, \cdot) \in C^\infty(\Omega; F) \text{ for all } \varepsilon \in]0,1]\}$
- if $p \in \mathbb{N}$, $K \Subset \Omega$ and $u \in \mathcal{E}_s[\Omega; F]$ $\big(u^{(p)} \in \mathcal{E}_s[\Omega; \mathcal{L}({}^p E; F)]\big)$,
  $u^{(p)}(\varepsilon, x) := [u(\varepsilon, \cdot)]^{(p)}(x)$ and $|u^{(p)}(\varepsilon, \cdot)|_{p,K} := \sup_{x\in K} |u^{(p)}(\varepsilon, x)|_p$
- $\mathcal{E}_{s,M}[\Omega; F]$, $\mathcal{N}_s[\Omega; F]$ and $\mathcal{G}_s(\Omega; F)$ are defined on page 2
- $\mathcal{G}_{s,\ell b}(\Omega; F)$ is defined on page 2
- $(f_1, \dots, f_\ell)$ denotes the class of the function

$$(\widehat{f}_1, \dots, \widehat{f}_\ell) \colon]0,1] \times \Omega \to F_1 \times \dots \times F_\ell, \quad (\varepsilon, x) \mapsto \Big(\widehat{f}_1(\varepsilon, x), \dots, \widehat{f}_\ell(\varepsilon, x)\Big)$$

- $\mathcal{E}_{s,M}[\Omega; \Omega'] := \{u \in \mathcal{E}_{s,M}[\Omega; F] \colon u(]0,1] \times \Omega) \subset \Omega'\}$
- $\mathcal{E}_{s,M,*}[\Omega; \Omega']$ and $\mathcal{E}_{s,M,*}[\Omega; \Omega']$ are defined on page 3
- If $(f, g) \in \mathcal{G}_{s,*}(\Omega; \Omega') \times \mathcal{G}_s(\Omega'; G)$, then $g \circ f := \widehat{g} \circ \widehat{f} + \mathcal{N}_s[\Omega; G]$
- $\mathcal{E}_{s,M}(F)$, $\mathcal{N}_s(F)$ and $\bar{F}_s$ are dfined on page 3
- Definition of $f \approx g$ for f and g in $\mathcal{G}_s(\Omega; \mathbb{K}^\ell)$ are provided on page 3
- $\nabla_t f := \Big(\frac{\partial f}{\partial t_1}, \dots, \frac{\partial f}{\partial t_m}\Big)$, $\operatorname{div}_x f := \frac{\partial f}{\partial x_1} + \dots + \frac{\partial f}{\partial x_n}$,
  $\partial_x^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, $\partial_x^n f := \frac{\partial^n f}{\partial x_1 \dots \partial x_n}$
- Definition of Dirac GFs in $\mathcal{G}_s(\mathbb{R}^n)$ is provided on page 4
- $\mathbb{Y}_n \colon \mathbb{R}^n \to \mathbb{R}$ denotes the Heaviside function in $\mathbb{R}^n$
- $\mathcal{H}(\mathbb{R}^n)$ is the set of Heaviside GFs in $\mathbb{R}^n$
- $\mathcal{H}_p(\mathbb{R}^n)$ is the set of H in $\mathcal{G}_{s,\ell b}(\mathbb{R}^n)$ so that has representative $\widehat{H}$ such that $\widehat{H}(\varepsilon, \cdot) \to \mathbb{Y}_n$ in $(\mathbb{R}^*)^n$ as $\varepsilon \downarrow 0$
- Property $(\mathcal{H}_r)^n$ is defined on page 5
- $\mathcal{H}_r(\mathbb{R}^n)$ is the set of elements of $\mathcal{G}_{s,\ell b}(\mathbb{R}^n)$ so that each one has representative verifying the property $(\mathcal{H}_r)^n$
- $\Lambda(n) := \{\varphi \in \mathcal{D}(\mathbb{R}^n) \colon \varphi \ge 0, \varphi(0) > 0, \operatorname{supp}(\varphi) \subset B_1[0] \text{ and } \int \varphi(\lambda)\, d\lambda = 1\}$
- If $\varphi \in \Lambda(n)$, then $\widehat{H}_\varphi \colon]0,1] \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\widehat{H}_\varphi(\varepsilon, \lambda_1, \dots, \lambda_n) := \int_{-\infty}^{\lambda_1} \dots \int_{-\infty}^{\lambda_n} \frac{1}{\varepsilon^n} \varphi\Big(\frac{t_1}{\varepsilon}, \dots, \frac{t_n}{\varepsilon}\Big) dt_1 \dots dt_n$$

- H_φ denotes the class of $\widehat{H}_\varphi$
- $\mathcal{H}_\Lambda^n(\mathbb{R}^n) := \{(K \circ p_1) \dots (K \circ p_n) \colon K \in \mathcal{H}_\Lambda(\mathbb{R})\}$
- $\mathcal{H}_\Lambda(\mathbb{R}^n) := \{H_\varphi \in \mathcal{G}_s(\mathbb{R}^n) \colon \varphi \in \Lambda(n)\}$
- If $y \in C^\infty(\mathbb{R}^m; \mathbb{R}^n)$, we have $y^* \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$: $(x, t) \mapsto x - y(t)$ and $y_* : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$: $(x, t) \mapsto (y^*(x, t), t)$; which are the functions associated with $y = (y_1, \dots y_n)$
- $\Omega^* := \Omega_-^+ \cup \Omega_+$, $\Omega_-^+ := \{(x, t) \colon y_j^*(x, t) < 0 \text{ for some } j = 1, \dots, n\}$, $\Omega_+ := \{(x, t) \colon y_j^*(x, t) > 0 \text{ for all } j = 1, \dots, n\}$, which are the sets associated with $y = (y_1, \dots y_n) \in C^\infty(\mathbb{R}^m; \mathbb{R}^n)$
- $\mathbb{Y}_{nm} \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ denotes the *Heaviside function in* $\mathbb{R}^n \times \mathbb{R}^m$
- $\mathcal{H}(\mathbb{R}^n \times \mathbb{R}^m)$ is the set of Heaviside GFs in $\mathbb{R}^n \times \mathbb{R}^m$

- $\mathcal{H}_p(\mathbb{R}^n \times \mathbb{R}^m)$ is the set of $H \in \mathcal{G}_{s,\ell b}(\mathbb{R}^n \times \mathbb{R}^m)$ so that $\widehat{H}(\varepsilon, \cdot) \to \mathbb{Y}_{nm}$ in $(\mathbb{R}^*)^n \times \mathbb{R}^m$ as $\varepsilon \downarrow 0$ for some representative $\widehat{H}$
- Property $(\mathcal{H}_r)^n_m$ is defined on page 9
- $\mathcal{H}_r(\mathbb{R}^n \times \mathbb{R}^m)$ is the set of elements in $\mathcal{G}_{s,\ell b}(\mathbb{R}^n \times \mathbb{R}^m)$ so that each one has representative verifying the property $(\mathcal{H}_r)^n_m$;
- $\mathcal{H}_\Lambda(\mathbb{R}^n \times \mathbb{R}^m) := \{H \circ \pi_m \colon H \in \mathcal{H}_\Lambda(\mathbb{R}^n)\} = \{H_\varphi \circ \pi_m \colon \varphi \in \Lambda(n)\}$
- $\mathcal{H}^{n,m}_\Lambda(\mathbb{R}^n \times \mathbb{R}^m) := \{H \circ \pi_m \colon H \in \mathcal{H}^n_\Lambda(\mathbb{R}^n)\}$
- if $\alpha = (\alpha_1, \dots, \alpha_\ell)$ and $\beta = (\beta_1, \dots, \beta_\ell)$ in $(\tilde{\mathbb{R}}_+)^\ell$:
$$I^\beta_\alpha := \prod_{i=1}^{\ell}]\alpha_i, \beta_i[\subset (\mathbb{R}^*_+)^\ell, \quad [\alpha, \beta] := \prod_{i=1}^{\ell} [\alpha_i, \beta_i], \quad]\alpha, \beta] := \prod_{i=1}^{\ell}]\alpha_i, \beta_i]$$
- Note that $I^\beta_\alpha = (\mathbb{R}^*_+)^\ell$ if $\alpha = 0$ and $\beta = (+\infty, \dots, +\infty)$
- If $\ell = 1$, then $I^\beta_\alpha =]\alpha, \beta[$. Also $I^\beta_\alpha = \mathbb{R}^*_+$, if $\alpha = 0$ and $\beta = +\infty$
- $\mathcal{E}_{s,M,\oslash}[\Omega; I^\beta_\alpha]$, $\mathcal{E}_{s,M,\otimes}[\Omega; I^\beta_\alpha] := \mathcal{E}_{s,M}[\Omega; I^\beta_\alpha] \cap \mathcal{E}_{s,M,\oslash}[\Omega; I^\beta_\alpha]$ and $\mathcal{G}_{s,\otimes}(\Omega; I^\beta_\alpha)$ are dfined on page 12
- $\mathcal{E}_{s,QM}[I^\beta_\alpha; G]$ and $C^\infty_{s,QM}[I^\beta_\alpha; G] := C^\infty(I^\beta_\alpha; G) \cap \mathcal{E}_{s,QM}[I^\beta_\alpha; G]$ are dfined on page 12
- If $f \in \mathcal{G}_{s,\otimes}(\Omega; I^\beta_\alpha)$ and $\varphi \in C^\infty_{s,QM}[I^\beta_\alpha; G]$, then $\varphi \circ f := \varphi \circ \widehat{f} + \mathcal{N}_s[\Omega; G]$;
- $\mathcal{E}_{s,M,\odot}[\Omega; \mathbb{R}^*]$, $\mathcal{E}_{s,M,\circ}[\Omega; \mathbb{R}^*] := \mathcal{E}_{s,M}[\Omega; \mathbb{R}^*] \cap \mathcal{E}_{s,M,\odot}[\Omega; \mathbb{R}^*]$ and $\mathcal{G}_{s,\circ}(\Omega; \mathbb{R}^*)$ are defined on page 13

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