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RANDOM DYNAMICAL SYSTEMS ON TIME SCALES

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Abstract. The purpose of this paper is to prove the existence and uniqueness of solution for random dynamic systems on time scales.

1. Introduction

The theory of dynamic systems on time scales allows us to study both continuous and discrete dynamic systems simultaneously. Since Hilger's initial work [10] there has been significant growth in the theory of dynamic systems on time scales, covering a variety of different qualitative aspects. We refer to the books [3, 4], and the papers [1, 2, 14, 15]. In recent years, some authors studied stochastic differential equations on time scales [5, 7, 13]. The main theoretical and practical aspects of probability theory and stochastic differential equations can be found in books [6, 12]. The organization of this paper is as follows. Section 2 presents a few definitions and concepts of time scales. Also, the notion of stochastic process on a time scale is introduced. In Section 3 we prove the existence and uniqueness of solution for the random dynamic systems on time scales.

Preliminaries. By a *time scale* $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$. Then $\mathbb{T}$ is a complete metric space with the metric defined by $d(t,s) := |t-s|$ for $t,s \in \mathbb{T}$. Since a time scale $\mathbb{T}$ is not connected in generally, we need the concept of jump operators. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. In this definition we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. The *graininess function* $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\sigma(t) > t$, we say t is a *right-scattered point*, while if $\rho(t) < t$, we say t is a *left-scattered point*. Points that are right-scattered and left-scattered at the same time will be called *isolated points*. A point $t \in \mathbb{T}$ such that $t < \sup \mathbb{T}$ and $\sigma(t) = t$, is called a *right-dense point*. A point $t \in \mathbb{T}$ such that $t > \inf \mathbb{T}$ and $\rho(t) = t$, is called a *left-dense point*. Points that are right-dense and left-dense at the same time will be called *dense points*. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$ if $\mathbb{T}$ has a left-scattered maximum m, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Given a time scale interval $[a,b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \le t \le b\}$, then $[a,b]_{\mathbb{T}}^\kappa$ denoted the interval $[a,b]_{\mathbb{T}}$ if $a < \rho(b) = b$ and denote the interval $[a,b)_{\mathbb{T}}$ if $a < \rho(b) < b$. In fact, $[a,b)_{\mathbb{T}} = [a,\rho(b)]_{\mathbb{T}}$. Also, for $a \in \mathbb{T}$, we define $[a,\infty)_{\mathbb{T}} = [a,\infty) \cap \mathbb{T}$. If $\mathbb{T}$ is a bounded time scale, then $\mathbb{T}$ can be identified with $[\inf \mathbb{T}, \sup \mathbb{T}]_{\mathbb{T}}$.

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If $t_0 \in \mathbb{T}$ and $\delta > 0$, then we define the following neighborhoods of t_0: $U_{\mathbb{T}}(t_0, \delta) := (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$, $U_{\mathbb{T}}^{+}(t_0, \delta) := [t_0, t_0 + \delta) \cap \mathbb{T}$, and $U_{\mathbb{T}}^{-}(t_0, \delta) := (t_0 - \delta, t_0] \cap \mathbb{T}$.

Definition 1.1 ([3])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* if its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$, and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Obviously, a continuous function is rd-continuous, and a rd-continuous function is regulated ([3, Theorem 1.60]).

Definition 1.2. A function $f : [a, b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is called Hilger continuous if f is continuous at each point (t, x) where t is right-dense, and the limits

$$\lim_{(s,y)\to(t^-,x)} f(s, y) \quad \text{and} \quad \lim_{y\to x} f(t, y)$$

both exist and are finite at each point (t, x) where t is left-dense.

Definition 1.3 ([3])**.** Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. Let $f^{\Delta}(t) \in \mathbb{R}$ (provided it exists) with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \tag{1.1}$$

for all $s \in U_{\mathbb{T}}(t, \delta)$. We call $f^{\Delta}(t)$ the *delta* (or *Hilger*) derivative (Δ-derivative for short) of f at t. Moreover, we say that f is delta differentiable (Δ-differentiable for short) on $\mathbb{T}^{\kappa}$ provided $f(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

The following result will be very useful.

Proposition 1.4 ([3, Theorem 1.16])**.** *Assume that $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$.*

(i) *If f is Δ-differentiable at t, then f is continuous at t.*
(ii) *If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t with*
$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$
(iii) *If f is Δ-differentiable at t and t is right-dense then*
$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$
(iv) *If f is Δ-differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.*

It is known [9] that for every $\delta > 0$ there exists at least one partition $P : a = t_0 < t_1 < \dots < t_n = b$ of $[a, b)_{\mathbb{T}}$ such that for each $i \in \{1, 2, \dots, n\}$ either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$. For given $\delta > 0$ we denote by $\mathcal{P}([a, b)_{\mathbb{T}}, \delta)$ the set of all partitions $P : a = t_0 < t_1 < \dots < t_n = b$ that possess the above property.

Let $f : \mathbb{T} \to \mathbb{R}$ be a bounded function on $[a, b)_{\mathbb{T}}$, and let $P : a = t_0 < t_1 < \dots < t_n = b$ be a partition of $[a, b)_{\mathbb{T}}$. In each interval $[t_{i-1}, t_i)_{\mathbb{T}}$,where $1 \leq i \leq n$, we choose an arbitrary point ξ_i and form the sum

$$S = \sum_{i=1}^{n} (t_i - t_{i-1}) f(\xi_i).$$

We call S a Riemann Δ-sum of f corresponding to the partition P.

Definition 1.5 ([8]). We say that f is Riemann Δ-integrable from a to b (or on $[a,b)_{\mathbb{T}}$) if there exists a number I with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann Δ-sum S of f corresponding to a partition $P \in \mathcal{P}([a,b)_{\mathbb{T}}, \delta)$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$, $i = 1, 2, \dots, n$. It is easily seen that such a number I is unique. The number I is the Riemann Δ-integral of f from a to b, and we will denote it by $\int_a^b f(t)\Delta t$.

Proposition 1.6 ([8, Theorem 5.8]). *A bounded function $f : [a,b)_{\mathbb{T}} \to \mathbb{R}$ is Riemann Δ-integrable on $[a,b)_{\mathbb{T}}$ if and only if the set of all right-dense points of $[a,b)_{\mathbb{T}}$ at which f is discontinuous is a set of Δ-measure zero.*

It is no difficult to see that every regulated function on a compact interval is bounded (see [3, Theorem 1.65]). Then we get that every regulated function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$, is Riemann Δ-integrable from a to b.

Proposition 1.7 ([11, Theorem 5.8]). *Assume that $a, b \in \mathbb{T}$, $a < b$ and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Then the integral has the following properties.*

(i) *If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right-hand side is the Riemann integral.*
(ii) *If $\mathbb{T}$ consists of isolated points, then*
$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)_{\mathbb{T}}} \mu(t) f(t).$$

If $f, g : \mathbb{T} \to \mathbb{R}$ are Riemann Δ-integrable on $[a,b)_{\mathbb{T}}$, then λf, $f + g$ and $|f|$ are are Riemann Δ-integrable on $[a,b)_{\mathbb{T}}$, and the following properties are true [3]:
$$\begin{gathered}
\int_a^b (\lambda f)(t)\Delta t = \lambda \int_a^b f(t)\Delta t, \quad \lambda \in \mathbb{R}, \\
\int_a^b (f+g)(t)\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t, \\
\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t \qquad (1.2) \\
|\int_a^b f(t)\Delta t| \le \int_a^b |f(t)|\Delta t, \\
\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t, \quad a < c < b,
\end{gathered}$$

Definition 1.8 ([3]). *A function $g : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $g^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$.*

One can show that each rd-continuous function has a Δ-antiderivative [3, Theorem 1.74].

Proposition 1.9 ([8, Theorem 4.1]). *Let $f : \mathbb{T} \to \mathbb{R}$ be Riemann Δ-integrable function on $[a,b)_{\mathbb{T}}$. If f has a Δ-antiderivative $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$, then $\int_a^b f(t)\Delta t = g(b) - g(a)$. In particular, $\int_t^{\sigma(t)} f(s)\Delta s = \mu(t) f(t)$ for all $t \in [a,b)_{\mathbb{T}}$ (see* [3, Theorem 1.75]*)*

Proposition 1.10 ([8, Theorem 4.3]). *Let $f : \mathbb{T} \to \mathbb{R}$ be a function which is Riemann Δ-integrable from a to b. For $t \in [a,b]_{\mathbb{T}}$, let $g(t) = \int_a^t f(t)\Delta t$. Then g is continuous on $[a,b]_{\mathbb{T}}$. Further, let $t_0 \in [a,b)_{\mathbb{T}}$ and let f be arbitrary at t_0 if t_0 is right-scattered, and let f be continuous at t_0 if t_0 is right-dense. Then g is Δ-differentiable at t_0 and $g^{\Delta}(t_0) = f(t_0)$.*

Lemma 1.11. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing function. If $s, t \in \mathbb{T}$ with $s \le t$, then*
$$\int_s^t g(\tau)\Delta\tau \le \int_s^t g(\tau)d\tau.$$

Stochastic process on time scales. Denote by $\mathcal{B}$ the σ-algebra of all Borel subsets of $\mathbb{R}$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability measure space. A function $X(\cdot) : \Omega \to \mathbb{R}$ is called a random variable if X is a measurable function from $(\Omega, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B})$; that is, $X^{-1}(B) := \{\omega \in \Omega; X(\omega) \in B\} \in \mathcal{F}$ for all $B \in \mathcal{B}$. A time scale stochastic process is a function $X(\cdot,\cdot) : [a,b]_{\mathbb{T}} \times \Omega \to \mathbb{R}$ such that $X(t,\cdot) : \Omega \to \mathbb{R}$ is a random variable for each $t \in \mathbb{T}$. For each point $\omega \in \Omega$, the function on $\mathbb{T}$ given by $t \mapsto X(t,\omega)$ is will be called a trajectory (or a sample path) of the time scale stochastic process $X(\cdot,\cdot)$ corresponding to ω. A time scale stochastic process $X(\cdot,\cdot)$ is said to be regulated (rd-continuous, continuous) if there exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ and such that the trajectory $t \mapsto X(t,\omega)$ is a regulated (rd-continuous, continuous) function on $[a,b]_{\mathbb{T}}$ for each $\omega \in \Omega_0$. Let $X(\cdot)$ and $Y(\cdot)$ be two random variables. If there exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ and such that $X(\omega) = Y(\omega)$ for all $\omega \in \Omega_0$, then we will write $X(\omega) =_P Y(\omega)$. Similarly for the inequalities. Let $X(\cdot,\cdot)$ and $Y(\cdot,\cdot)$ be two time scale stochastic processes. If there exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ and such that for each $\omega \in \Omega_0$ we have $X(t,\omega) = Y(t,\omega)$ for all $t \in [a,b]_{\mathbb{T}}$, then we will write $X(t,\omega) =_P Y(t,\omega)$, $t \in [a,b]_{\mathbb{T}}$. Similarly for the inequalities.

Lemma 1.12. *Let $X(\cdot,\cdot) : [a,b]_{\mathbb{T}} \times \Omega \to \mathbb{R}$ be a time scale stochastic process. If there exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that the function $t \mapsto X(t,\omega)$ is Riemann Δ-integrable on $[a,b)_{\mathbb{T}}$ for every $\omega \in \Omega_0$, then the function $Y(\cdot,\cdot) : [a,b]_{\mathbb{T}} \times \Omega \to \mathbb{R}$ given by*
$$Y(t,\omega) = \int_a^t X(s,\omega)\Delta s, \quad t \in [a,b]_{\mathbb{T}}$$
is a continuous time scale stochastic process.

Proof. From Proposition 1.10, it follows that the function $t \mapsto \int_a^t X(s,\omega)\Delta s$ is continuous for each $\omega \in \Omega_0$. Since the Riemann Δ-integral is a limit of the finite sum $S(\omega) = \sum_{i=1}^n (t_i - t_{i-1})X(\xi_i,\omega)$ of measurable functions, we have that $\omega \mapsto \int_a^t X(s,\omega)\Delta s$ is a measurable function. Therefore, $Y(\cdot,\cdot)$ is a continuous time scale stochastic process. □

2. Random initial value problem on time scales

In the following, consider an initial value problem of the form
$$\begin{gathered} X^{\Delta}(t,\omega) =_P f(t, X(t,\omega), \omega), \quad t \in [a,b]_{\mathbb{T}}^{\kappa} \\ X(a,\omega) =_P X_0(\omega), \end{gathered} \tag{2.1}$$
where $X_0 : \Omega \to \mathbb{R}$ is a random variable and $f : [a,b]_{\mathbb{T}}^{\kappa} \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies the following assumptions:

(H1) $f(t,x,\cdot):\Omega\to\mathbb{R}$ is a random variable for all $(t,x)\in[a,b]^{\kappa}_{\mathbb{T}}\times\mathbb{R}$,

(H2) with $P.1$, the function $f(\cdot,\cdot,\omega):[a,b]^{\kappa}_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$ is a Hilger continuous function at every point $(t,x)\in[a,b]^{\kappa}_{\mathbb{T}}\times\mathbb{R}$; that is, there exists $\Omega_0\subset\Omega$ with $P(\Omega_0)=1$ and such that for each $\omega\in\Omega_0$, the function $(t,x)\mapsto f(t,x,\omega)$ is Hilger continuous.

Definition 2.1. By a solution of (2.1) we mean a time scale stochastic process $X(\cdot,\cdot):[a,b]^{\kappa}_{\mathbb{T}}\times\Omega\to\mathbb{R}$ that satisfies conditions in (2.1). A solution $X(\cdot,\cdot)$ is unique if $X(t,\omega)=_P Y(t,\omega)$, $t\in[a,b]^{\kappa}_{\mathbb{T}}$ for any time scale stochastic process $Y(\cdot,\cdot):[a,b]^{\kappa}_{\mathbb{T}}\times\Omega\to\mathbb{R}$ which is a solution of (2.1).

Obviously, if there exists $\Omega_0\subset\Omega$ with $P(\Omega_0)=1$ and such that for each $\omega\in\Omega_0$ we have $|X(t,\omega)-Y(t,\omega)|=0$ for all $t\in[a,b]_{\mathbb{T}}$, then $X(t,\omega)=_P Y(t,\omega)$, $t\in[a,b]^{\kappa}_{\mathbb{T}}$; that is, if $|X(t,\omega)-Y(t,\omega)|=_P 0$ for all $t\in[a,b]^{\kappa}_{\mathbb{T}}$, then $X(t,\omega)=_P Y(t,\omega)$, $t\in[a,b]^{\kappa}_{\mathbb{T}}$.

Remark 2.2. We can consider the random differential equation (2.1) as a family (with respect to parameter ω) of deterministic differential equations, namely

$$\begin{gathered} X^{\Delta}(t,\omega)=f(t,X(t,\omega),\omega),\quad t\in[a,b]^{\kappa}_{\mathbb{T}}\\ X(a,\omega)=X_0(\omega). \end{gathered}\tag{2.2}$$

Generally, is not correct to solve each problem (2.2) to obtain the solutions of (2.1). Let us give two examples.

Example 2.3. Let $(\Omega,\mathcal{F},P)$ be a complete probability measure space. Consider an initial value problem of the form

$$\begin{gathered} X^{\Delta}(t,\omega)=K(\omega)X^2(t,\omega),\quad t\in[0,\infty)_{\mathbb{R}}\\ X(0,\omega)=1, \end{gathered}\tag{2.3}$$

where $K:\Omega\to(0,\infty)$ is a random variable. It is easy to see that, for each $\omega\in\Omega$, $X(t,\omega)=\frac{1}{1-K(\omega)t}$ is a solution of (2.3) on the interval $[0,1/K(\omega)]$. Since for each $a\ge 0$ we have that $P(1/K(\omega)>a)<1$, it follows that not all solutions $X(\cdot,\omega)$ are well defined on some common interval $[0,a)$.

Example 2.4. Let $(\Omega,\mathcal{F},P)$ be a complete probability measure space and let $\Omega_0\notin\mathcal{F}$. It is easy to check that, for each $\omega\in\Omega$, the function $X(\cdot,\cdot):[0,1]_{\mathbb{R}}\times\Omega\to\mathbb{R}$, given by

$$X(t,\omega)=\begin{cases}0 & \text{if } \omega\in\Omega_0\\ t^{3/2} & \text{if } \omega\in\Omega\setminus\Omega_0,\end{cases}$$

is a solution of the initial-value problem

$$\begin{gathered} X^{\Delta}(t,\omega)=\frac{3}{2}X(t,\omega),\quad t\in[0,\infty)_{\mathbb{R}}\\ X(0,\omega)=0. \end{gathered}$$

But $X(\cdot,\cdot)$ is not a stochastic process. Indeed, we have that

$$\{\omega\in\Omega;X(1,\omega)\in[-\frac{1}{2},\frac{1}{2}]\}=\Omega_0\notin\mathcal{F},$$

that is, $\omega\mapsto X(1,\omega)$ is not a measurable function.

Using Propositions 1.9 and 1.10 and [15, Lemma 2.3], it is easy to prove the following result.

Lemma 2.5. . *A time scale stochastic process $X(\cdot,\cdot):[a,b]_{\mathbb{T}}^{\kappa}\times\Omega\to\mathbb{R}$ is the solution of the problem (2.1) if and only if $X(\cdot,\cdot)$ is a continuous time scale stochastic process and it satisfies the following random integral equation*

$$X(t,\omega)=_P X_0(\omega)+\int_a^t f(s,X(s,\omega),\omega)\Delta s, t\in[a,b]_{\mathbb{T}}. \tag{2.4}$$

The following results is known as Gronwall's inequality on time scale and will be used in this paper.

Lemma 2.6 ([14, Lemma 3.1])**.** *Let an rd-continuous time scale stochastic processes $X(\cdot,\cdot),Y(\cdot,\cdot):[a,b]_{\mathbb{T}}^{\kappa}\times\Omega\to\mathbb{R}_+$ be such that*

$$X(t,\omega)\le_P Y(t,\omega)+\int_a^t q(s)X(s,\omega)\Delta s,\quad t\in[a,b]_{\mathbb{T}},$$

where $1+\mu(t)q(t)\neq 0$, for all $t\in[a,b]_{\mathbb{T}}$. Then we have

$$X(t,\omega)\le_P Y(t,\omega)+e_q(t,a)\int_a^t q(s)Y(s,\omega)\frac{1}{e_q(\sigma(s),a)}\Delta s,\quad t\in[a,b]_{\mathbb{T}}.$$

Theorem 2.7. *Let $f:[a,b]_{\mathbb{T}}^{\kappa}\times\mathbb{R}\times\Omega\to\mathbb{R}$ satisfy* (H1)–(H2) *and assume that there exists a rd-continuous time scale stochastic process $L(\cdot,\cdot):[a,b]_{\mathbb{T}}^{\kappa}\times\Omega\to\mathbb{R}$ such that*

$$|f(t,x,\omega)-f(t,y,\omega)|\le L(t,\omega)|x-y| \tag{2.5}$$

for every $t\in[a,b]_{\mathbb{T}}^{\kappa}$ and every $x,y\in\mathbb{R}$ with P.1. Let $X_0:\Omega\to\mathbb{R}$ a random variable such that

$$|f(t,X_0(\omega),\omega)|\le_P M,\quad t\in[a,b]_{\mathbb{T}}^{\kappa}, \tag{2.6}$$

where $M>0$ is a constant. Then problem (2.1) has a unique solution.

Proof. . To prove the theorem we apply the method of successive approximations (see [14]). For this, we define a sequence of functions $X_n(\cdot,\cdot):[a,b]_{\mathbb{T}}^{\kappa}\times\Omega\to\mathbb{R}$, $n\in\mathbb{N}$, as follows:

$$\begin{gathered}X_0(t,\omega)=X_0(\omega)\\ X_n(t,\omega)=X_0(\omega)+\int_a^t f(s,X_{n-1}(s,\omega),\omega)\Delta s,\quad n\ge 1,\end{gathered} \tag{2.7}$$

for every $t\in[a,b]_{\mathbb{T}}^{\kappa}$ and every $\omega\in\Omega$. First, using (2.6) and the Lemma 1.11, we observe that

$$\begin{aligned}|X_1(t,\omega)-X_0(t,\omega)|&\le\Big|\int_a^t f(s,X_0(\omega),\omega)\Delta s\Big|\le\int_a^t|f(s,X_0(\omega),\omega)|\Delta s\\&\le\int_a^t|f(s,X_0(\omega),\omega)|ds\le_P M(t-a)\\&\le M(b-a),\quad t\in[a,b]_{\mathbb{T}}.\end{aligned}$$

We prove by induction that for each integer $n\ge 2$ the following estimate holds

$$|X_n(t,\omega)-X_{n-1}(t,\omega)|\le_P M\widetilde{L}(\omega)\frac{(t-a)^n}{n!}\le M\widetilde{L}(\omega)\frac{(b-a)^n}{n!},\ t\in[a,b]_{\mathbb{T}}, \tag{2.8}$$

where $\widetilde{L}(\omega)=\sup_{[a,b]_{\mathbb{T}}}L(t,\omega)$. Suppose that (2.8) holds for $n=k\ge 2$. Then, using (2.5), (2.6) and Lemma 1.11, we obtain

$$|X_{k+1}(t,\omega)-X_k(t,\omega)|\le\int_a^t|f(s,X_k(s,\omega),\omega)-f(s,X_{k-1}(s,\omega),\omega)|\Delta s$$

$$
\begin{aligned}
&\le_P \widetilde{L}(\omega)\int_a^t |X_k(s,\omega)-X_{k-1}(s,\omega)|\Delta s\\
&\le_P \widetilde{L}(\omega)\frac{M}{k!}\int_a^t (s-a)^k\Delta s\\
&\le \widetilde{L}(\omega)\frac{M}{k!}\int_a^t (s-a)^k ds\\
&= M\widetilde{L}(\omega)\frac{(t-a)^{k+1}}{(k+1)!}\\
&\le M\widetilde{L}(\omega)\frac{(b-a)^{k+1}}{(k+1)!}, \quad t\in[a,b]_{\mathbb{T}}.
\end{aligned}
$$

Thus, (2.8) is true for $n=k+1$ and so (2.8) holds for all $n\ge 2$. Further, we show that for every $n\in\mathbb{N}$ the functions $X_n(\cdot,\omega):[a,b]_{\mathbb{T}}\to\mathbb{R}$ are continuous with P.1. Let $\varepsilon>0$ and $t,s\in[a,b]_{\mathbb{T}}$ be such that $|t-s|<\varepsilon/M$. We have

$$
\begin{aligned}
|X_1(t,\omega)-X_1(s,\omega)| &= |\int_a^t f(\tau,X_0(\omega),\omega)\Delta\tau-\int_a^s f(\tau,X_0(\omega),\omega)\Delta\tau|\\
&= |\int_s^t f(\tau,X_0(\omega),\omega)\Delta\tau|\\
&\le \int_s^t |f(\tau,X_0(\omega),\omega)|\Delta\tau\\
&\le \int_s^t |f(\tau,X_0(\omega),\omega)|d\tau\\
&\le_P M|t-s|<\varepsilon
\end{aligned}
$$

and so $t\mapsto X_1(t,\omega)$ is continuous with P.1. Since for each $n\ge 2$

$$
\begin{aligned}
&|X_n(t,\omega)-X_n(s,\omega)|\\
&= |\int_a^t f(\tau,X_{n-1}(\tau,\omega),\omega)\Delta\tau-\int_a^s f(\tau,X_{n-1}(\tau,\omega),\omega)\Delta\tau|\\
&\le \int_s^t |f(\tau,X_{n-1}(\tau,\omega),\omega)|\Delta\tau\\
&\le \int_s^t |f(\tau,X_0(\omega),\omega)|\Delta\tau+\int_s^t |f(\tau,X_{n-1}(\tau,\omega),\omega)-f(\tau,X_0(\omega),\omega)|\Delta\tau\\
&\le \int_s^t |f(\tau,X_0(\omega),\omega)|\Delta\tau\\
&\quad+\sum_{k=1}^{n-1}\int_s^t |f(\tau,X_k(\tau,\omega),\omega)-f(\tau,X_{k-1}(\tau,\omega),\omega)|\Delta\tau
\end{aligned}
$$

then, by induction, we obtain

$$
|X_n(t,\omega)-X_n(s,\omega)|\le_P M(1+\sum_{k=1}^{n-1}\frac{\widetilde{L}(\omega)^{k-1}(b-a)^k}{k!})|t-s|\to 0
$$

as $s\to t$ with P.1. Therefore, for every $n\in\mathbb{N}$ the function $X_n(\cdot,\omega):[a,b]_{\mathbb{T}}\times\Omega\to\mathbb{R}$ is continuous with P.1. Now, using Lemma 2.5 and (2.7), we deduce that the functions $X_n(t,\cdot):\Omega\to\mathbb{R}$ are measurable. Consequently, it follows that for every $n\in\mathbb{N}$ the function $X_n(\cdot,\cdot):[a,b]_{\mathbb{T}}\times\Omega\to\mathbb{R}$ is a time scale stochastic process.

Further, we shall show that the sequence $(X_n(t,\cdot))_{n\in\mathbb{N}}$ is uniformly convergent with P.1. Denote

$$Y_n(t,\omega)=|X_{n+1}(t,\omega)-X_n(t,\omega)|,\;\; n\in\mathbb{N}.$$

Since

$$Y_n(t,\omega)-Y_n(s,\omega)\le_P \widetilde{L}(\omega)\int_s^t|X_n(\tau,\omega)-X_{n-1}(\tau,\omega)|\Delta\tau$$

then, reasoning as above, we deduce that the functions $t\mapsto Y_n(t,\omega)$ are continuous with P.1. Now, using (2.8), we obtain

$$\sup_{t\in[a,b]_{\mathbb{T}}}|X_n(t,\omega)-X_m(t,\omega)|\le\sum_{k=m}^{n-1}\sup_{t\in[a,b]_{\mathbb{T}}}Y_k(t,\omega)\le_P M\sum_{k=m}^{n-1}\frac{\widetilde{L}(\omega)^k(b-a)^{k+1}}{(k+1)!}$$

for all $n>m>0$. Since the series $\sum_{n=1}^{\infty}\widetilde{L}(\omega)^{n-1}(b-a)^n/n!$ converges with P.1, then for each $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\sup_{t\in[a,b]_{\mathbb{T}}}|X_n(t,\omega)-X_m(t,\omega)|\le_P\varepsilon\quad\text{for all } n,m\ge n_0. \tag{2.9}$$

Hence, since $([a,b]_{\mathbb{T}},|\cdot|)$ is a complete metric space, it follows that there exists $\Omega_0\subset\Omega$ such that $P(\Omega_0)=1$ and for every $\omega\in\Omega_0$ the sequence $(X_n(t,\cdot))_{n\in\mathbb{N}}$ is uniformly convergent. For $\omega\in\Omega_0$ denote $\widetilde{X}(t,\omega)=\lim_{n\to\infty}X_n(t,\omega)$. Next, we define the function $X(\cdot,\cdot):[a,b]_{\mathbb{T}}\times\Omega\to\mathbb{R}$ as follows: $X(\cdot,\omega)=\widetilde{X}(\cdot,\omega)$ if $\omega\in\Omega_0$, and $X(\cdot,\omega)$ as an arbitrary function if $\omega\in\Omega\setminus\Omega_0$. Obviously, $X(\cdot,\omega)$ is continuos with P.1. Since, by Lemma 1.12 and (2.7), the functions $\omega\to X_n(\cdot,\omega)$ are measurable and $X(t,\omega)=\lim_{n\to\infty}X_n(t,\omega)$ for every $t\in[a,b]_{\mathbb{T}}$ with P.1, we deduce that $\omega\to X(t,\omega)$ is measurable for every $t\in[a,b]_{\mathbb{T}}$. Therefore, $X(\cdot,\cdot):[a,b]_{\mathbb{T}}\times\Omega\to\mathbb{R}$ is a continuous time scale stochastic process. We show that $X(\cdot,\cdot)$ satisfies the random integral equation (2.4). For each $n\in\mathbb{N}$ we put $G_n(t,\omega)=f(t,X_n(t,\omega),\omega)$, $t\in[a,b]_{\mathbb{T}}$, $\omega\in\Omega$. Then $G_n(t,\omega)$ is rd-continuous time scale stochastic process, and we have that

$$\sup_{t\in[a,b]_{\mathbb{T}}}|G_n(t,\omega)-G_m(t,\omega)|\le_P\widetilde{L}(\omega)\sup_{t\in[a,b]_{\mathbb{T}}}|X_n(t,\omega)-X_m(t,\omega)|,\quad t\in[a,b]_{\mathbb{T}}$$

for all $n,m\ge n_0$. Using (2.9) we infer that the sequence $(G_n(\cdot,\omega))_{n\in\mathbb{N}}$ is uniformly convergent with P.1. If we take $m\to\infty$, then for each $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that for every $n\ge n_0$ we have

$$\sup_{t\in[a,b]_{\mathbb{T}}}|G_n(t,\omega)-f(t,X(t,\omega),\omega)|\le_P\widetilde{L}(\omega)\sup_{t\in[a,b]_{\mathbb{T}}}|X_n(t,\omega)-X(t,\omega)|,\;\; t\in[a,b]_{\mathbb{T}}$$

and so $\lim_{n\to\infty}|G_n(t,\omega)-f(t,X(t,\omega),\omega)|=0$ for all $t\in[a,b]_{\mathbb{T}}$ with P.1. Also, it easy to see that

$$\sup_{t\in[a,b]_{\mathbb{T}}}|\int_a^t G_n(s,\omega)\Delta s-\int_a^t f(s,X(s,\omega),\omega)\Delta s|\le_P\widetilde{L}(\omega)\int_a^t|X_n(s,\omega)-X(s,\omega)|\Delta s.$$

Since the sequence $X(t,\omega)=\lim_{n\to\infty}X_n(t,\omega)$ uniformly with P.1, then it follows that

$$\lim_{n\to\infty}|\int_a^t G_n(s,\omega)\Delta s-\int_a^t f(s,X(s,\omega),\omega)\Delta s|=0\quad\forall t\in[a,b]_{\mathbb{T}}\text{ with } P.1.$$

Now, we have

$$\begin{aligned}
&\sup_{t\in[a,b]_{\mathbb{T}}}|X(t,\omega)-(X_0(\omega)+\int_a^t f(s,X(s,\omega),\omega)\Delta s)| \\
&\le \sup_{t\in[a,b]_{\mathbb{T}}}|X(t,\omega)-X_n(t,\omega)| \\
&\quad+\sup_{t\in[a,b]_{\mathbb{T}}}|X_n(t,\omega)-(X_0(\omega)+\int_a^t f(s,X_{n-1}(s,\omega),\omega)\Delta s)| \\
&\quad+\sup_{t\in[a,b]_{\mathbb{T}}}|\int_a^t f(s,X_{n-1}(s,\omega),\omega)\Delta s-\int_a^t f(s,X(s,\omega),\omega)\Delta s|.
\end{aligned}$$

Using the two previous convergence

$$\big|X(t,\omega)-\big(X_0(\omega)+\int_a^t f(s,X(s,\omega),\omega)\Delta s\big)\big|=0 \text{ for all } t\in[a,b]_{\mathbb{T}} \text{ with } P.1;$$

that is, $X(\cdot,\cdot)$ satisfies the random integral equation (2.4). Then, by Lemma 2.5, it follows that $X(\cdot,\cdot)$ is the solution of (2.1).

Finally, we show the uniqueness of the solution. For this, we assume that $X(\cdot,\cdot),Y(\cdot,\cdot):[a,b]_{\mathbb{T}}\times\Omega\to\mathbb{R}$ are two solutions of (2.4). Since

$$|X(t,\omega)-Y(t,\omega)|\le_P \int_a^t \widetilde{L}(\omega)|X(s,\omega)-Y(s,\omega)|ds,\quad t\in[a,b]_{\mathbb{T}},$$

from Lemma 2.6, it follows that $|X(t,\omega)-Y(t,\omega)|\le_P 0$, $t\in[a,b]_{\mathbb{T}}$ and so, the proof is complete. □

Let $\mathbb{T}$ be an upper unbounded time scale. Then under suitable conditions we can extend the notion of the solution of (2.1) from $[a,b]_{\mathbb{T}}^{\kappa}$ to $[a,\infty)_{\mathbb{T}}:=[a,\infty)\cap\mathbb{T}$, if we define f on $[a,\infty)_{\mathbb{T}}\times\mathbb{R}\times\Omega$ and show that the solution exists on each $[a,b]_{\mathbb{T}}$ where $b\in(a,\infty)_{\mathbb{T}}$, $a<\rho(b)$.

Theorem 2.8. *Assume that $f:[a,\infty)_{\mathbb{T}}\times\mathbb{R}\times\Omega\to\mathbb{R}$ satisfies the assumptions of Theorem 2.7 on each interval $[a,b]_{\mathbb{T}}$ with $b\in(a,\infty)_{\mathbb{T}}$, $a<\rho(b)$. If there is a constant $M>0$ such that $|f(t,x,\omega)|\le_P M$ for all $(t,x)\in[a,b)_{\mathbb{T}}\times\mathbb{R}$, then the problem* (2.1) *has a unique solution on $[a,\infty)_{\mathbb{T}}$.*

Proof. Let $X(t,\cdot)$ be the solution of (2.1) which exists on $[a,b)_{\mathbb{T}}$ with $b\in(a,\infty)_{\mathbb{T}}$, $a<\rho(b)$, and the value of b cannot be increased. First, we observe that b is a left-scattered point, then $\rho(b)\in(a,b)_{\mathbb{T}}$ and the solution $X(t,\cdot)$ exists on $[a,\rho(b)]_{\mathbb{T}}$. But then the solution $X(t,\cdot)$ exists also on $[a,b]_{\mathbb{T}}$, namely by putting

$$\begin{aligned}
X(b,\omega)&=_P X(\rho(b),\omega)+\mu(b)X^{\Delta}(\rho(b),\omega)\\
&=_P X(\rho(b),\omega)+\mu(b)f(\rho(b),X(\rho(b),\omega),\omega).
\end{aligned}$$

If b is a left-dense point, then their neighborhoods contain infinitely many points to the left of b. Then, for any $t,s\in(a,b)_{\mathbb{T}}$ such that $s<t$, we have

$$|X(t,\omega)-X(s,\omega)|\le\int_s^t|f(\tau,X(\tau,\omega),\omega)|\Delta\tau\le_P M|t-s|.$$

Taking limit as $s,t\to b^-$ and using Cauchy criterion for convergence, it follows $\lim_{t\to b^-}X(t,\omega)$ exists and is finite with $P.1$. Further, we define $X_b(\omega)=_P\lim_{t\to b^-}X(t,\omega)$ and consider the initial value problem

$$X^{\Delta}(t,\omega)=_P f(\tau,X(\tau,\omega),\omega),\quad t\in[b,b_1]_{\mathbb{T}},\quad b_1>\sigma(b),$$

$$X(b,\omega) =_P X_b(\omega).$$

By Theorem 2.7, one gets that $X(t,\omega)$ can be continued beyond b, contradicting our assumptions. Hence every solution $X(t,\omega)$ of e2.1 exists on $[a,\infty)_{\mathbb{T}}$ and the proof is complete. □

3. Random linear systems on time scales

Let $a:\Omega\to\mathbb{R}$ be a positively regressive random variable; that is, $1+\mu(t)a(\omega)>0$ for all $t\in\mathbb{T}$ and $\omega\in\Omega$. Then, by Lemma 1.12, the function $(t,\omega)\mapsto e_{a(\omega)}(t,t_0)$ defined by

$$e_{a(\omega)}(t,t_0) =_P \Big(\int_{t_0}^t \frac{\log(1+\mu(\tau)a(\omega))}{\mu(\tau)}\Delta\tau\Big),\quad t_0,t\in\mathbb{T},$$

is a continuous time scale stochastic process. For each fixed $\omega\in\Omega$, the sample path $t\mapsto e_{a(\omega)}(t,t_0)$ is the exponential function on time scales (see [3]). It easy to check that the stochastic process $(t,\omega)\mapsto e_{a(\omega)}(t,t_0)$ is a solution of the initial value problem (for deterministic case, see [3, Theorem 2.33])

$$\begin{gathered} X^{\Delta}(t,\omega) =_P a(\omega)X(t,\omega),\quad t\in[t_0,b]^{\kappa}_{\mathbb{T}}\\ X(t_0,\omega) =_P 1. \end{gathered}\tag{3.1}$$

If $a:\Omega\to\mathbb{R}$ is bounded with $P.1$ then, by the Theorems 2.7] and 2.8, it follows that (3.1) has a unique solution on $[t_0,\infty)_{\mathbb{T}}$.

Further, consider the following nonhomogeneous initial value problem

$$\begin{gathered} X^{\Delta}(t,\omega) =_P a(\omega)X(t,\omega)+h(t,\omega),\quad t\in[t_0,b]^{\kappa}_{\mathbb{T}}\\ X(t_0,\omega) =_P X_0(\omega), \end{gathered}\tag{3.2}$$

where $a:\Omega\to\mathbb{R}$ is a positively regressive random variable, $X_0:\Omega\to\mathbb{R}$ is a bounded random variable, and $h(,\cdot,):[a,b]^{\kappa}_{\mathbb{T}}\times\Omega\to\mathbb{R}$ is a rd-continuous time scale stochastic process.

Theorem 3.1. *Suppose that $a:\Omega\to\mathbb{R}$ is a positively regressive and bounded random variable, $X_0:\Omega\to\mathbb{R}$ is a bounded random variable, and $h(,\cdot,):[t_0,\infty)_{\mathbb{T}}\times\Omega\to\mathbb{R}$ is a rd-continuous time scale stochastic process. If there is a constant $\nu>0$ such that $|h(t,\omega)|\le_P\nu$ for all $t\in[t_0,b)_{\mathbb{T}}$ with $b\in(t_0,\infty)_{\mathbb{T}}$, $t_0<\rho(b)$, then the initial-value problem* (3.2) *has a unique solution on $[t_0,\infty)_{\mathbb{T}}$.*

Proof. First, we observe that we put $f(t,x,\omega):=a(\omega)x+h(t,\omega)$, then f satisfies the conditions (H_1) and (H_2). Moreover,

$$|f(t,x,\omega)-f(t,y,\omega)|\le_P|a(\omega)||x-y|$$

for every $t\in[t_0,\infty)_{\mathbb{T}}$ and every $x,y\in\mathbb{R}$. Therefore, by the Theorem 2.7, it follows that (3.2) has a unique solution on $[t_0,b]^{\kappa}_{\mathbb{T}}$. Further, let $X(t,\cdot)$ be the solution of (3.2) which exists on $[t_0,b)_{\mathbb{T}}$ with $b\in(t_0,\infty)_{\mathbb{T}}$, $t_0<\rho(b)$. Also, let $N>0$ be such that $|a(\omega)|\le_P N$. Then we have

$$\begin{gathered} |X(t,\omega)|\le|X(t_0,\omega)|+\int_{t_0}^t|a(\omega)X(s,\omega)|\Delta s+\int_{t_0}^t|h(s,\omega)|\Delta s\le_P\\ 1+\nu(t-t_0)+N\int_{t_0}^t|X(s,\omega)|\Delta s. \end{gathered}$$

Then, by the [3, Corollary 6.8], it follows that

$$|X(t,\omega)| \leq_P (1+\frac{\nu}{N})e_N(t,t_0) - \frac{\nu}{N} \leq (1+\frac{\nu}{N})e_N(b,t_0).$$

Hence $|f(t,X(t,\omega),\omega)| \leq_P M := \nu+(1+\frac{\nu}{N})e_N(b,t_0)$. Proceeding as in the proof of the Theorem 2.8 it follows that the unique solution of (3.2) exists on $[t_0,\infty)_{\mathbb{T}}$. $\square$

Theorem 3.2 (Variation of Constants)**.** *A continuous time scale stochastic process $X(\cdot,\cdot):[t_0,\infty)_{\mathbb{T}}\times\Omega\to\mathbb{R}$ is a solution of the initial-value problem* (3.2) *if and only if*

$$X(t,\omega) =_P e_{a(\omega)}(t,t_0)X_0(\omega) + \int_{t_0}^{t} e_{a(\omega)}(t,\sigma(s))h(s,\omega)\Delta s, t\in[t_0,\infty)_{\mathbb{T}}.$$

Proof. Multiplying $X^{\Delta}(t,\omega) =_P a(\omega)X(t,\omega)+h(t,\omega)$ by $e_{a(\omega)}(t_0,\sigma(t))$, we obtain that

$$X^{\Delta}(t,\omega)e_{a(\omega)}(t_0,\sigma(t)) - a(\omega)X(t,\omega)e_{a(\omega)}(t_0,\sigma(t)) =_P h(t,\omega)e_{a(\omega)}(t_0,\sigma(t));$$

that is,

$$[X(t,\omega)e_{a(\omega)}(t_0,t)]^{\Delta} =_P h(t,\omega)e_{a(\omega)}(t_0,\sigma(t)).$$

Integrating both sides of the last equality from t_0 to t, it follows that

$$X(t,\omega)e_{a(\omega)}(t_0,t) - X(t_0,\omega)e_{a(\omega)}(t_0,t_0) =_P \int_{t_0}^{t} e_{a(\omega)}(t_0,\sigma(s))h(s,\omega)\Delta s.$$

Multiplying the last equality by $e_{a(\omega)}(t,t_0)$, we obtain (3.2). $\square$

Corollary 3.3. *Let $X_0:\Omega\to\mathbb{R}$ be a bounded random variable. If the positively regressive random variable $a:\Omega\to\mathbb{R}$ is bounded with $P.1$, then the unique solution of the initial-value problem*

$$\begin{gathered} X^{\Delta}(t,\omega) =_P a(\omega)X(t,\omega), \quad t\in[t_0,\infty)_{\mathbb{T}} \\ X(t_0,\omega) =_P X_0(\omega) \end{gathered}$$

is given by

$$X(t,\omega) =_P e_{a(\omega)}(t,t_0)X_0(\omega), t\in[t_0,\infty)_{\mathbb{T}}.$$

Remark 3.4. Let $X_0:\Omega\to\mathbb{R}$ be a bounded random variable. If the positively regressive random variable $a:\Omega\to\mathbb{R}$ is bounded with $P.1$, then the unique solution of the initial-value problem

$$\begin{gathered} X^{\Delta}(t,\omega) =_P -a(\omega)X^{\sigma}(t,\omega), \quad t\in[t_0,\infty)_{\mathbb{T}} \\ X(t_0,\omega) =_P X_0(\omega) \end{gathered}$$

is given by

$$X(t,\omega) =_P e_{\ominus a(\omega)}(t,t_0)X_0(\omega), t\in[t_0,\infty)_{\mathbb{T}},$$

where $\ominus a(\omega) = -\frac{a(\omega)}{1+\mu(t)a(\omega)}$ (see [3]) and $X^{\sigma}(t,\omega) = X(\sigma(t),\omega)$. Indeed, we have (see [3])

$$\begin{aligned} X^{\Delta}(t,\omega) &=_P \Big(\frac{1}{e_{\ominus a(\omega)}(t,t_0)}\Big)^{\Delta} X_0(\omega) =_P -\frac{a(\omega)}{e_{a(\omega)}(\sigma(t),t_0)}X_0(\omega) \\ &=_P -a(\omega)e_{\ominus a(\omega)}(\sigma(t),t_0)X_0(\omega) =_P -a(\omega)X^{\sigma}(t,\omega). \end{aligned}$$

Theorem 3.5 (Variation of Constants)**.** *Suppose that $a:\Omega\to\mathbb{R}$ is a positively regressive and bounded random variable, $X_0:\Omega\to\mathbb{R}$ is a bounded random variable, and $h(,\cdot,):[t_0,\infty)_{\mathbb{T}}\times\Omega\to\mathbb{R}$ is a rd-continuous time scale stochastic process. If there is a constant $\nu>0$ such that $|h(t,\omega)|\leq_P \nu$ for all $t\in[t_0,b)_{\mathbb{T}}$ with $b\in(t_0,\infty)_{\mathbb{T}}$, $t_0<\rho(b)$, then the initial-value problem*

$$\begin{gathered} X^{\Delta}(t,\omega)=_P -a(\omega)X^{\sigma}(t,\omega)+h(t,\omega),\quad t\in[t_0,\infty)_{\mathbb{T}}\\ X(t_0,\omega)=_P X_0(\omega), \end{gathered} \tag{3.3}$$

has a unique solution on $[t_0,\infty)_{\mathbb{T}}$ given by

$$X(t,\omega)=_P e_{\ominus a(\omega)}(t,t_0)X_0(\omega)+\int_{t_0}^{t} e_{\ominus a(\omega)}(t,s)h(s,\omega)\Delta s, t\in[t_0,\infty)_{\mathbb{T}}. \tag{3.4}$$

Proof. Multiplying the both sides of the equation in (3.3) by $e_{a(\omega)}(t,t_0)$. Then we have

$$\begin{aligned} (e_{a(\omega)}(t,t_0)X(t,\omega))^{\Delta} &=_P e_{a(\omega)}(t,t_0)X^{\Delta}(t,\omega)+a(\omega)e_{a(\omega)}(t,t_0)X^{\sigma}(t,\omega)\\ &=_P e_{a(\omega)}(t,t_0)[X^{\Delta}(t,\omega)+a(\omega)X^{\sigma}(t,\omega)]\\ &=_P e_{a(\omega)}(t,t_0)h(t,\omega). \end{aligned}$$

Next, we integrate both sides from t_0 to t and we infer that

$$e_{a(\omega)}(t,t_0)X(t,\omega)-e_{a(\omega)}(t_0,t_0)X(t_0,\omega)=_P \int_{t_0}^{t} e_{a(\omega)}(s,t_0)h(s,\omega)\Delta s;$$

that is,

$$e_{a(\omega)}(t,t_0)X(t,\omega)=_P X_0(\omega)+\int_{t_0}^{t} e_{a(\omega)}(s,t_0)h(s,\omega)\Delta s.$$

Since

$$e_{a(\omega)}(t_0,t)=\frac{1}{e_{a(\omega)}(t,t_0)}=e_{\ominus a(\omega)}(t,t_0), e_{a(\omega)}(t_0,t)e_{a(\omega)}(t,t_0)=1$$

(see [3, Theorem 2.36]), then multiplying the both sides of the last equality by $e_{a(\omega)}(t_0,t)$, we obtain (3.4). □

Example 3.6. Let us consider $\Omega=(0,1)$, $\mathcal{F}$ the σ-algebra of all Borel subsets of Ω, P the Lebesgue measure on Ω, and the following initial-value problem

$$\begin{gathered} X^{\Delta}(t,\omega)=_P \omega X(t,\omega)+e_{\omega}(t,0),\quad t\in[0,\infty)_{\mathbb{T}}\\ X(0,\omega)=_P 1-\omega. \end{gathered} \tag{3.5}$$

Then, by the Theorems 2.8 and 3.1, the initial value problem (3.5) has a unique solution on $[0,\infty)_{\mathbb{T}}$, given by

$$X(t,\omega)=_P (1-\omega)e_{\omega}(t,0)+\int_0^{t} e_{\omega}(t,\sigma(s))e_{\omega}(s,0)\Delta s;$$

that is,

$$X(t,\omega)=_P e_{\omega}(t,0)\Big[1-\omega+\int_0^{t}\frac{1}{1+\mu(s)\omega}\Delta s\Big],\quad t\in[0,\infty)_{\mathbb{T}}.$$

Next, consider two particular cases.

If $\mathbb{T}=\mathbb{R}$, then $\mu(t)=0$ for all $t\in\mathbb{N}$, and $e_\omega(t,0)=e^{\omega t}$. Moreover, in this case we have

$$\int_0^t \frac{1}{1+\mu(s)\omega}\Delta s=\int_0^t ds=t.$$

It follows that the initial-value problem

$$\begin{gathered}X^\Delta(t,\omega)=_P \omega X(t,\omega)+e^{\omega t},\quad t\in[0,\infty)\\ X(0,\omega)=_P 1-\omega,\end{gathered}$$

has the solution $X(t,\omega)=(1-\omega+t)e^{\omega t}$, $t\in[0,\infty)$.

If $\mathbb{T}=\mathbb{N}$, then $\mu(n)=1$ for all $n\in\mathbb{N}$, and $e_\omega(n,0)=(1+\omega)^n$. Moreover, in this case we have

$$\int_0^t \frac{1}{1+\mu(s)\omega}\Delta s=\sum_{s\in[0,n)}\frac{1}{1+\omega}=\frac{n}{1+\omega}.$$

It follows that the difference initial-value problem

$$\begin{gathered}X_{n+1}(\omega)=_P (1+\omega)X_n(\omega)+(1+\omega)^n,\quad n\in\mathbb{N}\\ X_0(\omega)=_P 1-\omega,\end{gathered}$$

has the solution $X_n(\omega)=(1-\omega+\frac{n}{1+\omega})(1+\omega)^n$, $n\in\mathbb{N}$.

Example 3.7. Let us consider $\Omega=(0,1)$, $\mathcal{F}$ the σ-algebra of all Borel subsets of Ω, P the Lebesgue measure on Ω, and the initial-value problem

$$\begin{gathered}X^\Delta(t,\omega)=_P -\omega X^\sigma(t,\omega)+e_{\ominus\omega}(t,t_0),\quad t\in[t_0,\infty)_\mathbb{T}\\ X(t_0,\omega)=_P 1-\omega.\end{gathered}\tag{3.6}$$

The initial-value problem (3.6) has a unique solution on $[t_0,\infty)_\mathbb{T}$, given by

$$X(t,\omega)=_P (1-\omega)e_{\ominus\omega}(t,t_0)+\int_0^t e_{\ominus\omega}(t,s)e_{\ominus\omega}(s,t_0)\Delta s;$$

that is,

$$X(t,\omega)=_P (1-\omega-t_0+t)e_{\ominus\omega}(t,t_0),\ \ t\in[t_0,\infty)_\mathbb{T}.$$

If $\mathbb{T}=h\mathbb{N}$ with $h>0$, then $\mu(t)=h$ for all $t\in h\mathbb{N}$, and $e_{\ominus\omega}(t,0)=(1+\omega h)^{-t/h}$. It follows that the h-difference initial-value problem

$$\begin{gathered}X_{t+h}(\omega)=_P \frac{1}{1+\omega h}X_t(\omega)+h(1+\omega h)^{-t/h-1},\quad t\in h\mathbb{N}\\ X_0(\omega)=_P 1-\omega,\end{gathered}$$

has the unique solution $X_t(\omega)=_P (1-\omega+t)(1+\omega h)^{-t/h}$, $t\in h\mathbb{N}$.

If $\mathbb{T}=2^\mathbb{N}$, then $\mu(t)=t$ for all $t\in 2^\mathbb{N}$, and $e_{\ominus\omega}(t,0)=\prod_{s\in[0,t)}(1+\omega s)^{-1}$. It follows that the 2-difference initial value problem

$$\begin{gathered}X_t(\omega)=_P (1+\omega t)X_{2t}(\omega)-t\prod_{s\in[1,t)}(1+\omega s)^{-1},\quad t\in 2^\mathbb{N}\\ X_1(\omega)=_P 1-\omega,\end{gathered}$$

has the unique solution $X_t(\omega)=_P (1-\omega+t)\prod_{s\in[1,t)}(1+\omega s)^{-1}$, $t\in 2^\mathbb{N}$.

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