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LIMIT CYCLES OF THE GENERALIZED LIÉNARD DIFFERENTIAL EQUATION VIA AVERAGING THEORY

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ABSTRACT. We apply the averaging theory of first and second order to a generalized Liénard differential equation. Our main result shows that for any $n, m \geq 1$ there are differential equations $\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$, with f and g polynomials of degree n and m respectively, having at most $[n/2]$ and $\max\{[(n-1)/2] + [m/2], [n + (-1)^{n+1}/2]\}$ limit cycles, where $[\cdot]$ denotes the integer part function.

1. INTRODUCTION

One of the main topics in the theory of ordinary differential equations is the study of limit cycles: their existence, their number, and their stability. A limit cycle of a differential equation is a periodic orbit in the set of all isolated periodic orbits of the differential equation. The Second part of the 16th Hilbert's problem [11] is related to the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. Then there have been hundreds publications about the limit cycles of planar polynomial differential systems. The generalized polynomial Liénard differential equation was introduced in [14], and has the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1.1}$$

where the dot denotes differentiation with respect to time t, and $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively.

The Liénard equation, which is often taken as the typical example of nonlinear self-excited vibration problem, can be used to model resistor-inductor-capacitor circuits with nonlinear circuit elements. It can also be used to model certain mechanical systems which contain the nonlinear damping coefficients and the restoring force or stiffness. Limit cycles usually arise at a Hopf bifurcation in nonlinear systems with varying parameters. In mechanical systems, the varying parameter is frequently a damping coefficient (see [1, 6]). A lot of papers discuss the possible number of limit cycle of Liénard and generalized mixed Rayleigh-Liénard oscillators. Ding and Leung [6] investigated the generalized mixed Rayleigh-Liénard oscillator with highly nonlinear terms. They consider mainly the number of limit cycle bifurcation diagrams of these systems. For the subclass of polynomial vector fields (1.1) we have a simplified version of Hilbert's problem, see [15, 23].

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Many results on limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point, that are so called *small amplitude limit cycles*, see [16] and [20]. We denote by $\hat{H}(m,n)$ the maximum number of small amplitude limit cycles for systems of the form (1.1). The values of $\hat{H}(m,n)$ give a lower bound for the maximum number $H(m,n)$ (i.e. the Hilbert number) of limit cycles that the differential equation (1.1) with m and n fixed can have. For more information about the Hilbert's 16th problem and related topics see [12, 13].

Now we shall describe briefly the main results about the limit cycles on Liénard differential systems.

In 1928 Liénard [14] proved that if $m=1$ and $F(x)=\int_0^x f(s)ds$ is a continuous odd function , which has a unique root at $x=a$ and is monotone increasing for $x\geq a$, then equation (1.1) has a unique limit cycle.

In 1973 Rychkov [21] proved that if $m=1$ and $F(x)=\int_0^x f(s)ds$ is an odd polynomial of degree five, then equation (1.1) has at most two limit cycles.

In 1977 Lins, de Melo and Pugh [15] proved that $H(1,1)=0$ and $H(1,2)=1$.

In 1998 Coppel [5] proved that $H(2,1)=1$.

Dumortier, Li and Rousseau in [9] and [7] proved that $H(3,1)=1$.

In 1997 Dumortier and Chengzhi [8] proved that $H(2,2)=1$.

Blows, Lloyd [2] and Lynch ([17] and [19]) used inductive arguments to prove the following results.

- If g is odd then $\hat{H}(m,n)=[\frac{n}{2}]$.
- If f is even then $\hat{H}(m,n)=n$, whatever g is.
- If f is odd then $\hat{H}(m,2n+1)=[\frac{(m-2)}{2}]+n$.
- If $g(x)=x+g_e(x)$, where g_e is even then $\hat{H}(2m,2)=m$.

Christopher and Lynch [4] developed a new algebraic method for determining the Liapunov quantities of system (1.1) and proved the following:

- $\hat{H}(m,2)=[\frac{(2m+1)}{3}]$,
- $\hat{H}(2,n)=[\frac{(2n+1)}{3}]$,
- $\hat{H}(m,3)=2[\frac{(3m+2)}{8}]$ for all $1<m\leq 50$,
- $\hat{H}(3,n)=2[\frac{(3n+2)}{8}]$ for all $1<n\leq 50$,
- $\hat{H}(4,k)=\hat{H}(k,4)$ for $k=6,7,8,9$ and $\hat{H}(5,6)=\hat{H}(6,5)$.

In 1998 Gasull and Torregrosa [10] obtained upper bounds for $\hat{H}(7,6)$, $\hat{H}(6,7)$, $\hat{H}(7,7)$ and $\hat{H}(4,20)$.

In 2006 Yu and Han [25] proved that $\hat{H}(m,n)=\hat{H}(n,m)$ for $n=4$, $m=10,11,12,13$; $n=5$, $m=6,7,8,9$; $n=6$, $m=5,6$.

By using the averaging theory we shall study in this work the maximum number of limit cycles $\tilde{H}(m,n)$ which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Liénard differential equations of degree m and n as follows:

$$\begin{gathered}\dot{x}=y,\\ \dot{y}=-x-\sum_{k\geq 1}\epsilon^k(f_n^k(x,y)y+g_m^k(x)),\end{gathered}\tag{1.2}$$

where for every k the polynomial $g_m^k(x)$ has degree m, the polynomial $f_n^k(x,y)$ has degree n on x and y and ε is a small parameter, i.e. the maximal number of *medium amplitude limit cycles* which can bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, perturbed as in (1.2). In fact, we mainly shall compute lower estimations of $\tilde{H}(m,n)$. More precisely, we compute the maximum number of limit cycles $\tilde{H}_k(m,n)$ which bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, using the averaging theory of order k, for $k = 1, 2$. Of course $\tilde{H}_k(m,n) \le \tilde{H}(m,n) \le H(m,n)$.

In 2009, Llibre, Meureu and Teixeira [18] obtained lower estimates of $H(m,n)$ for all $m, n \ge 1$ for the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \sum_{k\ge 1} \epsilon^k (f_n^k(x)y + g_m^k(x)), \end{aligned} \tag{1.3}$$

and from these estimates they obtained that $\tilde{H}_k(m,n) \le \hat{H}(m,n)$ for $k = 1, 2, 3$ for the values which $\hat{H}(m,n)$ is known. The main result of this paper is the following.

Theorem 1.1. *If for every $k = 1, 2$, the polynomials $f_n^k(x,y)$ and $g_m^k(x)$ have degree n and m respectively, with $m, n \ge 1$, then for $|\varepsilon|$ sufficiently small, the maximum number of medium limit cycles of the polynomial Liénard differential systems* (1.2) *bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$,*

(a) *using average theory of first order: $\tilde{H}_1(m,n) = [n/2]$;*
(b) *using average theory of second order:*

$$\tilde{H}_2(m,n) = \max\{[(n-1)/2] + [m/2], [n + (-1)^{n+1}/2]\}.$$

We remark that in general, $\tilde{H}_k(m,n) \ne \tilde{H}_k(n,m)$ for $k = 1, 2$.

2. Averaging theory of first and second order

The averaging theory of first and second order for studying periodic orbits was developed in[3, 18]. It is summarized as follows. Consider the differential system

$$x'(t) = \epsilon F_1(t,x) + \epsilon^2 F_2(t,x) + \epsilon^3 R(t,x,\epsilon), \tag{2.1}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\epsilon_f, \epsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses hold:

(i) $F_1(t,.) \in C^1(D)$ for all $t \in \mathbb{R}$, $F_1, F_2, R, D_x F_1$ are locally Lipschitz with respect to x, and R is differentiable with respect to ϵ. We define

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$
$$F_{20}(z) = \frac{1}{T}\int_0^T [D_z F_1(s,z).y_1(s,z) + F_2(s,z)]ds,$$

where $y_1(s,z) = \int_0^s F_1(t,z)dt$.

(ii) For $V \subset D$ an open and bounded set and for each $\epsilon \in (-\epsilon_f, \epsilon_f)\backslash\{0\}$, there exists $a_\epsilon \in V$ such that $F_{10}(a_\epsilon) + \epsilon F_{20}(a_\epsilon) = 0$ and $d_B(F_{10} + \epsilon F_{20}, V, a_\epsilon) \ne 0$.

Then, for $|\epsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(.,\epsilon)$ of the system (2.1) such that $\varphi(0,\epsilon) = a_\epsilon$.

The expression $d_B(F_{10} + \epsilon F_{20}, V, a_\epsilon) \ne 0$ means that the Brouwer degree of the function $F_{10} + \epsilon F_{20} : V \to \mathbb{R}^n$ at the fixed point a_ϵ is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10}+\epsilon F_{20}$ at a_ϵ is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10}+\epsilon F_{20}$ are mainly the zeros of F_{10} for ϵ sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10}+\epsilon F_{20}$ are mainly the zeros of F_{20} for ϵ sufficiently small. In this case the previous result provides the *averaging theory of second order*.

For more information about the averaging theory see [22, 24].

3. Proof of statement (a) of Theorem 1.1

For applying the first-order averaging method, we write system (1.2) with $k=1$, in polar coordinates (r,θ) where $x=r\cos(\theta)$, $y=r\sin(\theta)$, $r>0$. In this way system (1.2) is written in the standard form for applying the averaging theory. If we write $f_n^1(x,y)=f(x,y)=\sum_{i+j=0}^{n} a_{ij}x^iy^j$ and $g_m^1(x)=g(x)=\sum_{i=0}^{m} b_ix^i$, system (1.2) becomes

$$\begin{aligned}\dot r &= -\epsilon[\sum_{i+j=0}^{n} a_{ij}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta)+\sum_{i=0}^{m} b_ir^i\cos^i(\theta)\sin(\theta)],\\ \dot\theta &= -1-\frac{\epsilon}{r}[\sum_{i+j=0}^{n} a_{ij}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta)+\sum_{i=0}^{m} b_ir^i\cos^{i+1}(\theta)].\end{aligned} \tag{3.1}$$

Taking θ as the new independent variable, system (3.1) becomes

$$\frac{dr}{d\theta}=\epsilon\Big(\sum_{i+j=0}^{n} a_{ij}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta)+\sum_{i=0}^{m} b_ir^i\cos^i(\theta)\sin(\theta)\Big)+O(\epsilon^2)$$

and

$$F_{10}(r)=\frac{1}{2\pi}\int_0^{2\pi}\Big(\sum_{i+j=0}^{n} a_{ij}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta)+\sum_{i=0}^{m} b_ir^i\cos^i(\theta)\sin(\theta)\Big)d\theta.$$

To calculate the exact expression of F_{10} we use the following formulas:

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)d\theta=\begin{cases}0 & \text{if } i \text{ is odd, or } j \text{ is odd}\\ \alpha_{ij} & \text{if } i \text{ is even and } j \text{ is even,}\end{cases}$$

$$\int_0^{2\pi}\cos^i(\theta)\sin(\theta)d\theta=0,\quad \text{for } i=0,1,\dots$$

Hence

$$F_{10}(r)=\frac{1}{2\pi}\sum_{i+j=0}^{n} a_{ij}\alpha_{ij}r^{i+j+1}\quad \text{when } i \text{ is even and } j \text{ is even.} \tag{3.2}$$

Then the polynomial $F_{10}(r)$ has at most $[\frac{n}{2}]$ positive roots, and we can choose the coefficients a_{ij} with i even and j even in such a way that $F_{10}(r)$ has exactly $[\frac{n}{2}]$ simple positive roots. Hence statement (a) of Theorem 1.1 is proved.

Example 3.1. We consider the system

$$\begin{aligned}\dot x &= y,\\ \dot y &= -x-\epsilon(3-x+2y+x^2-y^2-2xy^2)y+(x-y).\end{aligned} \tag{3.3}$$

The corresponding system (3.1) associated with (3.3) is

$$\begin{aligned}
\dot r &= -\epsilon r\sin(\theta)(2\sin(\theta)+\cos(\theta)-r(\cos(\theta)\sin(\theta)+2\sin(\theta)^2)\\
&\quad + r^2(\sin(\theta)\cos(\theta)^2-\sin(\theta)^3)-2r^3\cos(\theta)\sin(\theta)^3),\\
\dot\theta &= -1-\epsilon[2r\sin(\theta)\cos(\theta)-r^2\cos(\theta)^2\sin(\theta)\\
&\quad + 2r^2\sin(\theta)^2\cos(\theta)+r^3\sin(\theta)\cos(\theta)^3\\
&\quad - r^3\sin(\theta)^3\cos(\theta)-2r^4\cos(\theta)^2\sin(\theta)^3+r\cos(\theta)^2].
\end{aligned}$$

To look for limit cycles, we must solve the equation

$$F_{10}=\frac{1}{2\pi}\Big(2r\pi-\frac{1}{4}r^3\pi\Big)=0, \tag{3.4}$$

This equation possesses the positive root $r=2$. According with statement (a) of Theorem 1.1, that system (3.3) has exactly one limit cycle bifurcating from the periodic orbits of the linear differential system (3.3) with $\epsilon=0$, using the averaging theory of first order.

4. Proof of statement (b) of Theorem 1.1

For proving statement (b) of Theorem 1.1 we shall use the second-order averaging theory. If we write $f_n^1(x,y)=\sum_{i+j=0}^n a_{ij}x^iy^j$, $f_n^2(x,y)=\sum_{i+j=0}^n c_{ij}x^iy^j$, $g_m^1(x)=\sum_{i=0}^m b_ix^i$ and $g_m^2(x)=\sum_{i=0}^m d_ix^i$ then system (1.2) with $k=2$ in polar coordinates (r,θ), $r>0$ becomes

$$\begin{aligned}
\dot r &= -\epsilon\Big(\sum_{i+j=0}^n a_{ij}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta)+\sum_{i=0}^m b_ir^i\cos^i(\theta)\sin(\theta)\Big)\\
&\quad -\epsilon^2\Big(\sum_{i+j=0}^n c_{ij}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta)+\sum_{i=0}^m d_ir^i\cos^i(\theta)\sin(\theta)\Big),\\
\dot\theta &= -1-\frac{\epsilon}{r}\Big(\sum_{i+j=0}^n a_{ij}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta)+\sum_{i=0}^m b_ir^i\cos^{i+1}(\theta)\Big)\\
&\quad -\frac{\epsilon^2}{r}\Big(\sum_{i+j=0}^n c_{ij}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta)+\sum_{i=0}^m d_ir^i\cos^{i+1}(\theta)\Big).
\end{aligned} \tag{4.1}$$

Taking θ as the new independent variable in the system (4.1), it becomes

$$\frac{dr}{d\theta}=\epsilon F_1(\theta,r)+\epsilon^2F_2(\theta,r)+O(\epsilon^3),$$

where

$$F_1(\theta,r)=\sum_{i+j=0}^n a_{ij}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta)+\sum_{i=0}^m b_ir^i\cos^i(\theta)\sin(\theta),$$

$$\begin{aligned}
&F_2(\theta,r)\\
&=\Big[\sum_{i+j=0}^n c_{ij}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta)+\sum_{i=0}^m d_ir^i\cos^i(\theta)\sin(\theta)\Big]\\
&\quad -r\cos(\theta)\sin(\theta)\Big[\sum_{i+j=0}^n a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)+\sum_{i=0}^m b_ir^{i-1}\cos^i(\theta)\sin(\theta)\Big]^2.
\end{aligned}$$

Now we determine the corresponding function F_{20}. For this we put $F_{10}\equiv 0$ which is equivalent to $a_{ij}=0$ for all i and j even, and we compute

$$\frac{d}{dr}F_1(\theta,r)=\sum_{i+j=0}^{n}(i+j+1)a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+2}(\theta)+\sum_{i=1}^{m}ib_ir^{i-1}\cos^i(\theta)\sin(\theta),$$

and

$$\int_0^\theta F_1(\phi,r)d\phi=y_1^1+y_1^2$$

where

$$\begin{aligned}
y_1^1&=\int_0^\theta\sum_{i+j=0}^{n}a_{ij}r^{i+j+1}\cos^i(\phi)\sin^{j+2}(\phi)\\
&=a_{10}r^2\big(\alpha_{110}\sin(\theta)+\alpha_{210}\ \sin(3\theta)\big)+\dots+a_{lb}r^{l+b+1}\Big(\alpha_{1lb}\sin(\theta)+\alpha_{2lb}\sin(3\theta)\\
&\quad+\dots+\alpha_{\frac{(l+b+2)+1}{2}lb}\sin((l+b+2)\theta)\Big)+a_{01}r^2\big(\alpha_{101}+\alpha_{201}\cos(\theta)\\
&\quad+\alpha_{301}\cos(3\theta)\big)+\dots+a_{cd}r^{c+d+1}\Big(\alpha_{1cd}+\alpha_{2cd}\cos(\theta)+\alpha_{3cd}\cos(3\theta)+\dots\\
&\quad+\alpha_{\frac{(c+d+2)+3}{2}cd}\cos((c+d+2)\theta)\Big)+a_{11}r^3\big(\alpha_{111}+\alpha_{211}\cos(2\theta)\\
&\quad+\alpha_{311}\cos(4\theta)\big)+\dots+a_{ld}r^{l+d+1}\Big(\alpha_{1ld}+\alpha_{2ld}\cos(2\theta)+\alpha_{3ld}\cos(4\theta)+\dots\\
&\quad+\alpha_{\frac{(l+d+2)+2}{2}ld}\cos((l+d+2)\theta)\Big),
\end{aligned}$$

such that l is the greatest odd number and b is the greatest even number so that $l+b$ is less than or equal to n. c is the greatest even number and d is the greatest odd number so that $c+d$ is less than or equal to n. α_{ijk} are real constants exhibited during the computation of $\int_0^\theta\cos^i(\phi)\sin^{j+2}(\phi)d\phi$ for all i and j. and

$$y_1^2=\int_0^\theta\sum_{i=0}^{m}b_ir^i\cos^i(\phi)\sin(\phi)=b_0(1-\cos(\theta))+\dots+b_mr^m\frac{1}{m+1}(1-\cos^{m+1}(\theta).$$

We know from (3.2) that F_{10} is identically zero if and only if $a_{ij}=0$ for all i even and j even. Moreover

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)\sin((2k+1)\theta)d\theta=\begin{cases}0 & \text{if } i \text{ is odd and } j\in\mathbb{N},\\ A_{ij}^{2k+1} & \text{if } i \text{ is even and } j \text{ is odd,}\\ & \text{k=0,1,\dots}\end{cases}$$

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)d\theta\neq 0,\quad \text{if and only if } i \text{ is even and } j \text{ even,}$$

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)\cos((2k+1)\theta)d\theta=\begin{cases}0 & \text{if } j \text{ is odd and } i\in\mathbb{N},\\ B_{ij}^{2k+1} & \text{if } i \text{ is odd and } j \text{ is even,}\\ & k=0,1,\dots\end{cases}$$

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)\cos((2k)\theta)d\theta=0,\quad \text{for } i \text{ odd or } j \text{ odd, } k=0,1,\dots$$

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)\cos^{m+1}(\theta)d\theta=\begin{cases}0 & \text{if } j \text{ is odd and } i,m\in\mathbb{N},\\ M_{ij}^m & \text{if } i \text{ is odd, } j \text{ is even and } m \text{ is even,}\end{cases}$$

$$\int_0^{2\pi} \cos^i(\theta)\sin(\theta)\sin((2k+1)\theta)d\theta = \begin{cases} 0 & \text{if } i \text{ is odd, } k=0,1,\dots \\ N_i^{2k+1} & \text{if } i \text{ is even, } k=0,1,\dots \end{cases}$$

$$\int_0^{2\pi} \cos^i(\theta)\sin(\theta)d\theta = 0, \quad \forall i \in \mathbb{N},$$

$$\int_0^{2\pi} \cos^i(\theta)\sin(\theta)\cos((2k+1)\theta)d\theta = 0, \quad \forall i,k \in \mathbb{N},$$

$$\int_0^{2\pi} \cos^i(\theta)\sin(\theta)\cos((2k)\theta)d\theta = 0, \quad \forall i,k \in \mathbb{N},$$

$$\int_0^{2\pi} \cos^i(\theta)\sin(\theta)\cos^{m+1}(\theta)d\theta = 0, \quad \forall i,m \in \mathbb{N},$$

So

$$\begin{aligned}
&\int_0^{2\pi} \frac{d}{dr}F_1(\theta,r)y_1(\theta,r)d\theta \\
&= \int_0^{2\pi} [\sum_{i+j=0}^{n} (i+j+1)a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+2}(\theta) \\
&\quad + \sum_{i=1}^{m} ib_i r^{i-1}\cos^i(\theta)\sin(\theta)](y_1^1+y_1^2)d\theta \\
&= \sum_{i+j=0}^{n} (i+j+1)a_{ij}r^{i+j}\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)(y_1^1+y_1^2)d\theta \\
&\quad + \sum_{i=1}^{m} ib_i r^{i-1}\int_0^{2\pi}\cos^i(\theta)\sin(\theta)(y_1^1+y_1^2)d\theta \\
&= \sum_{i+j=1_{i\ even,\ j odd}}^{n} (i+j+1)a_{ij}r^{i+j}[a_{10}r^2(\alpha_{110}A_{ij}^1+\alpha_{210}A_{ij}^3)+\dots \\
&\quad + a_{lb}r^{l+b+1}(\alpha_{1lb}A_{ij}^1+\alpha_{2lb}A_{ij}^3+\dots+\alpha_{\frac{(l+b+2)+1}{2}lb}A_{ij}^{l+b+2})] \\
&\quad + \sum_{i+j=1, i \text{ odd}, j \text{ even}}^{n} (i+j+1)a_{ij}r^{i+j}[a_{01}r^2(\alpha_{201}B_{ij}^1+\alpha_{301}B_{ij}^3)+\dots \\
&\quad + a_{cd}r^{c+d+1}(\alpha_{2cd}B_{ij}^1+\alpha_{3cd}B_{ij}^3+\dots+\alpha_{\frac{(c+d+2)+3}{2}cd}B_{ij}^{c+d+2})] \\
&\quad + \sum_{i+j=1, i \text{ odd}, j \text{ even}, m \text{ even}}^{n} (i+j+1)a_{ij}r^{i+j}[-b_0M_{ij}^0-\dots-b_mr^m\frac{1}{m+1}M_{ij}^m] \\
&\quad + \sum_{i=2_{ieven}}^{m} ib_ir^{i-1}[a_{10}r^2(\alpha_{110}N_i^1+\alpha_{210}N_i^3)+\dots \\
&\quad + a_{lb}r^{l+b+1}(\alpha_{1lb}N_i^1+\alpha_{2lb}N_i^3+\dots+\alpha_{\frac{(l+b+2)+1}{2}lb}N_i^{l+b+2})].
\end{aligned}$$

Moreover,

$$\int_0^{2\pi} F_2(\theta,r)d\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi} [\sum_{i+j=0}^{n} c_{ij} r^{i+j+1} \cos^i(\theta)\sin^{j+2}(\theta) + \sum_{i=0}^{m} d_i r^i \cos^i(\theta)\sin(\theta)] d\theta \\
&\quad - \int_0^{2\pi} r\cos(\theta)\sin(\theta)\Big[\sum_{i+j=0}^{n} a_{ij} r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta) + \sum_{i=0}^{m} b_i r^{i-1}\cos^i(\theta)\Big]^2 d\theta,
\end{aligned}
$$

but

$$
\int_0^{2\pi} \cos^i(\theta)\sin^{j+2}(\theta) d\theta = \begin{cases} 0 & \text{if } i \text{ is odd or } j \text{ is odd,} \\ F_{ij} \neq 0 & \text{if } i \text{ is even and } j \text{ even.} \end{cases}
$$

Hence

$$
\begin{aligned}
&\int_0^{2\pi} F_2(\theta, r) d\theta \\
&= \sum_{i+j=0, i \text{ even}, j\text{even}}^{n} C_{ij} F_{ij} r^{i+j+1} \\
&\quad - 2 \sum_{i+j=1, i \text{ even}, j \text{ odd}}^{n} \sum_{l+k=1, l \text{ odd}, k \text{ even}}^{n} a_{ij} a_{lk} r^{i+j+l+k+1} \int_0^{2\pi} \cos^{i+l+1}(\theta)\sin^{j+k+3}(\theta) \\
&\quad - 2 \sum_{i+j=0, i \text{ even}, j \text{ even}}^{n} \sum_{l+k=2, l \text{ odd}, k \text{ odd}}^{n} a_{ij} a_{lk} r^{i+j+l+k+1} \int_0^{2\pi} \cos^{i+k+1}(\theta)\sin^{j+l+3}(\theta) \\
&\quad - 2 \sum_{k=1, k \text{ odd}}^{m} \sum_{i+j=0, i \text{ even}, j \text{ even}}^{n} b_k a_{ij} r^{k+i+j} \int_0^{2\pi} \cos^{k+i+1}(\theta)\sin^{j+2}(\theta) \\
&\quad - 2 \sum_{k=0, k \text{ even}}^{m} \sum_{i+j=0, i \text{ even}, j \text{ odd}}^{n} b_k a_{ij} r^{k+i+j} \int_0^{2\pi} \cos^{k+i+1}(\theta)\sin^{j+2}(\theta).
\end{aligned}
$$

Then $F_{20}(r)$ is the polynomial

$$
\begin{aligned}
&\sum_{i+j=1, i \text{ even}, j \text{ odd}}^{n} (i+j+1) a_{ij} r^{i+j} [a_{10} r^2 (\alpha_{110} A^1_{ij} + \alpha_{210} A^3_{ij}) + \dots \\
&\quad + a_{lb} r^{l+b+1} (\alpha_{1lb} A^1_{ij} + \alpha_{2lb} A^3_{ij} + \dots + \alpha_{\frac{(l+b+2)+1}{2} lb} A^{l+b+2}_{ij})] \\
&\quad + \sum_{i+j=1, i \text{ odd}, j \text{ even}}^{n} (i+j+1) a_{ij} r^{i+j} \big[a_{01} r^2 (\alpha_{201} B^1_{ij} + \alpha_{301} B^3_{ij}) + \dots \\
&\quad + a_{cd} r^{c+d+1} (\alpha_{2cd} B^1_{ij} + \alpha_{3cd} B^3_{ij} + \dots + \alpha_{\frac{(c+d+2)+3}{2} cd} B^{c+d+2}_{ij})\big] \\
&\quad + \sum_{i+j=1, i \text{ odd}, j \text{ even}, m \text{ even}}^{n} (i+j+1) a_{ij} r^{i+j} \big[-b_0 M^0_{ij} - \dots - b_m r^m \frac{1}{m+1} M^m_{ij}\big] \\
&\quad + \sum_{i=2, i \text{ even}}^{m} i b_i r^{i-1} \big[a_{10} r^2 (\alpha_{110} N^1_i + \alpha_{210} N^3_i) + \dots \\
&\quad + a_{lb} r^{l+b+1} (\alpha_{1lb} N^1_i + \alpha_{2lb} N^3_i + \dots + \alpha_{\frac{(l+b+2)+1}{2} lb} N^{l+b+2}_i)\big] \\
&\quad + \sum_{i+j=0, i \text{ even}, j \text{ even}}^{n} C_{ij} F_{ij} r^{i+j+1}
\end{aligned}
$$

$$
\begin{aligned}
&- 2 \sum_{i+j=1, i \text{ even}, j \text{ odd}}^{n} \sum_{l+k=1, l \text{odd}, k \text{ even}}^{n} a_{ij} a_{lk} r^{i+j+l+k+1} \int_0^{2\pi} \cos^{i+l+1}(\theta) \sin^{j+k+3}(\theta) \\
&- 2 \sum_{i+j=0, i \text{ even}, j \text{ even}}^{n} \sum_{l+k=2, l \text{odd}, k \text{odd}}^{n} a_{ij} a_{lk} r^{i+j+l+k+1} \int_0^{2\pi} \cos^{i+k+1}(\theta) \sin^{j+l+3}(\theta) \\
&- 2 \sum_{k=1, k \text{ odd}}^{m} \sum_{i+j=0, i \text{ even}, j \text{ even}}^{n} b_k a_{ij} r^{k+i+j} \int_0^{2\pi} \cos^{k+i+1}(\theta) \sin^{j+2}(\theta) \\
&- 2 \sum_{k=0, k \text{ even}}^{m} \sum_{i+j=0, i \text{ even}, j \text{ odd}}^{n} b_k a_{ij} r^{k+i+j} \int_0^{2\pi} \cos^{k+i+1}(\theta) \sin^{j+2}(\theta).
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
F_{20}(r) = & \sum_{i+j=1, i \text{ even}, j \text{ odd}}^{n} (i+j+1) a_{ij} r^{i+j} \big[a_{10} r^2 (\alpha_{110} A^1_{ij} + \alpha_{210} A^3_{ij}) + \dots \\
& + a_{lb} r^{l+b+1} (\alpha_{1lb} A^1_{ij} + \alpha_{2lb} A^3_{ij} + \dots + \alpha_{\frac{(l+b+2)+1}{2} lb} A^{l+b+2}_{ij}) \big] \\
& + \sum_{i+j=1, i \text{ odd}, j \text{ even}}^{n} (i+j+1) a_{ij} r^{i+j} \big[a_{01} r^2 (\alpha_{201} B^1_{ij} + \alpha_{301} B^3_{ij}) + \dots \\
& + a_{cd} r^{c+d+1} (\alpha_{2cd} B^1_{ij} + \alpha_{3cd} B^3_{ij} + \dots + \alpha_{\frac{(c+d+2)+3}{2} cd} B^{c+d+2}_{ij}) \big] \\
& + \sum_{i+j=1, i \text{ odd}, j \text{ even}, m \text{ even}}^{n} (i+j+1) a_{ij} r^{i+j} [-b_0 M^0_{ij} - \dots - b_m r^m \frac{1}{m+1} M^m_{ij}] \\
& + \sum_{i=2, i \text{ even}}^{m} i b_i r^{i-1} \big[a_{10} r^2 (\alpha_{110} N^1_i + \alpha_{210} N^3_i) + \dots \\
& + a_{lb} r^{l+b+1} (\alpha_{1lb} N^1_i + \alpha_{2lb} N^3_i + \dots + \alpha_{\frac{(l+b+2)+1}{2} lb} N^{l+b+2}_i) \big] \\
& + \sum_{i+j=0, i \text{ even}, j \text{ even}}^{n} C_{ij} F_{ij} r^{i+j+1} \\
& - 2 \sum_{i+j=1, i \text{ even}, j \text{ odd}}^{n} \sum_{l+k=1, l \text{ odd}, k \text{even}}^{n} a_{ij} a_{lk} r^{i+j+l+k+1} F_{(i+l+1)(j+k+1)} \\
& - 2 \sum_{i+j=0, i \text{ even}, j \text{ even}}^{n} \sum_{l+k=2, l \text{ odd}, k \text{ odd}}^{n} a_{ij} a_{lk} r^{i+j+l+k+1} F_{(i+k+1)(j+l+1)} \\
& - 2 \sum_{k=1, k \text{ odd}}^{m} \sum_{i+j=0, i \text{ even}, j \text{ even}}^{n} b_k a_{ij} r^{k+i+j} F_{(k+i+1)j} \\
& - 2 \sum_{k=0, k \text{ even}}^{m} \sum_{i+j=0, i \text{ even}, j \text{ odd}}^{n} b_k a_{ij} r^{k+i+j} F_{(k+i+1)j}.
\end{aligned}
$$

Note that to find the positive roots of F_{20} we must find the zeros of a polynomial in r^2 of degree equal to the

$$
\max \Big\{ \frac{i+j+l+b}{2}, \frac{i+j+c+d}{2}, \frac{i+j+m-1}{2}, \frac{l+b+m-1}{2},
$$

$$\frac{i+j}{2}, \frac{i+j+l+k}{2}, \frac{i+j+m-1}{2}\Big\}$$

we conclude that F_{20} has at most $\max\{[\frac{n-1}{2}]+[\frac{m}{2}],\,[n+\frac{(-1)^{n+1}}{2}]\}$ positive roots. Hence the statement (b) of Theorem 1.1 follows.

Example 4.1. We consider the system

$$\begin{gathered}\dot{x}=y,\\ \dot{y}=-x-\epsilon[(x+2y+xy-y^3-2xy^2)y+x]-\epsilon^2[(3y+xy-x^2+x^3)y+x].\end{gathered} \tag{4.2}$$

To look for the limit cycles, we must solve the equation

$$F_{20}=\frac{1}{2}\Big((\frac{-17}{96})r^7+(\frac{133}{96})r^5-(\frac{1}{3})r^3\Big)=0, \tag{4.3}$$

This equation has two positive roots $r_1 = 2.752278171$ and $r_2 = 0.4984920115$. According with statement (b) of Theorem 1.1, that system (4.2) has exactly two limit cycles bifurcating from the periodic orbits of the linear differential system (4.2) with $\epsilon = 0$, using the averaging theory of second order.

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