Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 28, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLVABILITY OF NONLOCAL PROBLEMS FOR SEMILINEAR ONE-DIMENSIONAL WAVE EQUATIONS

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Abstract. In this article, we prove theorems on existence, uniqueness, and nonexistence of solutions for nonlocal problems of a semilinear wave equations in one space variable.

1. Introduction

In a domain $\Omega : 0 < x < l, 0 < t < l$, we consider the question of finding a solution $u(x,t)$ to the nonlocal problem

$$L_\lambda u := u_{tt} - u_{xx} + \lambda f(x,t,u) = F(x,t), \quad (x,t) \in \Omega, \tag{1.1}$$

satisfying the homogeneous boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \le t \le l, \tag{1.2}$$

the initial condition

$$u(x,0) = \varphi(x), 0 \le x \le l, \tag{1.3}$$

and the nonlocal condition

$$K_\mu u_t := u_t(x,0) - \mu u_t(x,l) = \psi(x), 0 \le x \le l, \tag{1.4}$$

where f, F, φ, ψ are given continuous functions; λ and μ are given nonzero constants. The agreement conditions: $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$, $-\varphi''(0) + \lambda f(0,0,0) = F(0,0)$, $-\varphi''(l) + \lambda f(l,0,0) = F(l,0)$ represent necessary conditions for the solvability of (1.1)-(1.4).

There are many articles devoted to the study nonlocal problems for partial differential equations. In the case of abstract evolution equations and hyperbolic differential equations we refer the reader to [1, 2, 3, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 17, 20].

Definition 1.1. Let $f \in C(\overline{\Omega} \times \mathbb{R})$, $F \in C(\overline{\Omega})$ and functions $\varphi \in C^1([0,l]), \psi \in C([0,l])$ satisfy the agreement conditions $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$. Let $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 : x = 0, 0 \le t \le l$, $\Gamma_2 : x = l, 0 \le t \le l$. We call function u a strong generalized solution of (1.1)-(1.4) of the class C in the domain Ω, if $u \in C^0(\overline{\Omega}, \Gamma) := \{u \in C(\overline{\Omega}), u|_\Gamma = 0\}$ and there exists a sequence of functions $u_n \in C^2(\overline{\Omega}) \cap C^0(\overline{\Omega}, \Gamma)$, such that $u_n \to u$ and $L_\lambda u_n \to F$ in the space $C(\overline{\Omega})$, $u_n|_{t=0} \to \varphi$ in the space $C^1([0,l])$, and $K_\mu u_{nt} \to \psi$ in the space $C([0,l])$.

2000 *Mathematics Subject Classification.* 35L05, 35L20, 35L71.
Key words and phrases. Nonlocal problem; semilinear wave equation;
existence and nonexistence of solutions.
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Submitted November 14, 2011. Published February 15, 2012.

Remark 1.2. Note that a classical solution of (1.1)-(1.4) in the space $C^2(\overline{\Omega})$ represents a strong generalized solution of this problem of class C in the domain Ω in the sense of Definition 1.1. In turn, if the generalized solution of (1.1)-(1.4) of the class C in the domain Ω belongs to the space $C^2(\overline{\Omega})$, then it will be also a classical solution of this problem. Note that a strong generalized solution of (1.1)-(1.4) of the class C in the domain Ω satisfies the conditions (1.2), (1.3) in the ordinary classical sense.

Remark 1.3. Even in the linear case; i.e., for $\lambda = 0$, problem (1.1)-(1.4) is not always well-posed. For example, when $\lambda = 0$ and $|\mu| = 1$, the corresponding to (1.1)-(1.4) homogeneous problem has infinite set of linearly independent solutions (see the Lemma 3.3).

This work is organized as follows. In the Section 2 we study semilinear equation (1.1), when for $|\mu| < 1$ a priori estimate is valid for the strong generalized solution of (1.1)-(1.4) of the class C in the domain Ω in the sense of Definition 1.1. In the Section 3 we reduce problem (1.1)-(1.4) to an equivalent nonlinear integral equation. In the Section 4, base on the results obtained in previous sections, we prove theorems on existence and uniqueness of a solution of (1.1)-(1.4). Finally, in the Section 5, using the method of test-functions [18], we show that when the conditions of nonlinear term of (1.1), introduced in the Section 2, are violated then problem (1.1)-(1.4) may not have solution.

2. A priori estimate for the solution of (1.1)-(1.4)

Let

$$g(x,t,u) = \int_0^u f(x,t,s)ds, \quad (x,t,u) \in \overline{\Omega} \times \mathbb{R}. \tag{2.1}$$

Consider the following conditions imposed on function $g = g(x,t,u)$:

$$g(x,t,u) \geq -M_1, \quad (x,t,u) \in \overline{\Omega} \times \mathbb{R}, \tag{2.2}$$

$$g_t \in C(\overline{\Omega} \times \mathbb{R}), \quad g_t(x,t,u) \leq M_2, \quad (x,t,u) \in \overline{\Omega} \times \mathbb{R}, \tag{2.3}$$

where M_i is a non-negative constant for $i = 1,2$.

Lemma 2.1. *Let $\lambda > 0$, $|\mu| < 1$, $f \in C(\overline{\Omega} \times \mathbb{R})$, $F \in C(\overline{\Omega})$, $\varphi \in C^1([0,l])$, $\psi \in C([0,l])$, $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$, and the conditions* (2.2), (2.3) *be fulfilled. Then for the strong generalized solution $u = u(x,t)$ of* (1.1)-(1.4) *in class C in the domain Ω in the sense of Definition 1.1, following a priori estimate is valid:*

$$\begin{aligned} \|u\|_{C(\overline{\Omega})} \leq c_1 \|F\|_{C(\overline{\Omega})} &+ c_2 \|g(x,0,\varphi(x))\|^{1/2}_{C([0,l])} + c_3 \|\varphi\|_{C^1([0,l])} \\ &+ c_4 \|\psi\|_{C([0,l])} + c_5 \end{aligned} \tag{2.4}$$

with nonnegative constants $c_i = c_i(\lambda, \mu, l, M_1, M_2)$ independent of u, F, φ, ψ, and $c_i > 0$ for $i < 5$.

Proof. Let u be a strong generalized solution of (1.1)-(1.4) of class C in the domain Ω. In view of Definition 1.1 there exists a sequence of the functions $u_n \in C^2(\overline{\Omega}) \cap C^0(\overline{\Omega}, \Gamma)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{\Omega})} = 0, \quad \lim_{n\to\infty} \|L_\lambda u_n - F\|_{C(\overline{\Omega})} = 0, \tag{2.5}$$

$$\lim_{n\to\infty} \|u_n|_{t=0} - \varphi\|_{C^1([0,l])} = 0, \quad \lim_{n\to\infty} \|K_\mu u_{nt} - F\|_{C([0,l])} = 0, \tag{2.6}$$

and therefore
$$\lim_{n\to\infty} \|f(\cdot,\cdot,u_n(\cdot,\cdot)) - f(\cdot,\cdot,u(\cdot,\cdot))\|_{C(\overline{\Omega})} = 0.$$
Consider function $u_n \in C^2(\overline{\Omega}) \cap C^0(\overline{\Omega},\Gamma)$ as a solution of the problem
$$L_\lambda u_n = F_n, \tag{2.7}$$
$$u_n(0,t) = 0, u_n(l,t) = 0, 0 \le t \le l, \tag{2.8}$$
$$u_n(x,0) = \varphi_n(x), 0 \le x \le l, \tag{2.9}$$
$$K_\mu u_{nt} = \psi_n(x), 0 \le x \le l. \tag{2.10}$$
Here
$$F_n := L_\lambda u_n, \varphi_n := u_n|_{t=0},\ \psi_n(x) := K_\mu u_{nt}. \tag{2.11}$$
Multiplying both sides of the equation (2.7) by u_{nt} and integrating in the domain $\Omega_\tau := \{(x,t) \in \Omega : t < \tau\}$, $0 < \tau \le l$, due to the (2.1), we have
$$\begin{aligned} &\frac{1}{2}\int_{\Omega_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt - \int_{\Omega_\tau} \frac{\partial^2 u_n}{\partial x^2}\frac{\partial u_n}{\partial t}\,dx\,dt\\ &+ \lambda\int_{\Omega_\tau} \frac{d}{dt}\big(g(x,t,u_n(x,t)\big)\,dx\,dt - \lambda\int_{\Omega_\tau} g_t(x,t,u_n(x,t)\,dx\,dt\\ &= \int_{\Omega_\tau} F_n\frac{\partial u_n}{\partial t}\,dx\,dt. \end{aligned}\tag{2.12}$$
Let $\omega_\tau : 0 < x < l, t = \tau;\ 0 \le \tau \le l$ and denote by $\nu := (\nu_x,\nu_t)$ the unit vector of the outer normal to $\partial\Omega_\tau$. Since
$$\nu_x\big|_{\omega_\tau\cup\omega_0} = 0, \quad \nu_x\big|_{\Gamma_1} = -1, \quad \nu_x\big|_{\Gamma_2} = 1, \quad \nu_t\big|_{\Gamma} = 0, \quad \nu_t\big|_{\omega_\tau} = 1, \quad \nu_t\big|_{\omega_0} = -1,$$
taking into account the equalities (2.8) and integrating by parts, we obtain
$$\frac{1}{2}\int_{\Omega_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial t}\Big)^2 dx\,dt = \frac{1}{2}\int_{\partial\Omega_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\nu_t\,ds = \frac{1}{2}\int_{\omega_\tau} u_{nt}^2 dx - \frac{1}{2}\int_{\omega_0} u_{nt}^2 dx, \tag{2.13}$$
$$\begin{aligned} -\int_{\Omega_\tau}\frac{\partial^2 u_n}{\partial x^2}\frac{\partial u_n}{\partial t}\,dx\,dt &= \int_{\Omega_\tau}\Big[\frac{1}{2}(u_{nx}^2)_t - (u_{nx}u_{nt})_x\Big]\,dx\,dt\\ &= \frac{1}{2}\int_{\omega_\tau} u_{nx}^2 dx - \frac{1}{2}\int_{\omega_0} u_{nx}^2 dx, \end{aligned}\tag{2.14}$$
$$\begin{aligned} &\lambda\int_{\Omega_\tau}\frac{d}{dt}\big(g(x,t,u_n(x,t)\big)\,dx\,dt\\ &= \lambda\int_{\partial\Omega_\tau} g(x,t,u_n(x,t)\nu_t\,ds\\ &= \lambda\int_{\omega_\tau} g(x,t,u_n(x,t))dx - \lambda\int_{\omega_0} g(x,t,u_n(x,t))dx. \end{aligned}\tag{2.15}$$
In view of (2.13), (2.14), (2.15) from (2.12) we obtain
$$\begin{aligned} &\int_{\omega_\tau}[u_{nt}^2 + u_{nx}^2]dx\\ &= \int_{\omega_0}[u_{nt}^2 + u_{nx}^2]dx - 2\lambda\int_{\omega_\tau} g(x,t,u_n(x,t))dx + 2\lambda\int_{\omega_0} g(x,t,u_n(x,t))dx\\ &\quad + 2\lambda\int_{\Omega_\tau} g_t(x,t,u_n(x,t))\,dx\,dt + 2\int_{\Omega_\tau} F_n u_{nt}\,dx\,dt. \end{aligned}\tag{2.16}$$

Since $g \in C(\overline{\Omega} \times \mathbb{R})$, then due to (2.6), for any $\epsilon > 0$ there exists the number $N = N(\epsilon) > 0$ such that

$$\|g(x, 0, u_n(x, 0)\|_{C([0,l])} \le \|g(x, 0, \varphi(x))\|_{C([0,l])} + \epsilon,\ n > N. \tag{2.17}$$

Below we assume that $n > N$. Let

$$w_n(\tau) := \int_{\omega_\tau} [u_{nt}^2 + u_{nx}^2] dx. \tag{2.18}$$

Since $2F_n u_{nt} \le \epsilon_1^{-1} F_n^2 + \epsilon_1 u_{nt}^2$ for any $\epsilon_1 = \mathit{const} > 0$, then due to (2.2), (2.3), (2.17) and (2.18) from (2.16) it follows that

$$\begin{aligned} w_n(\tau) &\le w_n(0) + 2\lambda l M_1 + 2\lambda l \big(\|g(x, 0, \varphi(x))\|_{C([0,l])} + \epsilon\big) \\ &\quad + 2\lambda l M_2 + \epsilon_1 \int_{\Omega_\tau} u_{nt}^2 \, dx \, dt + \epsilon_1^{-1} \int_{\Omega_\tau} F_n^2 \, dx \, dt. \end{aligned} \tag{2.19}$$

Taking into account that

$$\int_{\Omega_\tau} u_{nt}^2 \, dx \, dt = \int_0^\tau \Big[\int_{\omega_s} u_{nt}^2 dx \Big] ds \le \int_0^\tau \Big[\int_{\omega_s} [u_{nt}^2 + u_{nx}^2] dx \Big] ds = \int_0^\tau w_n(s) ds,$$

from (2.19) we obtain

$$\begin{aligned} &w_n(\tau) \\ &\le \epsilon_1 \int_0^\tau w_n(s) ds + w_n(0) + 2\lambda l \big[M_1 + M_2 + \|g(x, 0, \varphi(x))\|_{C([0,l])} + \epsilon\big] \\ &\quad + \epsilon_1^{-1} \int_{\Omega_\tau} F_n^2 \, dx \, dt, \quad 0 < \tau \le l. \end{aligned} \tag{2.20}$$

Because $\Omega_\tau \subset \Omega$, by the Gronwall's Lemma [11, p. 13], from (2.20) it follows that for $0 < \tau \le l$,

$$\begin{aligned} w_n(\tau) \le \Big[&w_n(0) + 2\lambda l \big(M_1 + M_2 + \|g(x, 0, \varphi(x))\|_{C([0,l])} + \epsilon\big) \\ &+ \epsilon_1^{-1} l^2 \|F_n\|_{C(\overline{\Omega})}^2 \Big] e^{\epsilon_1 \tau}, \end{aligned} \tag{2.21}$$

Using the inequality

$$|a + b|^2 = a^2 + b^2 + 2ab \le a^2 + b^2 + \epsilon_2 a^2 + \epsilon_2^{-1} b^2 = (1 + \epsilon_2) a^2 + (1 + \epsilon_2^{-1}) b^2 \ \forall \epsilon_2 > 0,$$

from (2.10), we have

$$|u_{nt}(x, 0)|^2 = |\mu u_{nt}(x, l) + \psi_n(x)|^2 \le |\mu|^2 (1 + \epsilon_2) u_{nt}^2(x, l) + (1 + \epsilon_2^{-1}) \psi_n(x)^2. \tag{2.22}$$

From which we obtain

$$\begin{aligned} \int_{\omega_0} u_{nt}^2 dx &= \int_0^l |u_{nt}(x, 0)|^2 dx \\ &\le |\mu|^2 (1 + \epsilon_2) \int_0^l u_{nt}^2(x, l) dx + (1 + \epsilon_2^{-1}) \int_0^l \psi_n^2(x) dx \\ &= |\mu|^2 (1 + \epsilon_2) \int_{\omega_l} u_{nt}^2 dx + (1 + \epsilon_2^{-1}) l \|\psi_n\|_{C([0,l])}^2. \end{aligned} \tag{2.23}$$

In view of (2.18) from (2.21), we have

$$\int_{\omega_l} u_{nt}^2 dx \le w_n(l) \le \Big[\int_{\omega_0} \varphi_{nx}^2 dx + \int_{\omega_0} u_{nt}^2 dx + M_3 \Big] e^{\epsilon_1 l}, \tag{2.24}$$

where

$$M_3 = 2\lambda l\big(M_1 + M_2 + \|g(x,0,\varphi(x))\|_{C([0,l])} + \epsilon\big) + \epsilon_1^{-1} l^2 \|F_n\|^2_{C(\overline{\Omega})}. \tag{2.25}$$

From (2.23) and (2.24) it follows that

$$\begin{aligned} \int_{\omega_0} u_{nt}^2 dx \le |\mu|^2(1+\epsilon_2)\Big[\int_{\omega_0} \varphi_{nx}^2 dx + \int_{\omega_0} u_{nt}^2 dx + M_3\Big] e^{\epsilon_1 l} \\ + (1+\epsilon_2^{-1}) l \|\psi_n\|^2_{C([0,l])}. \end{aligned} \tag{2.26}$$

Because $|\mu| < 1$, then positive constants ϵ_1 and ϵ_2 can be chosen so small that

$$\mu_1 = |\mu|^2(1+\epsilon_2)e^{\epsilon_1 l} < 1. \tag{2.27}$$

Due to (2.27), from (2.26) we obtain

$$\begin{aligned} \int_{\omega_0} u_{nt}^2 dx \le (1-\mu_1)^{-1}\Big[|\mu|^2(1+\epsilon_2)\Big(\int_{\omega_0} \varphi_{nx}^2 dx + M_3\Big) e^{\epsilon_1 l} \\ + (1+\epsilon_2^{-1}) l \|\psi_n\|^2_{C([0,l])}\Big]. \end{aligned} \tag{2.28}$$

From (2.9) and (2.28) it follows that

$$\begin{aligned} w_n(0) &= \int_{\omega_0} [u_{nx}^2 + u_{nt}^2] dx \\ &\le \int_{\omega_0} \varphi_{nx}^2 dx + (1-\mu_1)^{-1}\Big[|\mu|^2(1+\epsilon_2)\Big(\int_{\omega_0} \varphi_{nx}^2 dx + M_3\Big) e^{\epsilon_1 l} \\ &\quad + (1+\epsilon_2^{-1}) l \|\psi_n\|^2_{C([0,l])}\Big] \\ &\le l\|\varphi_n\|^2_{C^1([0,l])} + (1-\mu_1)^{-1}\Big[|\mu|^2(1+\epsilon_2)\Big(l\|\varphi_n\|^2_{C^1([0,l])} + M_3\Big) e^{\epsilon_1 l} \\ &\quad + (1+\epsilon_2^{-1}) l \|\psi_n\|^2_{C([0,l])}\Big]. \end{aligned} \tag{2.29}$$

In view of (2.25) and (2.29), from (2.21) we obtain

$$\begin{aligned} w_n(\tau) \le \Big[& l\|\varphi_n\|^2_{C^1([0,l])} + (1-\mu_1)^{-1}\big\{|\mu|^2(1+\epsilon_2)\big(l\|\varphi_n\|^2_{C^1([0,l])} + M_3\big) e^{\epsilon_1 l} \\ & + (1+\epsilon_2^{-1}) l \|\psi_n\|^2_{C([0,l])}\big\} + M_3\Big] e^{\epsilon_1 \tau}, \quad 0 < \tau \le l. \end{aligned} \tag{2.30}$$

In view of (2.8), (2.18), using the Schwartz inequality, for any $(x,\tau) \in \Omega$ we have

$$\begin{aligned} |u_n(x,\tau)|^2 &= \Big(\int_0^x u_{nx}(\xi,\tau) d\xi\Big)^2 \le \int_0^x 1^2 d\xi \int_0^x u_{nx}^2(\xi,\tau) d\xi \\ &\le l \int_0^l u_{nx}^2(\xi,\tau) d\xi = l \int_{\omega_\tau} u_{nx}^2 dx \le l w_n(\tau), \end{aligned}$$

from which it follows that

$$|u_n(x,\tau)| \le [l w_n(\tau)]^{1/2} \quad \forall (x,\tau) \in \Omega. \tag{2.31}$$

Using the inequality

$$\Big(\sum_{i=1}^n a_i^2\Big)^{1/2} \le \sum_{i=1}^n |a_i|$$

and taking into account (2.25), from (2.30) and (2.31), we obtain

$$\begin{aligned} |u_n(x,\tau)| \le c_1\|F_n\|_{C(\overline{\Omega})} + c_2\|g(x,0,\varphi(x)\|^{1/2}_{C([0,l])} + c_3\|\varphi_n\|_{C^1([0,l])} \\ + c_4\|\psi_n\|_{C([0,l])} + \tilde{c}_5(\epsilon) \quad \forall (x,\tau)\in\Omega. \end{aligned} \tag{2.32}$$

Here

$$c_1 = \epsilon_1^{-\frac{1}{2}} l^{3/2}\alpha_1^{1/2}, \quad c_2 = (2\lambda\alpha_1)^{1/2} l, \quad \alpha_1 = (1-\mu_1)^{-1}\mu^2(1+\epsilon_2)e^{2\epsilon_1 l} + e^{\epsilon_1 l}, \tag{2.33}$$

$$c_3 = l^{1/2}\big[l + (1-\mu_1)^{-1}|\mu|^2 l(1+\epsilon_2)e^{\epsilon_1 l}\big]^{1/2} e^{\frac{1}{2}\epsilon_1 l}, \tag{2.34}$$

$$c_4 = (1-\mu_1)^{-\frac{1}{2}}(1+\epsilon_2^{-1})^{1/2} l e^{\frac{1}{2}\epsilon_1 l}, \quad \tilde{c}_5(\epsilon) = l(2\lambda\alpha_1)^{1/2}(M_1+M_2+\epsilon)^{1/2}, \tag{2.35}$$

where positive constants $\epsilon_1, \epsilon_2, \mu_1$ satisfy (2.27), and M_1, M_2 are from (2.2) and (2.3).

Since (2.32) is valid for any $\epsilon = const > 0$ and natural number $n > N(\epsilon)$, then, passing in the (2.32) to the limit for $n\to\infty$, in view of (2.5) and (2.6), we obtain a priori estimate (2.4) with constants c_1, c_2, c_3 and c_4 from (2.33)-(2.35), and for c_5 we have

$$c_5 := \lim_{\epsilon\to 0}\tilde{c}_5(\epsilon) = l(2\lambda\alpha_1)^{1/2}(M_1+M_2)^{1/2}. \tag{2.36}$$

This completes the proof. □

3. Reduction of (1.1)-(1.4) to a nonlinear integral equation

First let us consider in the domain $\Omega : 0 < x, t < l$ the linear mixed problem

$$u_{tt} - u_{xx} = F(x,t), \quad (x,t)\in\Omega, \tag{3.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0\le x\le l, \tag{3.2}$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0\le t\le l, \tag{3.3}$$

where $F\in C^1(\overline{\Omega}), u_0\in C^2([0,l]), u_1\in C^1([0,l])$ are given functions, satisfying the agreement conditions

$$u_0(0) = u_0(l) = u_1(0) = u_1(l) = 0, -u_0''(0) = F(0,0), -u_0''(l) = F(l,0).$$

For obtaining the solution $u\in C^2(\overline{\Omega})$ of (3.1)-(3.3) in convenient form we divide the domain Ω, being a quadrate with vertices in points $O(0,0)$, $A(0,l), B(l,l)$ and $C(l,0)$, into four right triangles $\triangle_1 = \triangle OO_1C, \triangle_2 = \triangle OO_1A$, $\triangle_3 = \triangle CO_1B$ and $\triangle_4 = \triangle O_1AB$, where point $O_1(l/2, l/2)$ is the center of quadrate Ω. In the triangle $\triangle_1 = \triangle OO_1C$ the solution of (3.1)-(3.3), as it is known, is given by the formula [4, p. 67]

$$\begin{aligned} u(x,t) = \frac{1}{2}[u_0(x+t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} u_1(\tau)d\tau \\ + \frac{1}{2}\int_{\Omega^1_{x,t}} F(\xi,\tau)d\xi d\tau, \quad (x,t)\in\triangle_1, \end{aligned} \tag{3.4}$$

where $\Omega^1_{x,t}$ is a triangle with vertices in points (x,t), $(t-x,0)$ and $(t+x,0)$.

For obtaining the solution of (3.1)-(3.3) in the other triangles $\triangle_2, \triangle_3$ and $\triangle_4$, we use the equality [4, p. 66]

$$u(P) = u(P_1) + u(P_2) - u(P_3) + \frac{1}{2}\int_{PP_1P_2P_3} F(\xi,\tau)d\xi\, d\tau, \tag{3.5}$$

which is valid for any rectangle $PP_1P_2P_3 \subset \overline{\Omega}$ characteristic for (3.1), where P, P_3 and P_1, P_2 are opposite vertices of this rectangle, besides, the ordinate of point P is greater than the ordinates of other points. Indeed, if point $(x,t) \in \triangle_2$, then, using the equality (3.5) for characteristic rectangle with vertices in points $P(x,t)$, $P_1(0,t-x), P_2(t,x)$, $P_3(t-x,0)$ and the formula (3.4) for point $P_2(t,x) \in \triangle_1$, and taking into account (3.3), we obtain

$$\begin{aligned} u(x,t) &= u(P_1) + u(P_2) - u(P_3) + \frac{1}{2}\int_{PP_1P_2P_3} F(\xi,\tau)d\xi d\tau \\ &= -u_0(t-x) + \frac{1}{2}[u_0(t-x) + u_0(t+x)] + \frac{1}{2}\int_{t-x}^{t+x} u_1(\tau)d\tau \\ &\quad + \frac{1}{2}\int_{\Omega^1_{t,x}} F(\xi,\tau)d\xi d\tau + \frac{1}{2}\int_{PP_1P_2P_3} F(\xi,\tau)d\xi d\tau \\ &= \frac{1}{2}[u_0(t+x) - u_0(t-x)] + \frac{1}{2}\int_{t-x}^{t+x} u_1(\tau)d\tau \\ &\quad + \frac{1}{2}\int_{\Omega^2_{x,t}} F(\xi,\tau)d\xi d\tau, \quad (x,t) \in \triangle_2. \end{aligned} \tag{3.6}$$

Here $\Omega^2_{x,t}$ is a quadrangle $PP_1\tilde{P}_2P_3$, where $\tilde{P}_2 = \tilde{P}_2(t+x,0)$. Analogously,

$$\begin{aligned} u(x,t) &= \frac{1}{2}[u_0(x-t) - u_0(2l-t-x)] + \frac{1}{2}\int_{x-t}^{2l-t-x} u_1(\tau)d\tau \\ &\quad + \frac{1}{2}\int_{\Omega^3_{x,t}} F(\xi,\tau)d\xi d\tau, \quad (x,t) \in \triangle_3, \end{aligned} \tag{3.7}$$

$$\begin{aligned} u(x,t) &= -\frac{1}{2}[u_0(t-x) + u_0(2l-t-x)] + \frac{1}{2}\int_{t-x}^{2l-t-x} u_1(\tau)d\tau \\ &\quad + \frac{1}{2}\int_{\Omega^4_{x,t}} F(\xi,\tau)d\xi d\tau, \quad (x,t) \in \triangle_4. \end{aligned} \tag{3.8}$$

Here $\Omega^3_{x,t}$ is a quadrangle with vertices $P^3(x,t), P^3_1(l,t+x-l), P^3_2(x-t,0)$ and $P^3_3(2l-t-x,0)$, while $\Omega^4_{x,t}$ is a pentagon with vertices $P^4(x,t), P^4_1(0,t-x)$, $P^4_2(t-x,0), P^4_3(2l-t-x,0)$ and $P^4_4(l,t+x-l)$.

Remark 3.1. Note that for $F \in C(\overline{\Omega})$, $u_0 \in C^1([0,l]), u_1 \in C([0,l])$, satisfying the agreement conditions $u_0(0) = u_0(l) = u_1(0) = u_1(l) = 0$, the function $u \in C^1(\overline{\Omega})$ represented in Ω by formulas (3.4), (3.6)-(3.8) is a generalized solution of (3.1)–(3.3) of the class C^1.

Further, using formulas (3.4), (3.6)-(3.8) let us solve a linear problem corresponding to (1.1)-(1.4); i.e., when in (1.1) the parameter $\lambda = 0$ and the problem has the form

$$L_0u := u_{tt} - u_{xx} = F(x,t), \quad (x,t) \in \Omega, \tag{3.9}$$

$$u(0,t) = 0, u(l,t) = 0, \quad 0 \le t \le l, \tag{3.10}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{3.11}$$

$$K_\mu u_t := u_t(x,0) - \mu u_t(x,l) = \psi(x), \quad 0 \le x \le l. \tag{3.12}$$

Indeed, differentiating the equality (3.8) on t, we have

$$\begin{aligned} u_t(x,t) = &-\frac{1}{2}[u_0'(t-x) - u_0'(2l-t-x)] \\ &- \frac{1}{2}[u_1(2l-t-x) + u_1(t-x)] + F_1(x,t), \end{aligned} \tag{3.13}$$

where

$$F_1(x,t) := \frac{\partial}{\partial t}\Big[\frac{1}{2}\int_{\Omega^4_{x,t}} F(\xi,\tau)d\xi d\tau\Big]. \tag{3.14}$$

From (3.13) for $t = l$ we obtain

$$u_t(x,l) = -u_1(l-x) + F_1(x,l), \quad 0 \le x \le l. \tag{3.15}$$

Substituting (3.15) in (3.12), with respect to unknown function $u_1(x) = u_t(x,0)$, we obtain the functional equation

$$u_1(x) + \mu u_1(l-x) = \psi_1(x),\ 0 \le x \le l. \tag{3.16}$$

Here

$$\psi_1(x) := \psi(x) + \mu F_1(x,l),\ 0 \le x \le l. \tag{3.17}$$

Putting in (3.16) the value $l - x$ instead of x we obtain

$$\mu u_1(x) + u_1(l-x) = \psi_1(l-x),\ 0 \le x \le l. \tag{3.18}$$

For $|\mu| \ne 1$, eliminating $u_1(l-x)$ from the system (3.16), (3.18), we have

$$u_1(x) = (1-\mu^2)^{-1}(\psi_1(x) - \mu\psi_1(l-x)), \tag{3.19}$$

Substituting in (3.4), (3.6)-(3.8) the function $\varphi(x)$ from (3.11) instead the function $u_0(x)$, and the right side part of (3.19) instead of the function $u_1(x)$, fulfilling the certain conditions of smoothness and agreement imposed on the functions F, φ and ψ, we obtain a unique solution $u = u(x,t)$ of (3.9)-(3.12).

Remark 3.2. It is easy to see that for $|\mu| = 1$ the homogeneous equation, corresponding to (3.18), has an infinite set of linearly independent solutions, which for $\mu = 1$ are arbitrary odd functions u_1^0 with respect to point $x = l/2$; i.e., $u_1^0(l/2+\xi) = -u_1^0(l/2-\xi)$, $|\xi| \le l/2$, while for $\mu = -1$ they are arbitrary even functions u_1^0 with respect to point $x = l/2$; i.e., $u_1^0(l/2+\xi) = u_1^0(l/2-\xi)$, $|\xi| \le l/2$.

Due to (3.4), (3.6)-(3.8), (3.19) and Remark 3.2 the following lemma is valid.

Lemma 3.3. *Let $F \in C^1(\overline{\Omega})$, $\varphi \in C^2([0,l])$, $\psi \in C^1([0,l])$ and the agreement conditions $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$, $-\varphi''(0) = F(0,0), -\varphi''(l) = F(l,0)$ be fulfilled. Then for $|\mu| \ne 1$, problem (3.9)-(3.12) has an unique solution $u \in C^2(\overline{\Omega})$, which is given by formulas (3.4), (3.6)-(3.8), where instead of $u_0(x)$ must be put $\varphi(x)$ from (3.11) and instead of the function $u_1(x)$ must be put the function from (3.19). These formulas, when $|\mu| \ne 1$ and $F \in C(\overline{\Omega})$, $\varphi \in C^1([0,l]), \psi \in C([0,l])$ with the agreement conditions $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$, give the function $u \in C^1(\overline{\Omega})$, being a generalized solution of (3.9)-(3.12) of the class C^1. Finally, for $|\mu| = 1$ the homogeneous problem, corresponding to (3.9)-(3.12), has an infinite set of linearly independent solutions of the class $C^2(\overline{\Omega})$, which are given by formulas (3.4), (3.6)-(3.8), where $F = 0, u_0 = 0$, and $u_1 \in C^1([0,l]), u_1(0) = u_2(l) = 0$ is an arbitrary odd (even) function with respect to point $x = l/2$ in the case $\mu = 1$ ($\mu = -1$).*

Remark 3.4. In view of Lemma 3.3 and formulas (3.4), (3.6)-(3.8), (3.14), (3.17) and (3.19), it is easy to see that for $|\mu| \neq 1$ a unique solution $u \in C^2(\overline{\Omega})$ of (3.9)-(3.12) can be represented in the form

$$u(x,t) = \big(l_0^{-1}(\varphi,\psi)\big)(x,t) + \big(L_0^{-1}F\big)(x,t), \quad (x,t) \in \overline{\Omega}, \tag{3.20}$$

where $l_0^{-1}(\varphi,\psi)$ represents a solution of (3.9)-(3.12) for $F=0$, and $L_0^{-1}F$ is also a solution to this problem for $\varphi = 0, \psi = 0$. Note that for $\varphi \in C^1([0,l]), \psi \in C([0,l])$ and $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$ the function $l_0^{-1}(\varphi,\psi)$ is a generalized solution of (3.9)-(3.12) of the class $C^1(\overline{\Omega})$ for $F=0$, and when $F \in C(\overline{\Omega})$ the function $L_0^{-1}F$ is a generalized solution of this problem of the class $C^1(\overline{\Omega})$ for $\varphi = 0, \psi = 0$. In this case the linear operator $l_0^{-1} : C^{01}([0,l]) \times C^0([0,l]) \to C(\overline{\Omega})$ is continuous, where $C^{0k}([0,l]) := \{\chi \in C^k([0,l]) : \chi(0) = \chi(l) = 0\}$; i.e.,

$$\|l_0^{-1}(\varphi,\psi)\|_{C(\overline{\Omega})} \le \tilde{c}\|(\varphi,\psi)\|_{C^{01}([0,l])\times C^0([0,l])} \tag{3.21}$$

for all $(\varphi,\psi) \in C^{01}([0,l]) \times C^0([0,l])$ with positive constant $\tilde{c}$, not depending on (φ,ψ).

Remark 3.5. Using standard reasoning one may show that the operator L_0^{-1} from (3.20), being a linear integral operator, acts continuously from the space $C(\overline{\Omega})$ into the space $C^1(\overline{\Omega})$; i.e.,

$$\|L_0^{-1}F\|_{C^1(\overline{\Omega})} \le c_0\|F\|_{C(\overline{\Omega})} \quad \forall F \in C(\overline{\Omega}) \tag{3.22}$$

with positive constant c_0, not depending on F.

Remark 3.6. Since the space $C^1(\overline{\Omega})$ is compactly embedded into the space $C(\overline{\Omega})$ [9, p. 135], in view of (3.22) the operator $L_0^{-1} : C(\overline{\Omega}) \to C(\overline{\Omega})$ from (3.20) is a linear compact operator. One may come to the same conclusion noting that the continuous operator $L_0^{-1} : C(\overline{\Omega}) \to C(\overline{\Omega})$ maps bounded in $C(\overline{\Omega})$ sets into equicontinuous sets, and further using the criterium of precompactness of a set in the space $C(\overline{\Omega})$ [19, p. 414].

Remark 3.7. If $u \in C^2(\overline{\Omega})$ is a classical solution of (1.1)-(1.4), then due to the representation (3.20) it will satisfy the nonlinear integral equation

$$\begin{aligned} &u(x,t) + \lambda\big(L_0^{-1}f\big|_{u=u(x,t)}\big)(x,t)\\ &= \big(l_0^{-1}(\varphi,\psi)\big)(x,t) + \big(L_0^{-1}F\big)(x,t), \quad (x,t) \in \overline{\Omega}. \end{aligned} \tag{3.23}$$

Lemma 3.8. *Let $f \in C^1(\overline{\Omega} \times \mathbb{R})$. A function $u \in C(\overline{\Omega})$ is a strong generalized solution of* (1.1)-(1.4) *of the class C in the domain Ω in the sense of Definition 1.1 if and only if it is a continuous solution of the nonlinear integral equation* (3.23).

Proof. Let $u \in C(\overline{\Omega})$ be a solution of (3.23). Since $F \in C(\overline{\Omega})$ ($\varphi \in C^1([0,l])$, $\psi \in C([0,l])$, $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$), and the space $C^2(\overline{\Omega})$ ($C^k([0,l])$) is dense in $C(\overline{\Omega})$ ($C^{k_1}([0,l]), k_1 < k$) [21, p. 37], then there exist the sequences of functions $F_n \in C^2(\overline{\Omega}), \varphi_n \in C^2([0,l])$ and $\psi_n \in C^1([0,l])$ such that $\varphi_n(0) = \varphi_n(l) = \psi_n(0) = \psi_n(l) = 0$, $-\varphi''(0) + \lambda f(0,0,0) = F_n(0,0)$, $-\varphi_n''(l) + \lambda f(l,0,0) = F_n(l,0)$, and

$$\begin{gathered} \lim_{n\to\infty} \|F_n - F\|_{C(\overline{\Omega})} = 0, \quad \lim_{n\to\infty} \|\varphi_n - \varphi\|_{C^1([0,l])} = 0,\\ \lim_{n\to\infty} \|\psi_n - \psi\|_{C([0,l])} = 0. \end{gathered} \tag{3.24}$$

Since $u \in C(\overline{\Omega})$ represents a solution of the integral equation (3.23), as it is easy to verify $u|_\Gamma = 0$; i.e., $u \in C^0(\overline{\Omega}, \Gamma)$, and therefore there exists the sequence of functions $w_n \in C^2(\overline{\Omega}) \cap C^0(\overline{\Omega}, \Gamma)$ such that

$$\lim_{n\to\infty} \|w_n - u\|_{C(\overline{\Omega})} = 0. \tag{3.25}$$

Let

$$u_n = -\lambda L_0^{-1} f\big|_{u=w_n} + l_0^{-1}(\varphi_n, \psi_n) + L_0^{-1} F_n. \tag{3.26}$$

Since $w_n|_\Gamma = 0$, $-\varphi_n''(0) + \lambda f(0,0,0) = F_n(0,0)$, $-\varphi_n''(l) + \lambda f(l,0,0) = F_n(l,0)$, it is obvious that $-\varphi_n''(0) = (-\lambda f|_{u=w_n} + F_n)(0,0)$ and $-\varphi_n''(l) = (-\lambda f|_{u=w_n} + F_n)(l,0)$. Therefore, since $f \in C^1(\overline{\Omega} \times \mathbb{R})$, $w_n \in C^2(\overline{\Omega}), F_n \in C^2(\overline{\Omega})$ and $\big(-\lambda f|_{u=w_n} + F_n\big) \in C^1(\overline{\Omega})$, then in view of Remark 3.4 the function u_n from (3.26) belongs to the space $C^2(\overline{\Omega}) \cap C^0(\overline{\Omega}, \Gamma)$, and

$$u_n\big|_{t=0} = \varphi_n, K_\mu u_n = \psi_n. \tag{3.27}$$

From (3.21), (3.22), (3.24)-(3.27) it follows immediately that

$$u_n(x,t) \to \Big[-\lambda\big(L_0^{-1} f|_{u=u(x,t)}\big)(x,t) + \big(l_0^{-1}(\varphi,\psi)\big)(x,t) + \big(L_0^{-1}F\big)(x,t)\Big]$$

in the space $C(\overline{\Omega})$. Also, from (3.23) it follows that

$$-\lambda\big(L_0^{-1} f|_{u=u(x,t)}\big)(x,t) + \big(l_0^{-1}(\varphi,\psi)\big)(x,t) + \big(L_0^{-1}F\big)(x,t) = u(x,t).$$

Therefore,

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{\Omega})} = 0. \tag{3.28}$$

Due to Remark 3.4, from (3.26) it follows that $L_0 u_n = -\lambda f|_{u=w_n} + F_n$ and, therefore,

$$\begin{aligned} L_\lambda u_n &= L_0 u_n + \lambda f|_{u=u_n} = -\lambda f|_{u=w_n} + F_n + \lambda f|_{u=u_n} \\ &= -\lambda(f(\cdot, w_n) - f(\cdot, u)) + \lambda(f(\cdot, u_n) - f(\cdot, u)) + F_n. \end{aligned} \tag{3.29}$$

Since $f \in C(\overline{\Omega} \times \mathbb{R})$, in view of (3.25), (3.28) from (3.29), we have

$$\lim_{n\to\infty} \|L_\lambda u_n - F_n\|_{C(\overline{\Omega})} = 0. \tag{3.30}$$

Due to (3.24) and (3.27), we have

$$\lim_{n\to\infty} \|u_n|_{t=0} - \varphi\|_{C^1([0,l])} = 0, \quad \lim_{n\to\infty} \|K_\mu u_n - \psi\|_{C([0,l])} = 0. \tag{3.31}$$

Therefore, from (3.28), (3.30) and (3.21) we conclude that a continuous solution $u \in C(\overline{\Omega})$ of the nonlinear integral equation (3.23) is also a strong generalized solution of (1.1)-(1.4) of the class C in the domain Ω in the sense of Definition 1.1. The inverse is obvious. $\square$

4. Existence and uniqueness of the solution to (1.1) - (1.4)

Rewrite the equation (3.23) in the form

$$u = Tu := -\lambda\Big(L_0^{-1} f|_{u=u(x,t)}\Big)(x,t) + l_0^{-1}(\varphi,\psi) + L_0^{-1}F, \tag{4.1}$$

where operator $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ is continuous and compact, since the operator $N : C(\overline{\Omega}) \to C(\overline{\Omega})$, acting according to the formula $Nu := -\lambda f(x,t,u)$ is bounded and continuous, and the linear operator $L_0^{-1} : C(\overline{\Omega}) \to C(\overline{\Omega})$ due to Remark 3.6, it is compact. Here we take into account that the component $T_1 u := l_0^{-1}(\varphi,\psi) + L_0^{-1}F$ of the operator T from (4.1) is constant, and therefore, continuous and compact operator, acting in the space $C(\overline{\Omega})$. At the same time, according to Lemmas 2.1 and 3.8, and also (2.2), (2.3), (2.33)-(2.36) for any parameter $\tau \in [0,1]$ and every solution $u \in C(\overline{\Omega})$ of the equation $u = \tau T u$ it is valid a priori estimate (2.4) with the same constants c_i, $i = 1, \dots, 5$, not depending on u, F, φ, ψ and τ. Therefore, according to the Leray-Schauder theorem [22, p. 375], the equation (4.1) for the conditions of the Lemmas 2.1 and 3.8 has at least one solution $u \in C(\overline{\Omega})$. In this way, due to the Lemmas 2.1, 3.8 and also Remark 3.6, we have proved the following theorem.

Theorem 4.1. *Let $\lambda > 0$, $|\mu| < 1$, $f \in C^1(\overline{\Omega} \times \mathbb{R})$, $F \in C(\overline{\Omega})$, $\varphi \in C^1([0,l])$, $\psi \in C([0,l])$, $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$ and the conditions* (2.2), (2.3) *be fulfilled. Then* (1.1)-(1.4) *has at least one strong generalized solution of the class C in the domain Ω in the sense of Definition 1.1.*

Remark 4.2. Since (3.23) can be rewritten in the form of (3.20):

$$u(x,t) = \big(l_0^{-1}(\varphi,\psi)\big)(x,t) + \big(L_0^{-1}\big(-\lambda f|_{u=u(x,t)} + F\big)(x,t), \quad (x,t) \in \overline{\Omega},$$

in view of Lemma 3.3 and Remark 3.4, the generalized solution u of the class C, the existence of which is asserted in the Theorem 4.1, belongs to the class $C^1(\overline{\Omega})$. Moreover, if we require in addition that $F \in C^1(\overline{\Omega})$, $\varphi \in C^2([0,l])$, $\psi \in C^1([0,l])$ and $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$, $-\varphi''(0) + \lambda f(0,0,0) = F(0,0)$, $-\varphi''(l) + \lambda f(l,0,0) = F(l,0)$, then this solution will belong to the class $C^2(\overline{\Omega})$; i.e., it will be a classical solution of (1.1)-(1.4).

Remark 4.3. Let us consider some classes of functions $f = f(x,t,u)$ frequently encountered in applications and which satisfy the conditions (2.2), (2.3):

1. $f(x,t,u) = f_0(x,t)\psi(u)$, where $f_0, \frac{\partial}{\partial t} f_0 \in C(\overline{\Omega})$ and $\psi \in C(\mathbb{R})$. In this case $g(x,t,u) = f_0(x,t)\int_0^u \psi(s)ds$ and when $f_0 \geq 0$, $\frac{\partial}{\partial t} f_0 \leq 0$, $\int_0^u \psi(s)ds \geq -M$, M is a non-negative constant, the conditions (2.2), (2.3) will be fulfilled.
2. $f(x,t,u) = f_0(x,t)|u|^\alpha sgn\, u$, where $f_0, \frac{\partial}{\partial t} f_0 \in C(\overline{\Omega})$ and $\alpha > 1$. In this case $g(x,t,u) = f_0(x,t)\frac{|u|^{\alpha+1}}{\alpha+1}$ and when $f_0 \geq 0$, $\frac{\partial}{\partial t} f_0 \leq 0$, the conditions (2.2), (2.3) will be fulfilled.
3. $f(x,t,u) = f_0(x,t)e^u$, where $f_0, \frac{\partial}{\partial t} f_0 \in C(\overline{\Omega})$. In this case $g(x,t,u) = f(x,t,u)$ and when $f_0 \geq 0$, $\frac{\partial}{\partial t} f_0 \leq 0$, the conditions (2.2), (2.3) will be also fulfilled.

Therefore, if function $f \in C^1(\overline{\Omega} \times \mathbb{R})$ belongs to the one of the classes considered above, then according to the Theorem 4.1, problem (1.1)-(1.4) is solvable in the class C in the sense of Definition 1.1.

Remark 4.4. Let us consider the example of the function f, which is also often encountered in applications, when at least one of the conditions (2.2) and (2.3) is violated. Such function is

$$f(x,t,u) = f_0(x,t)|u|^\alpha, \quad \alpha > 1, \tag{4.2}$$

where $f_0, \frac{\partial}{\partial t} f_0 \in C(\overline{\Omega})$ and $f_0 \neq 0$. In this case due to (2.1) we have $g(x,t,u) = f_0(x,t)\frac{|u|^\alpha u}{\alpha+1}$, and since $\alpha > 1$ and $f_0 \neq 0$, then the condition (2.2) will be violated. If $\frac{\partial}{\partial t} f_0 \neq 0$, then the condition (2.3) will be also violated. Below we show that when (2.2) and (2.3) are violated then the problem (1.1)-(1.4) may be insoluble.

Let us consider the uniqueness of the solution of (1.1)-(1.4). Let the function f satisfy the Lipshitz local condition on the set $\overline{\Omega} \times \mathbb{R}$ with respect to variable u; i.e.,

$$|f(x,t,u_2) - f(x,t,u_1)| \leq M(R)|u_2 - u_1|, \quad (x,t) \in \overline{\Omega},\ |u_i| \leq R,\ i = 1,2, \tag{4.3}$$

where $M = M(R)$ is a non-negative constant, it is nondecreasing function of variable R.

Theorem 4.5. *Let $|\mu| < 1$, $F \in C(\overline{\Omega}); \varphi \in C^1([0,l]), \psi \in C([0,l])$, $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$, function $f \in C(\overline{\Omega} \times \mathbb{R})$ and satisfy the condition (4.3). Then there exists a positive number $\lambda_0 = \lambda_0(F, f, \varphi, \mu, l)$ such that for $0 < \lambda < \lambda_0$, problem (1.1)-(1.4) can not have more than one strong generalized solution of the class C in the domain Ω in the sense of Definition 1.1.*

Proof. Suppose that (1.1)-(1.4) has two strong generalized solutions u_1 and u_2 of the class C in the domain Ω. According to Definition 1.1 there exists a sequence of functions $u_{jn} \in C^2(\overline{\Omega}) \cap C^0(\overline{\Omega}, \Gamma)$ such that

$$\lim_{n\to\infty} \|u_{jn} - u_j\|_{C(\overline{\Omega})} = 0, \quad \lim_{n\to\infty} \|L_\lambda u_{jn} - F\|_{C(\overline{\Omega})} = 0, \tag{4.4}$$

$$\lim_{n\to\infty} \|u_{jn}|_{t=0} - \varphi\|_{C^1([0,l])} = 0, \quad \lim_{n\to\infty} \|K_\mu u_{jnt} - \psi\|_{C([0,l])} = 0, \tag{4.5}$$

for $j = 1, 2$. Let $v_n := u_{2n} - u_{1n}$. It is easy to see that the function $v_n \in C^2(\overline{\Omega}) \cap C^0(\overline{\Omega}, \Gamma)$ represents a classical solution of the problem

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\Big) v_n = \big(F_n + g_n\big)(x,t), \quad (x,t) \in \Omega, \tag{4.6}$$

$$v_n(0,t) = 0, v_n(l,t) = 0, 0 \le t \le l, \tag{4.7}$$

$$v_n(x,0) = \varphi_n(x), 0 \le x \le l, \tag{4.8}$$

$$K_\mu v_{nt} := v_{nt}(x,0) - \mu v_{nt}(x,l) = \psi_n(x), 0 \le x \le l. \tag{4.9}$$

Here

$$g_n := \lambda(f(x,t,u_{1n}) - f(x,t,u_{2n})), \tag{4.10}$$

$$F_n := L_\lambda u_{2n} - L_\lambda u_{1n}, \tag{4.11}$$

$$\varphi_n := v_n|_{t=0}, \tag{4.12}$$

$$\psi_n := K_\mu v_{nt}. \tag{4.13}$$

From the proof of Lemma 2.1 it follows easily that a priori estimate (2.4) is valid in the linear case too; i.e., when in (1.1) the parameter $\lambda = 0$. In this case due to (2.33)-(2.36), determining the constants c_i, we have $c_2 = c_5 = 0$ and the estimate (2.4) takes the form

$$\|u\|_{C(\overline{\Omega})} \le c_1 \|F\|_{C(\overline{\Omega})} + c_3 \|\varphi\|_{C^1([0,l])} + c_4 \|\psi\|_{C([0,l])}, \tag{4.14}$$

where the constants c_1, c_3 and c_4 do not depend on the parameter λ and the functions u, F, φ, ψ.

In view of (4.14) for the solution $v_n \in C^2(\overline{\Omega}) \cap C^0(\overline{\Omega}, \Gamma)$ of (4.6)-(4.9), the following estimate is valid

$$\|v_n\|_{C(\overline{\Omega})} \le c_1 \|F_n + g_n\|_{C(\overline{\Omega})} + c_3 \|\varphi_n\|_{C^1([0,l])} + c_4 \|\psi_n\|_{C([0,l])}. \tag{4.15}$$

From (4.4), (4.5) and (4.11)-(4.13) it follows that

$$\lim_{n\to\infty} \|F_n\|_{C(\overline{\Omega})} = 0, \quad \lim_{n\to\infty} \|\varphi_n\|_{C^1([0,l])} = 0, \quad \lim_{n\to\infty} \|\psi_n\|_{C([0,l])} = 0. \tag{4.16}$$

Due to a priori estimate (2.4) for the solutions u_1 and u_2 of (1.1)-(1.4), we have

$$\|u_j\|_{C(\overline{\Omega})} \le m_3 + \lambda^{1/2} m_4, \quad j = 1, 2, \tag{4.17}$$

where according to (2.33)-(2.36) positive constants $m_i = m_i(\mu, l, M_1, M_2, F, \varphi, \psi)$, $i = 3, 4$, do not depend on λ.

Let us fix arbitrarily the number $\lambda_1 > 0$ and put $M_0 = M(m_3 + \lambda_1^{1/2} m_4 + 1)$, where $M = M(R)$ is nondecreasing function from (4.3). In view of (4.4) for any $\epsilon > 0$ there exists number $N > 0$ such that $\|u_{jn}\|_{C(\overline{\Omega})} \le \|u_j\|_{C(\overline{\Omega})} + \epsilon$, $j = 1, 2$, for $n > N$, and, therefore, for $0 < \lambda < \lambda_1$, taking into account (4.17), we have

$$\|u_{jn}\|_{C(\overline{\Omega})} \le m_3 + \lambda^{1/2} m_4 + \epsilon \le m_3 + \lambda_1^{1/2} m_4 + \epsilon, \quad j = 1, 2;\, n > N. \tag{4.18}$$

From (4.3), (4.10) and (4.18) for $0 < \lambda < \lambda_1$ and $\epsilon = 1$ it follows that

$$\|g_n\|_{C(\overline{\Omega})} \le \lambda \|f(x, t, u_{1n}) - f(x, t, u_{2n})\|_{C(\overline{\Omega})} \le \lambda M_0 \|v_n\|_{C(\overline{\Omega})}, \tag{4.19}$$

for $n > N$. Due to (4.15) and (4.19) we have

$$\|v_n\|_{C(\overline{\Omega})} \le c_1 \|F_n\|_{C(\overline{\Omega})} + \lambda c_1 M_0 \|v_n\|_{C(\overline{\Omega})} + c_3 \|\varphi_n\|_{C^1([0,l])} + c_4 \|\psi_n\|_{C([0,l])},$$

for $n > N$, whence for $\lambda_0 := min\big(\lambda_1, \frac{1}{c_1 M_0}\big)$ and $0 < \lambda < \lambda_0$ it follows that

$$\|v_n\|_{C(\overline{\Omega})} \le (1 - \lambda c_1 M_0)^{-1} \big[c_1 \|F_n\|_{C(\overline{\Omega})} + c_3 \|\varphi_n\|_{C^1([0,l])} + c_4 \|\psi_n\|_{C([0,l])}\big], \tag{4.20}$$

for $n > N$. From (4.4) we find that

$$\lim_{n\to\infty} \|v_n\|_{C(\overline{\Omega})} = \|u_2 - u_1\|_{C(\overline{\Omega})}.$$

Also, in view of (4.16) and (4.20) we have

$$\lim_{n\to\infty} \|v_n\|_{C(\overline{\Omega})} = 0.$$

Thus $\|u_2 - u_1\|_{C(\overline{\Omega})} = 0$; i.e., $u_2 = u_1$, which leads to contradiction, the proof is complete. $\square$

Since the function $f \in C^1(\overline{\Omega} \times \mathbb{R})$ satisfies condition (4.3), then from theorems 4.1 and 4.5, we have the following theorem.

Theorem 4.6. *Let $|\mu| < 1$, $f \in C^1(\overline{\Omega} \times \mathbb{R})$, $F \in C(\overline{\Omega})$, $\varphi \in C^1([0, l])$, $\psi \in C([0, l])$, $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$, and the conditions* (2.2), (2.3) *be fulfilled. Then there exists a positive number $\lambda_0 = \lambda_0(F, \varphi, \psi, \mu, l)$ such that for $0 < \lambda < \lambda_0$ the problem* (1.1)-(1.4) *has a unique strong generalized solution of the class C in the domain Ω in the sense of Definition 1.1.*

5. Cases of nonexistence of solutions to (1.1)-(1.4)

Below, using the method of test-functions [18], we show that when condition (2.2) or (2.3) is violated, problem (1.1)-(1.4) may have no strong generalized solution of the class C in the domain Ω, in the sense of Definition 1.1.

Lemma 5.1. *Let u is a strong generalized solution of* (1.1)-(1.4) *of the class C in the domain Ω in the sense of Definition 1.1. Then the integral equation*

$$\int_\Omega u \Box v \, dx\, dt = -\lambda \int_\Omega f(x, t, u) v \, dx\, dt + \int_\Omega F v \, dx\, dt \tag{5.1}$$

is valid for any test function v such that

$$v \in C^2(\overline{\Omega}), v|_{\partial\Omega} = v_t|_{\partial\Omega} = v_x|_{\partial\Omega} = 0, \tag{5.2}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$.

Proof. According to the definition of a strong generalized solution u of (1.1)-(1.4) of the class C in the domain Ω in the sense of Definition 1.1 there exists a sequence of functions $u_n \in C^2(\overline{\Omega}) \cap C^0(\overline{\Omega}, \Gamma)$ such that the equalities (2.5), (2.6) are valid and also, as an implication, the equality

$$\lim_{n\to\infty} \|f(x,t,u_n) - f(x,t,u)\|_{C(\overline{\Omega})} = 0. \tag{5.3}$$

Let $F_n := L_\lambda u_n$. Multiply both parts of the equality $L_\lambda u_n = F_n$ by the function v and integrate the received equality in the domain Ω. By integration by parts of the left side of this equality and due to (5.2) we have

$$\int_\Omega u_n \Box v\,dx\,dt + \lambda \int_\Omega f(x,t,u_n)v\,dx\,dt = \int_\Omega F_n v\,dx\,dt. \tag{5.4}$$

In view of (2.5) and (5.3), passing in the equality (5.4) to the limit for $n \to \infty$, we obtain (5.1). The proof is complete. $\square$

Consider the following condition imposed on function f:

$$f(x,t,u) \le -|u|^{\alpha+1}, \quad (x,t,u) \in \overline{\Omega} \times \mathbb{R}, \tag{5.5}$$

where α is a positive constant. It is easy to verify that when (5.5) is fulfilled, condition (2.2) is violated.

Let us introduce a function $v_0 = v_0(x,t)$ such that

$$v_0 \in C^2(\overline{\Omega}), v_0|_\Omega > 0, v_0|_{\partial\Omega} = v_{0x}|_{\partial\Omega} = v_{0t}|_{\partial\Omega} = 0 \tag{5.6}$$

and

$$æ_0 = \int_\Omega \frac{|\Box v_0|^{p'}}{|v_0|^{p'-1}}\,dx\,dt < +\infty, \quad p' = 1 + \frac{1}{\alpha}. \tag{5.7}$$

Simple verification shows that for function v_0, satisfying conditions (5.6) and (5.7), can be chosen as

$$v_0(x,t) = [xt(l-x)(l-t)]^k, \quad (x,t) \in \Omega,$$

for k a sufficiently large constant.

Due to (5.5) and (5.6) from (5.1), where instead of v is chosen v_0, in the case $\lambda > 0$, we have

$$\lambda \int_\Omega |u|^p v_0\,dx\,dt \le \int_\Omega |u||\Box v_0|\,dx\,dt - \int_\Omega F v_0\,dx\,dt, p = \alpha + 1. \tag{5.8}$$

Theorem 5.2. *Let $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfy (5.5), and $F = \gamma F^0$, where $F^0 \in C(\overline{\Omega})$, $F^0 \ge 0$ and $F^0 \ne 0$. The functions φ, ψ satisfy the conditions from Definition 1.1. Then for $\lambda > 0$ there exists the number $\gamma_0 = \gamma_0(F^0, \alpha, \lambda) > 0$, such that for $\gamma > \gamma_0$, problem (1.1)-(1.4) does not have a strong generalized solution of the class C in the domain Ω in the sense of Definition 1.1.*

Proof. If in the Young's inequality with the parameter $\epsilon > 0$,

$$ab < \frac{\epsilon}{p} a^p + \frac{1}{p'\epsilon^{p'-1}} b^{p'}; \quad a, b \ge 0, \frac{1}{p} + \frac{1}{p'} = 1, p = \alpha + 1 > 1$$

we take $a = |u|v_0^{\frac{1}{p}}, b = \frac{|\Box v_0|}{v_0^{\frac{1}{p}}}$, then, since $\frac{p'}{p} = p' - 1$, we obtain

$$|u||\Box v_0| = |u|v_0^{\frac{1}{p}} \frac{|\Box v_0|}{v_0^{\frac{1}{p}}} \le \frac{\epsilon}{p}|u|^p v_0 + \frac{1}{p'\epsilon^{p'-1}} \frac{|\Box v_0|^{p'}}{v_0^{p'-1}}. \tag{5.9}$$

Since $F = \gamma F^0$ and due to (5.9), from (5.8) it follows that

$$\big(\lambda - \frac{\epsilon}{p}\big) \int_\Omega |u|^p v_0 \,dx\,dt \le \frac{1}{p'\epsilon^{p'-1}} \int_\Omega \frac{|\Box v_0|^{p'}}{v_0^{p'-1}} \,dx\,dt - \gamma \int_\Omega F^0 v_0 \,dx\,dt,$$

whence for $\epsilon < \lambda p$, we obtain

$$\int_\Omega |u|^p v_0 \,dx\,dt \le \frac{p}{(\lambda p - \epsilon)p'\epsilon^{p'-1}} \int_\Omega \frac{|\Box v_0|^{p'}}{v_0^{p'-1}} \,dx\,dt - \frac{p\gamma}{\lambda p - \epsilon} \int_\Omega F^0 v_0 \,dx\,dt. \quad (5.10)$$

Taking into account that $p' = \frac{p}{p-1}$, $p = \frac{p'}{p'-1}$ and

$$\min_{0<\epsilon<\lambda p} \frac{p}{(\lambda p - \epsilon)p'\epsilon^{p'-1}} = \frac{1}{\lambda^p},$$

which is achieved at $\epsilon = \lambda$, from (5.10) it follows that

$$\int_\Omega |u|^p v_0 \,dx\,dt \le \frac{1}{\lambda^{p'}} \int_\Omega \frac{|\Box v_0|^{p'}}{v_0^{p'-1}} \,dx\,dt - \frac{p'\gamma}{\lambda} \int_\Omega F^0 v_0 \,dx\,dt. \quad (5.11)$$

In view of the conditions imposed on function F^0 and $v_0|_\Omega > 0$ we have

$$0 < æ_1 := \int_\Omega F^0 v_0 \,dx\,dt < +\infty. \quad (5.12)$$

Denoting the right part of the inequality (5.11) by $\chi = \chi(\gamma)$, which is a linear function with respect to the parameter γ, from (5.7) and (5.12) we have

$$\chi(\gamma) \begin{cases} < 0 & \text{for } \gamma > \gamma_0 \\ > 0 & \text{for } \gamma < \gamma_0, \end{cases} \quad (5.13)$$

where

$$\chi(\gamma) = \frac{æ_0}{\lambda^{p'}} - \frac{p'\gamma}{\lambda} æ_1, \quad \gamma_0 = \frac{\lambda æ_0}{\lambda^{p'} p' æ_1}.$$

There remains only to note that the left-hand side of (5.11) is nonnegative, whereas the right-hand side, due to (5.13), is negative for $\gamma > \gamma_0$. Thus, for $\gamma > \gamma_0$, problem (1.1)-(1.4) does not have a strong generalized solution of the class C in the domain Ω in the sense of Definition 1.1. The proof is complete. □

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