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BOUNDED AND LARGE RADIALLY SYMMETRIC SOLUTIONS FOR SOME (p,q)-LAPLACIAN STATIONARY SYSTEMS

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Abstract. This article concerns radially symmetric positive solutions of second-order quasilinear elliptic systems. In terms of the growth of the variable potential functions, we establish conditions such that the solutions are either bounded or blow up at infinity.

1. Introduction

Existence and nonexistence of solutions of second-order quasilinear elliptic systems of the form

$$\begin{aligned} \operatorname{div}(|\nabla u|^{p-2}\nabla u) &= \varphi(|x|)g_1(v)g_2(u), \quad \text{in } \mathbb{R}^n,\\ \operatorname{div}(|\nabla v|^{q-2}\nabla v) &= \psi(|x|)f_1(u)f_2(v), \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

have been intensively studied in the previous few years. See, for example, [1, 2, 9, 13, 14, 16, 17, 19, 20] and the reference therein. Problem (1.1) arises in the theory of quasiregular and quasiconformal mappings as well as in the study of non-Newtonian fluids. In the latter case, the pair (p,q) is a characteristic of the medium. Media with $(p,q) > (2,2)$ are called dilatant fluids and those with $(p,q) < (2,2)$ are called pseudoplastics. If $(p,q) = (2,2)$, they are Newtonian fluids. When $p = q = 2$ system (1.1) becomes

$$\begin{aligned} \Delta u &= \varphi(|x|)g_1(v)g_2(u), \quad \text{in } \mathbb{R}^n,\\ \Delta v &= \psi(|x|)f_1(u)f_2(v), \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{1.2}$$

for which the existence and non-existence of positive radial entire large or bounded solutions has been extensively studied. When $f_2 = g_2 = 1$, $g_1(v) = v^\alpha$, $f_1(u) = u^\beta$, $0 < \alpha \le \beta$, Lair and Wood [11] considered the existence and nonexistence of entire positive radial solutions to (1.2) under the conditions of integrability or nonintegrability of the functions $r \to r\varphi(r)$ and $r \to r\psi(r)$ on $(0,\infty)$. Their results were extended by Cîrstea and Rădulescu [5], Wang and Wood [18], Ghergu and Râdulescu [7], Peng and Song [15], Ghanmi, Mâagli, Râdulescu and Zeddini [6], Li, Zhang, Zhang [12] and Zhang [21]. Many generalizations of these results have been extended to system (1.1). See, for example, [17, 20]. Our purpose is to generalize the results of [6, 21] to systems (1.1) under the hypotheses that

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the radial potentials φ, ψ are nonnegative continuous functions on $(0,\infty)$ and the nonlinearities f_i, g_i $(i=1,2)$ are nonnegative, continuous and nondecreasing on $[0,\infty)$. In all the results, we establish in this paper we study only positive radial solutions in the sense of distributions, especially because of the physical meaning of the corresponding unknowns.

To discuss the existence of positive radial solutions to this class of nonlinear systems, we are first concerned with the following two systems of differential equations

$$\begin{gathered}
\frac{1}{A}(A\phi_p(y'))' = \varphi(t)g_1(z)g_2(y), \quad \text{in } (0,\infty),\\
\frac{1}{B}(B\phi_q(z'))' = \psi(t)f_1(y)f_2(z), \quad \text{in } (0,\infty),\\
y(0)=a>0, \quad z(0)=b>0,\\
\lim_{t\to 0} A(t)\phi_p(y'(t)) = \lim_{t\to 0} B(t)\phi_q(z'(t)) = 0,
\end{gathered} \tag{1.3}$$

and

$$\begin{gathered}
\frac{1}{A}(A\phi_p(y'))' = \varphi(t)g_1(z)g_2(y)\,, \quad \text{in } (0,\infty),\\
\frac{1}{B}(B\phi_q(z'))' = \psi(t)f_1(y)f_2(z),, \quad \text{in } (0,\infty),\\
y(\infty) = \lim_{t\to\infty} y(t) = c>0, \quad z(\infty) = \lim_{t\to\infty} z(t) = d>0,\\
\lim_{t\to 0} A(t)\phi_p(y'(t)) = \lim_{t\to 0} B(t)\phi_q(z'(t)) = 0\,,
\end{gathered} \tag{1.4}$$

where $p,q>1$, $\phi_k(x) = |x|^{k-2}x$ for $k=p,q$ and A,B are continuous functions in $[0,\infty)$, differentiable and positive in $(0,\infty)$ and satisfy the following growth hypotheses:

$$\int_0^1 \Big[\frac{1}{A(t)}\int_0^t A(s)\,ds\Big]^{1/(p-1)} dt < \infty, \quad \int_0^1 \Big[\frac{1}{B(t)}\int_0^t B(s)\,ds\Big]^{1/(q-1)} dt < \infty\,.$$

In particular, these assumptions are fulfilled if A and B are nondecreasing.

In the sequel, we denote by $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$ and we remark that ϕ_k is a multiplicative function for $k=p,q$. Namely $\phi_k(xy) = \phi_k(x)\phi_k(y)$ for $x>0$ and $y>0$. Moreover $\phi_{p'}$ and $\phi_{q'}$ are respectively the inverse functions of ϕ_p and ϕ_q.

For any nonnegative measurable functions φ in $(0,\infty)$, we define

$$\begin{gathered}
K_p\varphi(t) = \int_0^t \phi_{p'}\Big(\frac{1}{A(r)}\int_0^r A(s)\varphi(s)ds\Big)dr\,,\\
S_q\varphi(t) = \int_0^t \phi_{q'}\left(\frac{1}{B(r)}\int_0^r B(s)\varphi(s)ds\right)dr\,,\\
G_p\varphi(t) = \int_t^\infty \phi_{p'}\Big(\frac{1}{A(r)}\int_0^r A(s)\varphi(s)ds\Big)dr\,,\\
H_q\varphi(t) = \int_t^\infty \phi_{q'}\Big(\frac{1}{B(r)}\int_0^r B(s)\varphi(s)ds\Big)dr\,.
\end{gathered}$$

Finally, we define for $\beta>0$ the function F_β on $[\beta,\infty)$ by

$$F_\beta(t) = \int_\beta^t \frac{ds}{\phi_{p'}(g_1(s)g_2(s)) + \phi_{q'}(f_1(s)f_2(s))}$$

and we note that F_β has an inverse function F_β^{-1} on $[\beta,\infty)$.

2. Main results

We are first concerned with the existence of a positive solution of the system (1.3). For this purpose, we assume that φ, ψ, f_i, g_i $(i = 1, 2)$ satisfy the following hypotheses.

(H1) $\varphi, \psi : (0, \infty) \to [0, \infty)$ are continuous functions satisfying

$$\int_0^1 \Big[\frac{1}{A(t)} \int_0^t A(s)\varphi(s)\,ds\Big]^{1/(p-1)} dt < \infty,$$
$$\int_0^1 \Big[\frac{1}{B(t)} \int_0^t B(s)\psi(s)\,ds\Big]^{1/(q-1)} dt < \infty.$$

(H2) The functions f_i, g_i: $[0, \infty) \to [0, \infty)$ are nondecreasing continuous, positive on $(0, \infty)$.

Our existence result for (1.3) is the following

Theorem 2.1. *Under the hypotheses* (H1)–(H2) *and*

(H3) $K_p\varphi(t) + S_q\psi(t) < F_{a+b}(\infty)$ *for all* $t > 0$,

System (1.3) *has a positive solution* $(y, z) \in \big(C([0, \infty)) \cap C^1((0, \infty))\big)^2$ *satisfying for each* $t \in [0, \infty)$

$$a + \phi_{p'}(g_1(b)g_2(a))K_p\varphi(t) \le y(t) \le F^{-1}_{a+b}[K_p\varphi(t) + S_q\psi(t)],$$
$$b + \phi_{q'}(f_1(a)f_2(b))S_q\psi(t) \le z(t) \le F^{-1}_{a+b}[K_p\varphi(t) + S_q\psi(t)].$$

As a consequence of this result we obtain the following

Corollary 2.2. *Under the hypotheses* (H1)–(H3) *and*

(H4) $K_p\varphi(\infty) < \infty$ *and* $S_q\psi(\infty) < \infty$,

System (1.3) *has a positive bounded solution* $(y, z) \in \big(C([0, \infty)) \cap C^1((0, \infty))\big)^2$.

Corollary 2.3. *Under the hypotheses* (H1)–(H3) *and*

(H5) $K_p\varphi(\infty) = S_q\psi(\infty) = \infty$,

System (1.3) *has a positive solution* $(y, z) \in \big(C([0, \infty)) \cap C^1((0, \infty))\big)^2$ *satisfying* $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = \infty$.

Next, we investigate the existence of positive solution to (1.4).

Theorem 2.4. *Under hypotheses* (H1), (H2), (H4) *and*

(H6) *There exist* $c > 0$ *and* $d > 0$ *such that*

$$c - \phi_{p'}(g_1(d)g_2(c))K_p\varphi(\infty) > 0, \quad d - \phi_{q'}(f_1(c)f_2(d))S_q\psi(\infty) > 0\,,$$

Problem (1.4) *has a positive bounded solution*

$$(y, z) \in \big(C([0, \infty)) \cap C^1((0, \infty))\big) \times \big(C([0, \infty)) \cap C^1((0, \infty))\big)$$

satisfying, for each $t \in [0, \infty)$,

$$c - \phi_{p'}(g_1(d)g_2(c))G_p\varphi(t) \le y(t) \le c,$$
$$d - \phi_{q'}(f_1(c)f_2(d))H_q\psi(t) \le z(t) \le d.$$

Remark 2.5. Let $g_1(t) = t^{\alpha_1}$, $g_2(t) = t^{\alpha_2}$, $f_1(t) = t^{\beta_1}$ and $f_2(t) = t^{\beta_2}$ with $\alpha_i, \beta_i \ge 0$. Then, the condition (H6) is satisfied for infinitely many positive real numbers c, d if $\alpha_1\beta_1 \ne (p - 1 - \alpha_2)(q - 1 - \beta_2)$.

Now, we give our existence results for (1.1).

Theorem 2.6. *Assume that* (H2) *is satisfied and that* (H1) *and* (H3) *are satisfied with* $A(t) = B(t) = t^{n-1}$. *Then* (1.1) *has infinitely many positive continuous radial solutions* (u, v). *Moreover,*

- *If*
$$\int_0^\infty \phi_{p'}\Big(\frac{1}{r^{n-1}}\int_0^r s^{n-1}\varphi(s)ds\Big)dr = \int_0^\infty \phi_{q'}\Big(\frac{1}{r^{n-1}}\int_0^r s^{n-1}\psi(s)ds\Big)dr = \infty,$$
then these solutions are large; i.e., $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = \infty$.
- *If*
$$\int_0^\infty \phi_{p'}\Big(\frac{1}{r^{n-1}}\int_0^r s^{n-1}\varphi(s)ds\Big)dr < \infty$$
and
$$\int_0^\infty \phi_{q'}\left(\frac{1}{r^{n-1}}\int_0^r s^{n-1}\psi(s)ds\right)dr < \infty,$$
then u *and* v *are bounded.*

Next, we replace hypothesis (H3) by hypothesis (H6) to obtain the existence of positive continuous bounded radial solutions to (1.1).

Theorem 2.7. *Let* f_i, g_i, *satisfying* (H2) *and assume that* (H1), (H4), (H6) *are satisfied with* $A(t) = B(t) = t^{n-1}$. *Then* (1.1) *has a positive radial bounded solution* (u, v) *with*
$$\lim_{|x|\to\infty} u(x) = \text{const} > 0, \qquad \lim_{|x|\to\infty} v(x) = \text{const} > 0.$$

3. Proof of main results

Proof of Theorem 2.1. Let $(y_k)_{k\geq 0}$ and $(z_k)_{k\geq 0}$ be sequences of positive continuous functions defined on $[0, \infty)$ by
$$y_0(t) = a, \quad z_0(t) = b,$$
$$y_{k+1}(t) = a + \int_0^t \phi_{p'}\Big(\frac{1}{A(r)}\int_0^r A(s)\varphi(s)g_1(z_k(s))g_2(y_k(s))ds\Big)dr$$
$$z_{k+1}(t) = b + \int_0^t \phi_{q'}\Big(\frac{1}{B(r)}\int_0^r B(s)\psi(s)f_1(y_k(s))f_2(z_k(s))ds\Big)dr.$$
Clearly $y_k, z_k \in C([0,\infty)) \cap C^1((0,\infty))$ and positive, so we deduce from the monotonicity of f_i, g_i, $\phi_{p'}$ and $\phi_{q'}$ that $(y_k)_{k\geq 0}$ and $(z_k)_{k\geq 0}$ are nondecreasing sequences and for each $k \in \mathbb{N}$, the functions $t \to y_k(t)$ and $t \to z_k(t)$ are nondecreasing. Hence, for each $t \in (0, \infty)$,
$$\begin{aligned}
&y'_{k+1}(t)\\
&= \phi_{p'}\Big(\frac{1}{A(t)}\int_0^t A(s)\varphi(s)g_1(z_k(s))g_2(y_k(s))ds\Big)\\
&\leq \phi_{p'}(g_1(z_k(t))g_2(y_k(t)))\phi_{p'}\Big(\frac{1}{A(t)}\int_0^t A(s)\varphi(s)ds\Big)\\
&\leq \phi_{p'}(g_1(z_{k+1}(t) + y_{k+1}(t))g_2(y_{k+1}(t) + z_{k+1}(t)))\phi_{p'}\Big(\frac{1}{A(t)}\int_0^t A(s)\varphi(s)ds\Big)\\
&\leq [\phi_{p'}((g_1(z_{k+1}(t) + y_{k+1}(t))g_2(y_{k+1}(t) + z_{k+1}(t)))
\end{aligned}$$

$$+\phi_{q'}((f_1(z_{k+1}(t)+y_{k+1}(t))f_2(y_{k+1}(t)+z_{k+1}(t)))]\phi_{p'}\Big(\frac{1}{A(t)}\int_0^t A(s)\varphi(s)ds\Big)$$

Which implies, by putting $w_k = y_k + z_k$, that

$$\begin{aligned}&\frac{y'_{k+1}(t)}{\phi_{p'}((g_1(w_{k+1}(t))g_2(w_{k+1}(t)))+\phi_{q'}((f_1(w_{k+1}(t))f_2(w_{k+1}(s)))}\\&\leq \phi_{p'}\Big(\frac{1}{A(t)}\int_0^t A(s)\varphi(s)ds\Big),\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\frac{z'_{k+1}(t)}{\phi_{p'}((g_1(w_{k+1}(t))g_2(w_{k+1}(t)))+\phi_{q'}((f_1(w_{k+1}(t))f_2(w_{k+1}(t)))}\\&\leq \phi_{q'}\Big(\frac{1}{B(t)}\int_0^t B(s)\psi(s)ds\Big)\end{aligned}$$

Consequently,

$$\int_0^t \frac{w'_k(s)ds}{\phi_{p'}(g_1(w_k(t))g_2(w_k(s)))+\phi_{q'}((f_1(w_k(s))f_2(w_k(s)))}\leq K_p\varphi(t)+S_q\psi(t),$$

which gives

$$\int_{a+b}^{w_k(t)}\frac{ds}{\phi_{q'}(f_1(s)f_2(s))+\phi_{p'}(g_1(s)g_2(s))}\leq K_p\varphi(t)+S_q\psi(t).$$

Namely

$$F_{a+b}(y_k(t)+z_k(t))\leq K_p\varphi(t)+S_q\psi(t).$$

Which by hypothesis (H3) implies

$$y_k(t)+z_k(t)\leq F_{a+b}^{-1}(K_p\varphi(t)+S_q\psi(t)).$$

Therefore, the sequences $(y_k)_{k\geq 0}$ and $(z_k)_{k\geq 0}$ converge locally uniformly to two functions y and z that satisfy for each $t\in[0,\infty)$,

$$\begin{gathered}y(t)=a+\int_0^t\phi_{p'}\Big(\frac{1}{A(r)}\int_0^r A(s)\varphi(s)g_1(z(s))g_2(y(s))ds\Big)dr,\\ z(t)=b+\int_0^t\phi_{q'}\Big(\frac{1}{B(r)}\int_0^r B(s)\psi(s)f_1(y(s))f_2(z(s))ds\Big)dr\end{gathered}$$

Hence, $y,z\in C([0,\infty))\cap C^1((0;\infty))$ and (y,z) is a solution of (1.3) satisfying

$$\begin{gathered}a+\phi_{p'}(g_1(b)g_2(a))K_p\varphi(t)\leq y(t)\leq F_{a+b}^{-1}(K_p\varphi(t)+S_q\psi(t)),\\ b+\phi_{q'}(f_1(a)f_2(b))S_q\psi(t)\leq z(t)\leq F_{a+b}^{-1}(K_p\varphi(t)+S_q\psi(t)).\end{gathered}$$

To state another corollary of Theorem 2.1, we consider two continuous functions $h,k:[0,\infty)\to[0,\infty)$ and study the existence of positive solutions for the system

$$\begin{gathered}\frac{1}{A}(A\phi_p(y'))'+h(y)|y'|^p=\varphi(t)g_1(z)g_2(y),\quad \text{in } (0,\infty),\\ \frac{1}{B}(B\phi_q(z'))'+k(z)|z'|^q=\psi(t)f_1(y)f_2(z),\quad \text{in } (0,\infty),\\ y(0)=a>0,\quad z(0)=b>0,\\ \lim_{t\to 0}A(t)\phi_p(y'(t))=\lim_{t\to 0}B(t)\phi_q(z'(t))=0.\end{gathered}\tag{3.1}$$

To this aim, we define

$$\rho_1(t) = \int_0^t \exp\Big(\frac{1}{p-1}\int_0^\zeta h(s)ds\Big)d\zeta, \quad \rho_2(t) = \int_0^t \exp\Big(\frac{1}{q-1}\int_0^\zeta k(s)ds\Big)d\zeta.$$

Clearly ρ_1, ρ_2 are bijections from $[0,\infty)$ to itself. Let M_1, M_2, N_1 and N_2 be the functions defined on $[0,\infty)$ by $M_1\circ\rho_2 = g_1$, $M_2\circ\rho_1 = ({\rho_1}')^{p-1}g_2$, $N_1\circ\rho_1 = f_1$ and $N_2\circ\rho_2 = ({\rho_2}')^{q-1}f_2$.

Corollary 3.1. *Under the hypotheses* (H1), (H2) *and*

(H3') *for all* $t>0$,
$$K_p\varphi(t) + S_q\psi(t) < \int_{\rho_1(a)+\rho_2(b)}^\infty \frac{dt}{\phi_{p'}(M_1(t)M_2(t)) + \phi_{q'}(N_1(t)N_2(t))},$$

System (3.1) *has a positive solution* $(y,z)\in\big(C([0,\infty))\cap C^1((0,\infty))\big)\times\big(C([0,\infty))\cap C^1((0,\infty))\big)$. *Moreover, when* $K_p\varphi(\infty)<\infty$ *and* $S_q\psi(\infty)<\infty$, y *and* z *are bounded; when* $K_p\varphi(\infty) = S_q\psi(\infty) = \infty$, $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = \infty$.

Proof. Put $Y=\rho_1(y)$ and $Z=\rho_2(z)$. Then (y,z) is a solution of (3.1) if and only if (Y,Z) is a solution of

$$\begin{gathered}
\frac{1}{A}(A\phi_p(Y'))' = \varphi\, M_1(Z)M_2(Y), \quad \text{in } (0,\infty),\\
\frac{1}{B}(B\phi_q(Z'))' = \psi\, N_1(Y)N_2(Z), \quad \text{in } (0,\infty),\\
Y(0)=\rho_1(a)>0, \quad Z(0)=\rho_2(b)>0,\\
\lim_{t\to0} A(t)\phi_p(Y'(t)) = \lim_{t\to0} B(t)\phi_q(Z'(t)) = 0,
\end{gathered}$$

So the result follows from Theorem 2.1. $\square$

Next, we aim to prove Theorem 2.4. We note that the proof established in [6] for the case $p=q=2$ and $g_2=f_2=1$ can not be adapted. So we will use a fixed point argument.

Proof of Theorem 2.4. Let $C_0([0,\infty)) = \{\omega\in C([0,\infty),\mathbb{R}) : \lim_{t\to\infty}|\omega(t)| = 0\}$. Clearly $C_0([0,\infty))$ is a Banach space endowed with the uniform norm $\|\omega\|_\infty = \sup_{t\in[0,\infty)}|\omega(t)|$.

To apply the Schauder fixed point theorem, we put $c_1 = \phi_{p'}(g_1(d)g_2(c))K_p\varphi(\infty)$, $d_1 = \phi_{q'}(f_1(c)f_2(d))S_q\psi(\infty)$ and we consider the nonempty closed convex set

$$\Lambda = \{(\omega,\,\theta)\in(C_0([0,\,\infty)))^2 : -c_1\le\omega\le0 \text{ and } -d_1\le\theta\le0\}.$$

Consider the operator T defined on Λ by $T(\omega,\theta) = (\widetilde{\omega},\widetilde{\theta})$, where

$$\begin{aligned}
\widetilde{\omega}(t) &= -G_p(\varphi\, g_1(\theta+d)g_2(\omega+c))(t)\\
&= -\int_t^\infty \phi_{p'}\Big(\frac{1}{A(r)}\int_0^r A(s)\varphi(s)g_1(\theta(s)+d)g_2(\omega(s)+c)ds\Big)dr\\
\widetilde{\theta}(t) &= -H_q(\psi\, f_1(\omega+c)f_2(\theta+d))(t)\\
&= -\int_t^\infty \phi_{q'}\left(\frac{1}{B(r)}\int_0^r B(s)\psi(s)f_1(\omega(s)+c)f_2(\theta(s)+d)ds\right)dr.
\end{aligned}$$

First, we show that $T\Lambda\subset\Lambda$. Let $(\omega,\theta)\in\Lambda$, then using hypotheses (H1), (H2) and (H4) we deduce that $(\widetilde{\omega},\widetilde{\theta})\in C([0,\,\infty))$. Moreover, since $\lim_{t\to\infty} G_p\varphi(t) =$

$\lim_{t\to\infty} G_q\psi(t) = 0$, it follows that $\lim_{t\to\infty} |\widetilde{\omega}(t)| = \lim_{t\to\infty} |\widetilde{\theta}(t)| = 0$. Which implies that $\widetilde{\omega}, \widetilde{\theta} \in C_0([0, \infty))$. Using again the monotonicity of f_i, g_i we deduce that $(\widetilde{\omega}, \widetilde{\theta}) \in \Lambda$ and consequently $T\Lambda \subset \Lambda$.

Secondly, we will prove that $T\Lambda$ is relatively compact in $(C_0([0,\infty)))^2$. Clearly $T\Lambda$ is uniformly bounded in $(C_0([0,\infty)))^2$. Let us prove that $T\Lambda$ is equicontinuous on $[0,\infty)$ and satisfy the property $\lim_{t\to\infty} \sup_{(\omega,\,\theta)\in\Lambda} |\widetilde{\omega}(t)| + |\widetilde{\theta}(t)| = 0$ known as equidecay property to 0 at infinity. Let $t_1, t_2 \in [0,\infty]$ with $t_1 < t_2$. Then for each $(\omega,\, \theta) \in \Lambda$ we have

$$
\begin{aligned}
|\widetilde{\omega}(t_1) - \widetilde{\omega}(t_2)| &= \int_{t_1}^{t_2} \phi_{p'}\Big(\frac{1}{A(r)} \int_0^r A(s)\varphi(s) g_1(\theta(s)+d) g_2(\omega(s)+c) ds\Big) dr \\
&\le \phi_{p'}(g_1(d)g_2(c)) \int_{t_1}^{t_2} \phi_{p'}\Big(\frac{1}{A(r)} \int_0^r A(s)\varphi(s) ds\Big) dr
\end{aligned}
$$

and

$$
|\widetilde{\theta}(t_1) - \widetilde{\theta}(t_2)| \le \phi_{q'}(f_1(c)f_2(d)) \int_{t_1}^{t_2} \phi_{q'}\Big(\frac{1}{B(r)} \int_0^r B(s)\psi(s) ds\Big) dr.
$$

Since, the functions $r \mapsto \phi_{p'}\big(\frac{1}{A(r)} \int_0^r A(s)\varphi(s) ds\big)$ and $r \mapsto \phi_{q'}\big(\frac{1}{B(r)} \int_0^r B(s)\psi(s)\, ds\big)$ are integrable on $(0,\infty)$ by hypothesis (H4), we deduce that $T\Lambda$ is equicontinous on $[0,\infty)$ and equidecays to 0 at infinity. Hence it follows by Ascoli's theorem, [8, p.185], that $T\Lambda$ is relatively compact in $(C_0([0,\infty)))^2$.

Finally, we prove the continuity of T in Λ. Let $(\omega_m, \theta_m)_m$ be a sequence in Λ which converges uniformly on $[0,\infty)$ to $(\omega,\theta) \in \Lambda$. Using the continuity of f_i, g_i and the dominated convergence theorem, we deduce that $(\widetilde{\omega_m})$ and $(\widetilde{\theta_m})$ converge pointwise respectively to $\widetilde{\omega}$ and $\widetilde{\theta}$. Now, since $T\Lambda$ is equicontinuous on $[0,\infty)$, then $(\widetilde{\omega_m})$ and $(\widetilde{\theta_m})$ converge uniformly on each compact of $[0,\infty)$ respectively to $\widetilde{\omega}$ and $\widetilde{\theta}$. This together with the fact that $\widetilde{\omega}, \widetilde{\theta} \in C_0([0,\infty))$ and $(\widetilde{\omega_m}, \widetilde{\theta_m})$ have the equidecay property imply that $(\widetilde{\omega_m})$ converges uniformly on $[0,\infty)$ to $\widetilde{\omega}$ and $(\widetilde{\theta_m})$ converges uniformly on $[0,\infty)$ to $\widetilde{\theta}$. This proves the continuity of T.

Therefore, there exists $(\omega,\theta) \in \Lambda$ such that $T(\omega,\theta) = (\omega,\theta)$ by the Schauder fixed point theorem. Put $y = \omega + c$ and $z = \theta + d$. Then y, z satisfy the integral equations

$$
\begin{aligned}
y(t) &= c - \int_t^{\infty} \phi_{p'}\Big(\frac{1}{A(r)} \int_0^r A(s)\varphi(s) g_1(z(s)) g_2(y(s)) ds\Big) dr \\
z(t) &= d - \int_t^{\infty} \phi_{q'}\Big(\frac{1}{B(r)} \int_0^r B(s)\psi(s) f_1(y(s)) f_2(z(s)) ds\Big) dr.
\end{aligned}
$$

Clearly $(y,z) \in \big(C([0,\infty)) \cap C^1((0,\infty))\big)^2$, satisfying for each $t \in [0,\infty)$

$$
\begin{gathered}
c - \phi_{p'}(g_1(d)g_2(c)) G_p\varphi(t) \le y(t) \le c, \\
d - \phi_{q'}(f_1(c)f_2(d)) H_q\psi(t) \le v(t) \le d
\end{gathered}
$$

and (y,z) is a positive bounded solution of (1.4). □

Proof of Theorems 2.6 and 2.7. We first observe that (u,v) is a positive radial entire solution of (1.1) if and only if the function $(y(t), z(t)) = (u(x), v(x))$, $t = |x|$, satisfies the system of second order ordinary differential equations

$$\begin{gathered}\frac{1}{t^{n-1}}(t^{n-1}\phi_p(y'))' = \varphi(t)g_1(z)g_2(y), \quad t>0,\\ \frac{1}{t^{n-1}}(t^{n-1}\phi_p(z'))' = \psi(t)f_1(y)f_2(z), \quad t>0,\\ y'(0)=0, \quad z'(0)=0.\end{gathered} \tag{3.2}$$

Hence the result follows from Theorem 2.1 with $A(t)=B(t)=t^{n-1}$. Since infinitely many positive real numbers a,b can be chosen in (1.3), then we can construct an infinitude of positive radial solutions to (1.1). This completes the proof. $\square$

Next, we consider some continuous functions $\lambda,\mu:[0,\infty)\to[0,\infty)$ and $\varphi,\psi:(0,\infty)\to[0,\infty)$ satisfying:

(H7)
$$\begin{gathered}\int_0^1 \phi_{p'}\Big(r^{1-n}\exp\Big(-\int_0^r \lambda(\zeta)\,d\zeta\Big)\int_0^r s^{n-1}\exp\Big(\int_0^s \lambda(\zeta)d\zeta\Big)\varphi(s)ds\Big)dr<\infty,\\ \int_0^1 \phi_{q'}\Big(r^{1-n}\exp\Big(-\int_0^r \mu(\zeta)d\zeta\Big)\int_0^r s^{n-1}\exp\Big(\int_0^s \mu(\zeta)d\zeta\Big)\psi(s)ds\Big)dr<\infty.\end{gathered}$$

and we define

$$\begin{gathered}K_p^\lambda\varphi(t)=\int_0^t \phi_{p'}\Big(\frac{1}{\exp\big(\int_0^r\lambda(s)ds\big)r^{n-1}}\int_0^r \exp\Big(\int_0^s\lambda(\varsigma)d\varsigma\Big)s^{n-1}\varphi(s)ds\Big)dr,\\ S_q^\mu\psi(t)=\int_0^t \phi_{q'}\Big(\frac{1}{\exp\Big(\int_0^r\mu(s)ds\Big)r^{n-1}}\int_0^r \exp\Big(\int_0^s\mu(\varsigma)d\varsigma\Big)s^{n-1}\psi(s)ds\Big)dr.\end{gathered}$$

Corollary 3.2. *Let f_i,g_i satisfying* (H2) *and let $\lambda,\mu:[0,\infty)\to[0,\infty)$ and $\varphi,\psi:(0,\infty)\to[0,\infty)$ be continuous functions satisfying* (H7). *Assume further that*

(H8) *there exist $a,b>0$ such that $K_p^\lambda\varphi(t)+S_q^\mu\psi(t)<F_{a+b}(\infty)$ for all $t>0$,*

then the problem

$$\begin{gathered}\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda(|x|)|\nabla u|^{p-1}=\varphi(|x|)g_1(v)g_2(u), \quad \text{in } \mathbb{R}^n,\\ \operatorname{div}(|\nabla v|^{q-2}\nabla v)+\mu(|x|)|\nabla v|^{q-1}=\psi(|x|)f_1(u)f_2(v), \quad \text{in } \mathbb{R}^n,\end{gathered} \tag{3.3}$$

has infinitely many positive radial solutions (u,v). Moreover,

- (i) *If $K_p^\lambda\varphi(t)<\infty=S_q^\mu\psi(t)=\infty$, then these solutions are large.*
- (ii) *If $K_p^\lambda\varphi(t)<\infty$ and $S_q^\mu\psi(t)<F_{a+b}(\infty)$, then these solutions are bounded.*

Proof. Let $A(t)=t^{n-1}\exp\big(\int_0^t\lambda(s)ds\big)$ and $B(t)=t^{n-1}\exp\big(\int_0^t\mu(s)ds\big)$. Then, from Theorem 2.1, the system

$$\begin{gathered}\frac{1}{t^{n-1}}(t^{n-1}\phi_p(y'))'+\lambda(t)\phi_p(y') = \varphi(t)g_1(z)g_2(y), \quad t>0,\\ \frac{1}{t^{n-1}}(t^{n-1}\phi_q(z'))'+\mu(t)\phi_q(z') = \psi(t)f_1(y)f_2(z), \quad t>0,\\ y'(0)=0, \quad z'(0)=0,\end{gathered} \tag{3.4}$$

has infinitely many positive solutions $(y,z)\in(C([0,\infty))\times C^1((0,\infty)))^2$. Put $u(x)=y(t)$, $v(x)=z(t)$, with $t=|x|$. Then (u,v) are positive solutions of (3.3). $\square$

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