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EXISTENCE OF SOLUTIONS FOR DISCONTINUOUS $p(x)$-LAPLACIAN PROBLEMS WITH CRITICAL EXPONENTS

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ABSTRACT. In this article, we study the existence of solutions to the problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{p^*(x)-2}u + f(u) \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{gathered}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p(x)$ is a continuous function with $1 < p(x) < N$ and $p^*(x) = \frac{Np(x)}{N-p(x)}$. Applying nonsmooth critical point theory for locally Lipschitz functionals, we show that there is at least one nontrivial solution when λ less than a certain number, and f maybe discontinuous.

1. Introduction and statement of main results

In recent years, the study of problems in differential equations involving variable exponents has been a topic of interest. This is due to their applications in image restoration, mathematical biology, the study of dielectric breakdown, electrical resistivity, polycrystal plasticity, the growth of heterogeneous sandpiles and fluid dynamics, etc. We refer the reader to [4, 5, 6, 12, 14, 20, 26] and references therein for more information.

In this article, we discuss the existence of solutions to the problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{p^*(x)-2}u + f(u) \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p(x)$ is a continuous function defined on $\overline{\Omega}$ with $1 < p(x) < N$, $p^*(x) = \frac{Np(x)}{N-p(x)}$, and $\lambda > 0$. The function $f(u)$ can have discontinuities, so that functionals associated with (1.1) may not be differentiable, and standard variational techniques can not be applied. There are many publications for the case when $p(x)$ is a constant function; see for example [1, 2, 3, 9, 24]. For the existence of solutions for $p(x)$-Laplacian problems we refer the reader to [7, 11, 13, 16, 19, 22].

The existence of solutions for $p(x)$-Laplacian problems with critical growth is relatively new. In 2012, Bonder and Silva [8] extended the concentration-compactness

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principle of Lions to the variable exponent spaces and proved the existence of solutions to the problem

$$
\begin{gathered}
-\Delta_{p(x)}u = |u|^{q(x)-2}u + \lambda(x)|u|^{r(x)-2}u \quad x \in \Omega, \\
u = 0 \quad x \in \partial\Omega.
\end{gathered}
$$

Where Ω is a smooth bounded domain in $\mathbb{R}^N$, with $q(x) \le p^*(x)$ and the set $\{q(x) = p^*(x)\} \neq \emptyset$, we can find a similar result in [15]. Fu [17] studied the existence of solutions for $p(x)$-Laplacian equationd involving the critical exponent and obtained a sequence of radially symmetric solutions.

In the present paper, we study the discontinuous $p(x)$-Laplacian problems with critical growth for (1.1). To handle the gaps at the discontinuity points, our approach uses nonsmooth critical point theory for locally Lipschitz functionals, we obtain some general results for the simple case when f has only one point of discontinuity.

Because f is discontinuous, we say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a solution of the multivalued problem associated to (1.1) if u satisfies

$$
-\Delta_{p(x)}u - \lambda|u|^{p^*(x)-2}u \in \widehat{f}(u) \quad \text{a.e. in } \Omega,
$$

where $\widehat{f}(u)$ is the multivalued function $\widehat{f}(u) = [\underline{f}(u), \overline{f}(u)]$ with

$$
\underline{f}(t) = \liminf_{s\to t} f(s), \quad \overline{f}(t) = \limsup_{s\to t} f(s).
$$

In this article, we assume $f : \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying:

(F1) $f(t) = 0$ if $t \le 0$ and for all $t \in \mathbb{R}$, there exist the limits:

$$
f(t+0) = \lim_{\delta\to 0^+} f(t+\delta); \quad f(t-0) = \lim_{\delta\to 0^+} f(t-\delta).
$$

(F2) there exist $C_1, C_2 > 0$ such that $|f(t)| \le C_1 + C_2|t|^{q(x)-1}$, where $q(x) \in C(\overline{\Omega})$ such that $p(x) < q(x) < p^*(x)$.

(F3) $f(t) = o(|t|^{p(x)-1})$ as $t \to 0$.

(F4) $\underline{f}(t)t \ge q_- F(t) > 0$, for all $t \in \mathbb{R}\backslash\{0\}$, where $F(t) = \int_0^t f(s)ds$.

Note that by hypothesis (F1),

$$
\overline{f}(u) = \max\{f(u-0), f(u+0)\}, \quad \underline{f}(u) = \min\{f(u-0), f(u+0)\}.
$$

Theorem 1.1. *Suppose f satisfies* (F1)-(F4). *Then there exists $\lambda_0 > 0$ such that* (1.1) *has a nontrivial solution for every $\lambda \in (0, \lambda_0)$.*

One of the main motivations is to consider the particular case associated with (1.1),

$$
\begin{gathered}
-\Delta_{p(x)}u = \lambda|u|^{p^*(x)-2}u + bh(u-a)|u|^{q(x)-2}u \quad x \in \Omega, \\
u = 0 \quad x \in \partial\Omega,
\end{gathered} \tag{1.2}
$$

where $h(t) = 0$ if $t \le 0$ and $h(t) = 1$ if $t > 0$, a and b are positive real parameters, $p(x) < q(x) < p^*(x)$. As a direct consequence of Theorem 1.1, we have

Theorem 1.2. *For every $a, b > 0$, there exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, Equation* (1.2) *has a nontrivial solution satisfying* $\text{meas}\{x \in \Omega : u(x) > a\} > 0$.

The rest of this article is organized as follows: In section 2 we introduce some necessary preliminary knowledge; in section 3 contains the proof of our main results.

2. Preliminaries

First, we recall some definitions and properties of generalized gradient of locally Lipschitz functionals, which will be used later. Let X be a Banach space, X^* be its topological dual and $\langle\cdot,\cdot\rangle$ be the duality. A functional $I : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $u \in X$ there exists a neighborhood U of u and a constant $K > 0$ depending on U such that

$$|I(u) - I(v)| \le K\|u - v\|, \quad \forall u, v \in U.$$

For a locally Lipschitz functional I, we define the generalized directional derivative at $u \in X$ in the direction $v \in X$ by

$$I^0(u; v) = \limsup_{h\to 0,\, \delta\downarrow 0} \frac{I(u + h + \delta v) - I(u + h)}{\delta}.$$

It is easy to show that $I^0(u; v)$ is subadditive and positively homogeneus. The generalized gradient of I at u is the set

$$\partial I(u) = \{w \in X^* : I^0(u; v) \ge \langle w, v\rangle, \forall v \in X\}.$$

Then, for each $v \in X$, $I^0(u; v) = \max\{\langle\omega, v\rangle : \omega \in \partial I(u)\}$. A point $u \in X$ is a critical point of I if $0 \in \partial I(u)$. It is easy to see that if $u \in X$ is a local minimum or maximum, then $0 \in \partial I(u)$.

Next, we recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an arbitrary domain with smooth boundary. Set

$$C_+(\overline{\Omega}) = \{p(x) \in C(\overline{\Omega}) : p(x) > 1, \forall x \in \overline{\Omega}\},$$
$$p_+ = \max_{x\in\Omega} p(x), \quad p_- = \min_{x\in\Omega} p(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u \in M(\Omega) : \int_\Omega |u(x)|^{p(x)}dx < \infty\},$$

with the norm

$$|u|_{p(x)} = \inf\{\mu > 0 : \int_\Omega |\frac{u}{\mu}|^{p(x)}dx \le 1\},$$

where $M(\Omega)$ is the set of all measurable real functions defined on Ω.

Define the space

$$W_0^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Proposition 2.1 ([18, 21]). *There is a constant $C > 0$ such that for all $u \in W_0^{1,p(x)}(\Omega)$,*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}.$$

So $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$. Hence we will use the norm $\|u\| = |\nabla u|_{p(x)}$ for all $u \in W_0^{1,p(x)}(\Omega)$.

Proposition 2.2 ([18, 21]). *Set $\rho(u) = \int_\Omega |u|^{p(x)}dx$. For $u, u_n \in L^{p(x)}(\Omega)$, we have:*

(1) $|u|_{p(x)} < 1$ $(= 1; > 1) \Leftrightarrow \rho(u) < 1$ $(= 1; > 1)$.

(2) *If* $|u|_{p(x)} > 1$, *then* $|u|_{p(x)}^{p_-} \le \rho(u) \le |u|_{p(x)}^{p_+}$.
(3) *If* $|u|_{p(x)} < 1$, *then* $|u|_{p(x)}^{p_+} \le \rho(u) \le |u|_{p(x)}^{p_-}$.
(4) $\lim_{n\to\infty} u_n = u \Leftrightarrow \lim_{n\to\infty} \rho(u_n - u) = 0$.
(5) $\lim_{n\to\infty} |u_n|_{p(x)} = \infty \Leftrightarrow \lim_{n\to\infty} \rho(u_n) = \infty$.

Proposition 2.3 ([18]). *If $q(x) \in C_+(\Omega)$ and $q(x) < p^*(x)$ for any $x \in \Omega$, the imbedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$ is compact.*

Proposition 2.4 ([21]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$,*

$$\int_\Omega |uv|dx \le (\frac{1}{p_-} + \frac{1}{q_-})|u|_{p(x)}|v|_{q(x)}.$$

Proposition 2.5 ([16]). *If $|u|^{q(x)} \in L^{\frac{s(x)}{q(x)}}(\Omega)$, where $q(x), s(x) \in L_+^\infty(\Omega)$, $q(x) \le s(x)$, then $u \in L^{s(x)}(\Omega)$ and there is a number $\overline{q} \in [q_-, q_+]$ such that $||u|^{q(x)}|_{\frac{s(x)}{q(x)}} = (|u|_{s(x)})^{\overline{q}}$.*

Let $I_\lambda(u) : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ be the energy functional defined as

$$I_\lambda(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \lambda \int_\Omega \frac{1}{p^*(x)}|u|^{p^*(x)}dx - \int_\Omega F(u)dx, \tag{2.1}$$

denote $\Phi(u) = \int_\Omega F(u)dx$. We say that $I_\lambda(u)$ satisfies the nonsmooth $(PS)_c$ condition, if any sequence $\{u_n\} \subseteq X$ such that $I_\lambda(u_n) \to c$ and $m(u_n) = \min\{\|w\|_{X^*} : w \in \partial I_\lambda(u_n)\} \to 0$, as $n \to \infty$, possesses a convergent subsequence. To prove our main results, we use the generalizations of the mountain pass theorem [10].

Theorem 2.6. *Let X be a reflexive Banach space, $I : X \to \mathbb{R}$ is locally Lipschitz functional which satisfies the nonsmooth $(PS)_c$ condition, $I(0) = 0$ and there are $\rho, r > 0$ and $e \in X$ with $\|e\| > r$, such that*

$$I(u) \ge \beta \quad \text{if } \|u\| = r \quad \text{and} \quad I(e) \le 0.$$

Then there exists $u \in X$ such that $0 \in \partial I(u)$ and $I(u) = c$. Where

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t)),$$
$$\Gamma = \{\gamma \in C([0,1], X) | \gamma(0) = 0, \gamma(1) = e\}.$$

Recall the concentration-compactness principle for variable exponent spaces. This will be the keystone that enable us to verify that I_λ satisfies the nonsmooth $(PS)_c$ condition.

Proposition 2.7 ([8]). *Let $\{u_n\}$ converge weakly to u in $W_0^{1,p(x)}(\Omega)$ such that $|u_n|^{p^*(x)}$ and $|\nabla u_n|^{p(x)}$ converge weakly to nonnegative measures ν and μ on $\mathbb{R}^N$ respectively. Then, for some countable set J, we have:*

(i) $\nu = |u|^{p^*(x)} + \sum_{j\in J} \nu_j \delta_{x_j}$,
(ii) $\mu \ge |\nabla u|^{p(x)} + \sum_{j\in J} \mu_j \delta_{x_j}$,
(iii) $S\nu_j^{\frac{1}{p^*(x_j)}} \le \mu_j^{\frac{1}{p(x_j)}}$,

where $x_j \in \Omega$, δ_{x_j} is the Dirac measure at x_j, ν_j and μ_j are constants and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S = \inf \Big\{ \frac{\|u\|_{1,p(x)}}{|u|_{p^*(x)}} : u \in W_0^{1,p(x)}(\Omega), u \neq 0 \Big\}.$$

3. Proof of main results

In this section, we denote by $u_n \rightharpoonup u$ the weak convergence of a sequence u_n to u in $W_0^{1,p(x)}(\Omega)$, and $o(1)$ denote a real vanishing sequence, C and $C_i, i = 1, 2, \dots$ are positive constants, $|A|$ is the Lebesgue measure of A and $p'(x)$ as the conjugate function of $p(x)$. $u \in W_0^{1,p(x)}(\Omega)$ is called a solution of (1.1) if u is a critical point of $I_\lambda(u)$ and satisfies

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - \lambda |u|^{p^*(x)-2}u \in [\underline{f}(u), \overline{f}(u)] \quad \text{a.e. } x \in \Omega.$$

Lemma 3.1. *The function $\Phi(u)$ is locally Lipschitz on $W_0^{1,p(x)}(\Omega)$.*

Proof. By (F2), Proposition 2.4 and 2.5, for all $u, v \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned}
|\Phi(u) - \Phi(v)| &\leq \int_\Omega \Big| \int_u^v |f(t)| dt \Big| dx \\
&\leq \int_\Omega \Big| \int_u^v |C_1 + C_2 |t|^{q(x)-1}| dt \Big| dx \\
&\leq (|C_1|_{\frac{q(x)}{q(x)-1}} + C_3 |u|_{q(x)}^{\overline{q}-1} + C_3 |v|_{q(x)}^{\overline{q}-1}) |u - v|_{q(x)}.
\end{aligned}$$

From Proposition 2.3, we obtain that there is a neighborhood $U \subset W_0^{1,p(x)}(\Omega)$ of u such that

$$|\Phi(u) - \Phi(v)| \leq K \|u - v\|,$$

where $K > 0$ depends on $\max\{\|u\|, \|v\|\}$. So, $\Phi(u)$ is locally Lipschitz in $W_0^{1,p(x)}(\Omega)$. The proof is complete. □

From Lemma 3.1, by Chang's results we have that $I_\lambda(u)$ is locally Lipschitz and $\omega \in \partial I_\lambda(u)$ if and only if there is $\overline{\omega} \in W^{-1,p'(x)}(\Omega)$ such that for all $\varphi \in W_0^{1,p(x)}(\Omega)$,

$$\langle \omega, \varphi \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx - \lambda \int_\Omega |u|^{p^*(x)-2} u \varphi dx - \int_\Omega \overline{\omega} \varphi dx, \tag{3.1}$$

and

$$\overline{\omega}(x) \in [\underline{f}(u(x)), \overline{f}(u(x))] \quad \text{a.e. } x \in \Omega. \tag{3.2}$$

Lemma 3.2. *Assume* (F1), (F2). *Let $\{u_n\}$ be a bounded sequence in $W_0^{1,p(x)}(\Omega)$ such that $I_\lambda(u_n) \to c$ and $m(u_n) \to 0$. Then there exists a subsequence (denoted again by u_n) and some $u \in W_0^{1,p(x)}(\Omega)$, such that*

$$|\nabla u_n|^{p(x)-2}\nabla u_n \rightharpoonup |\nabla u|^{p(x)-2}\nabla u \quad \textit{weakly in } [L^{\frac{p(x)}{p(x)-1}}(\Omega)]^N.$$

Proof. The proof is similar to that of [25, Theorem 1]. Because $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, there exist a subsequence and $u \in W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $u_n \to u$ in $L^{p(x)}(\Omega)$ as $n \to \infty$.

We claim that the set J given by Proposition 2.7 is finite. Choose a function $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi(x) \leq 1$, $\varphi(x) \equiv 1$ on $B(0,1)$ and $\varphi(x) \equiv 0$ on

$\mathbb{R}^N \setminus B(0,2)$. Let $\varphi_{j,\varepsilon}(x) = \varphi(\frac{x-x_j}{\varepsilon})$, for any $x \in \mathbb{R}^N$, $\epsilon > 0$ and $j \in J$. It is clear that $\{\varphi_{j,\varepsilon} u_n\} \subset W_0^{1,p(x)}(\Omega)$ for any $j \in J$, and is bounded in $W_0^{1,p(x)}(\Omega)$. Take $\varphi = \varphi_{j,\varepsilon} u_n$ in $\langle \omega_n, \varphi \rangle$, we obtain

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot u_n \nabla \varphi_{j,\varepsilon} dx + \int_\Omega |\nabla u_n|^{p(x)} \varphi_{j,\varepsilon} dx \\ &- \lambda \int_\Omega |u_n|^{p^*(x)} \varphi_{j,\varepsilon} dx - \int_\Omega \overline{\omega}_n \varphi_{j,\varepsilon} u_n dx = o(1). \end{aligned} \tag{3.3}$$

Taking into account that $\omega_n \in \partial I_\lambda(u_n)$, by (F2) and $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, we infer that $\overline{\omega}_n$ is bounded in $W^{-1,p'(x)}(\Omega)$, and so there exists $\overline{\omega}_0 \in W^{-1,p'(x)}(\Omega)$ such that

$$\overline{\omega}_n \rightharpoonup \overline{\omega}_0 \quad \text{in } W^{-1,p'(x)}(\Omega) \quad \text{and} \quad \overline{\omega}_0 \in [\underline{f}(u), \overline{f}(u)]. \tag{3.4}$$

Let $n \to \infty$ in (3.3), by Proposition 2.7, we have

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot u_n \nabla \varphi_{j,\varepsilon} dx \\ &= \lambda \int_\Omega \varphi_{j,\varepsilon} d\nu - \int_\Omega \varphi_{j,\varepsilon} d\mu + \int_\Omega \overline{\omega}_0 \varphi_{j,\varepsilon} u \, dx. \end{aligned} \tag{3.5}$$

By Hölder inequality it is easy to check that

$$\begin{aligned} 0 &\le |\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi_{j,\varepsilon} \cdot u_n \, dx| \\ &\le \Big(\int_\Omega |\nabla u_n|^{p_+} dx \Big)^{\frac{p_+-1}{p_+}} \Big(\int_\Omega |u_n|^{p_+} |\nabla \varphi_{j,\varepsilon}|^{p_+} dx \Big)^{\frac{1}{p_+}} \\ &\quad + \Big(\int_\Omega |\nabla u_n|^{p_-} dx \Big)^{\frac{p_--1}{p_-}} \Big(\int_\Omega |u_n|^{p_-} |\nabla \varphi_{j,\varepsilon}|^{p_-} dx \Big)^{\frac{1}{p_-}} \\ &\le C_4 (\int_\Omega |\nabla u_n|^{p_+} dx)^{\frac{p_+-1}{p_+}} \Big(\int_{B(x_j,2\varepsilon)} |u_n|^{(p_+)^*} dx \Big)^{\frac{1}{(p_+)^*}} \\ &\quad + C_5 \Big(\int_\Omega |\nabla u_n|^{p_-} dx \Big)^{\frac{p_--1}{p_-}} \Big(\int_{B(x_j,2\varepsilon)} |u_n|^{(p_-)^*} dx \Big)^{\frac{1}{(p_-)^*}} \quad \to 0, \quad \text{as } \epsilon \to 0. \end{aligned}$$

From (3.5), as $\epsilon \to 0$, we obtain $\lambda \nu_j = \mu_j$. From Proposition 2.7, we conclude that

$$\nu_j = 0 \quad \text{or} \quad \nu_j \ge S^N \max\{\lambda^{-\frac{N}{p_+}}, \lambda^{-\frac{N}{p_-}}\}. \tag{3.6}$$

It implies that J is a finite set.

Without loss of generality, let $J = \{1, 2, \dots, m\}$. For any $\delta > 0$, we denote $\Omega_\delta = \{x \in \Omega | \operatorname{dist}(x, x_j) > \delta\}$. Choose R large enough such that $\overline{\Omega} \subset \{x \in \mathbb{R}^N | |x| < R\}$, $\psi(x) \in C^\infty(\mathbb{R}^N)$, $0 \le \psi(x) \le 1$, $\psi(x) \equiv 0$ on $B(0,2R)$ and $\psi(x) \equiv 1$ on $\mathbb{R}^N \setminus B(0,3R)$. Take $\epsilon > 0$ small enough such that $B(x_i, \epsilon) \cap B(x_j, \epsilon) = \emptyset$, $\forall i, j \in J$, $i \ne j$ and $\cup_{j=1}^m B(x_j, \epsilon) \subset B(0, 2R)$. We take

$$\psi_\epsilon(x) = 1 - \sum_{j=1}^m \varphi_{j,\varepsilon} - \psi(x), \quad x \in \mathbb{R}^N.$$

Then $\psi_\epsilon(x) \in C^\infty(\mathbb{R}^N)$, $\operatorname{supp} \psi_\epsilon \subset B(0,3R)$ with $\psi_\epsilon(x) = 0$ on $\cup_{j=1}^m B(x_j, \epsilon/2)$ and $\psi_\epsilon(x) = 1$ on $(\mathbb{R}^N \setminus B(x_j, \epsilon)) \cap B(0, 2R)$.

As $\{\psi_\epsilon u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, let $\varphi=\psi_\epsilon u_n$ in $\langle\omega_n,\varphi\rangle$, we obtain

$$\begin{aligned}&\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\cdot u_n\nabla\psi_\epsilon dx+\int_\Omega|\nabla u_n|^{p(x)}\psi_\epsilon dx\\&-\lambda\int_\Omega|u_n|^{p^*(x)}\psi_\epsilon dx-\int_\Omega\overline{\omega}_n\psi_\epsilon u_n dx=o(1).\end{aligned}$$

By (3.4) and $u_n\rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, we can easily obtain

$$\lim_{n\to\infty}\int_\Omega\overline{\omega}_n\psi_\epsilon u_n dx=\int_\Omega\overline{\omega}_0\psi_\epsilon u dx.$$

Since $\psi_\epsilon(x)=0$ on $\cup_{j=1}^m B(x_j,\frac{\epsilon}{2})$ and $\nu=|u|^{p^*(x)}+\sum_{j\in J}\nu_j\delta_{x_j}$, we obtain

$$\lim_{n\to\infty}\int_\Omega|u_n|^{p^*(x)}\psi_\epsilon dx=\int_\Omega\psi_\epsilon d\nu=\int_\Omega|u|^{p^*(x)}\psi_\epsilon dx.$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega|\nabla u_n|^{p(x)}\psi_\epsilon dx&=\lim_{n\to\infty}(-\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\cdot u_n\nabla\psi_\epsilon dx)\\&\quad+\lambda\int_\Omega|u|^{p^*(x)}\psi_\epsilon dx+\int_\Omega\overline{\omega}_0\psi_\epsilon u dx.\end{aligned}\tag{3.7}$$

In the same way, taking $\varphi=\psi_\epsilon u$ in $\langle\omega_n,\varphi\rangle$, we obtain

$$\begin{aligned}&\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\cdot u\nabla\psi_\epsilon dx+\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla u\cdot\psi_\epsilon dx\\&-\lambda\int_\Omega|u_n|^{p^*(x)-2}u_n u\psi_\epsilon dx-\int_\Omega\overline{\omega}_n\psi_\epsilon u dx=o(1).\end{aligned}$$

Thus

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla u\cdot\psi_\epsilon dx\\&=\lim_{n\to\infty}(-\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\cdot u\nabla\psi_\epsilon dx)+\lambda\int_\Omega|u|^{p^*(x)}\psi_\epsilon dx+\int_\Omega\overline{\omega}_0\psi_\epsilon u dx.\end{aligned}\tag{3.8}$$

So, from (3.7) and (3.8), as $n\to\infty$, we have

$$\begin{aligned}0&\le\int_{\Omega_\delta}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)dx\\&\le\int_\Omega\psi_\epsilon(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)dx\\&=\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi_\epsilon\cdot(u-u_n)dx\\&\quad+\int_\Omega\psi_\epsilon|\nabla u|^{p(x)-2}\nabla u\cdot(\nabla u-\nabla u_n)dx+o(1)\\&\le\|\nabla\psi_\epsilon\|_\infty\cdot||\nabla u_n|^{p(x)-1}|_{p'(x)}\cdot|u-u_n|_{p(x)}\\&\quad+\int_\Omega\psi_\epsilon|\nabla u|^{p(x)-2}\nabla u\cdot(\nabla u_n-\nabla u)dx+o(1).\end{aligned}$$

Thus we have

$$\lim_{n\to\infty}\int_{\Omega_\delta}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)dx=0.\tag{3.9}$$

Denote
$$g_n(x) = (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u),$$
then $g_n(x) \ge 0$, and by (3.9), $g_n(x) \to 0$ a.e. on Ω_δ. Let E be a compact subset of Ω_δ, suppose $g_n(x) \to 0$ a.e. on E. If ∇u_n were not convergence to ∇u everywhere on E, there would at least exist $x_0 \in E$ such that
$$\lim_{n\to\infty} \nabla u_n(x_0) \neq \nabla u(x_0).$$
Then
$$\begin{aligned}|\nabla u_n(x_0)|^{p(x_0)} &= |\nabla u_n(x_0)|^{p(x_0)-2}\nabla u_n(x_0)\nabla u(x_0)\\ &\quad + |\nabla u(x_0)|^{p(x_0)-2}\nabla u_n(x_0)\nabla u(x_0) - |\nabla u(x_0)|^{p(x_0)} + g_n(x_0).\end{aligned}$$
By the interpolation inequality,
$$\begin{aligned}||\nabla u_n(x_0)|^{p(x_0)-2}\nabla u_n(x_0)\nabla u(x_0)| &\le |\nabla u_n(x_0)|^{p(x_0)-1} \cdot |\nabla u(x_0)|\\ &\le \epsilon_1 |\nabla u_n(x_0)|^{p(x_0)} + C_{\epsilon_1}|\nabla u(x_0)|^{p(x_0)},\end{aligned}$$
and
$$\begin{aligned}||\nabla u(x_0)|^{p(x_0)-2}\nabla u_n(x_0)\nabla u(x_0)| &\le |\nabla u(x_0)|^{p(x_0)-1} \cdot |\nabla u_n(x_0)|\\ &\le \epsilon_2 |\nabla u(x_0)|^{p(x_0)} + C_{\epsilon_2}|\nabla u_n(x_0)|^{p(x_0)}.\end{aligned}$$
We choose ϵ_1, ϵ_2 properly, because $g_n(x_0)$ is bounded, then $|\nabla u(x_0)| \le C$. Let $\nabla u(x_0) = \eta$, so we can assume $\nabla u_n(x_0) \to \overline{\eta} \neq \eta$. Thus
$$g_n(x_0) \to (|\overline{\eta}|^{p(x_0)-2}\overline{\eta} - |\eta|^{p(x_0)-2}\eta)(\overline{\eta} - \eta) > 0.$$
This contradicts $g_n(x_0) \to 0$. Hence, $\nabla u_n(x_0) \to \nabla u(x_0)$ everywhere on E. So $\nabla u_n(x_0) \to \nabla u(x_0)$ a.e. on Ω_δ. Since δ is arbitrary, we obtain $\nabla u_n(x_0) \to \nabla u(x_0)$ a.e. on Ω. Since $\{|\nabla u_n|^{p(x)-2}\nabla u_n\}$ is integrable in $L^1(\Omega)$, we obtain that as $n \to \infty$,
$$|\nabla u_n|^{p(x)-2}\nabla u_n \rightharpoonup |\nabla u|^{p(x)-2}\nabla u \quad \text{weakly in } [L^{\frac{p(x)}{p(x)-1}}(\Omega)]^N.$$
The proof is complete $\square$

Lemma 3.3. *Suppose f satisfies* (F2), (F4). *Then I_λ satisfies the nonsmooth $(PS)_c$ condition provided $c < (\frac{1}{p_+} - \frac{1}{q_-})S^N \max\{\lambda^{1-\frac{N}{p_+}}, \lambda^{1-\frac{N}{p_-}}\}$.*

Proof. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be such that $I_\lambda(u_n) \to c$ and $m(u_n) \to 0$ as $n \to \infty$. We must show the existence of a subsequence of $\{u_n\}$ which converges strongly in $W_0^{1,p(x)}(\Omega)$. First, we show that $\{u_n\}$ is bounded. We know that
$$\begin{aligned}I_\lambda(u_n) &= \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \lambda\int_\Omega \frac{1}{p^*(x)}|u_n|^{p^*(x)}dx - \int_\Omega F(u_n)dx\\ &\ge \frac{1}{p_+}\int_\Omega |\nabla u_n|^{p(x)}dx - \frac{\lambda}{p_-^*}\int_\Omega |u_n|^{p^*(x)}dx - \int_\Omega F(u_n)dx.\end{aligned} \tag{3.10}$$
Let $\omega_n \in \partial I_\lambda(u_n)$ such that $\|\omega_n\| = m(u_n) = o(1)$. From (3.1) we have
$$\langle \omega_n, u_n\rangle = \int_\Omega |\nabla u_n|^{p(x)}dx - \lambda\int_\Omega |u_n|^{p^*(x)}dx - \int_\Omega \overline{\omega}_n u_n dx, \tag{3.11}$$

where $\overline{\omega}_n(x) \in [\underline{f}(u_n), \overline{f}(u_n)]$ for a.e. $x \in \Omega$. By (F4) we obtain

$$\frac{1}{q_-}\overline{\omega}_n u_n \geq \frac{1}{q_-}\underline{f}(u_n)u_n \geq F(u_n). \tag{3.12}$$

From (3.10), (3.11) and (3.12), we obtain

$$C_6(1+\|u_n\|) \geq I_\lambda(u_n) - \frac{1}{q_-}\langle \omega_n, u_n\rangle \geq (\frac{1}{p_+} - \frac{1}{q_-})\int_\Omega |\nabla u_n|^{p(x)}dx. \tag{3.13}$$

If $\|u_n\| > 1$, by Proposition 2.2, we obtain

$$(\frac{1}{p_+} - \frac{1}{q_-})\|u_n\|^{p_-} \leq C_6(1+\|u_n\|).$$

Thus $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Then there exist a subsequence and $u \in W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, so we know that $\{|u_n|^{p^*(x)-2}u_n\varphi\}$ is uniformly integrable in $L^1(\Omega)$. By this fact, Lemma 3.2 and $m(u_n) \to 0$, taking $n \to \infty$ in $\langle \omega_n, \varphi\rangle$, we have

$$0 = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi dx - \lambda\int_\Omega |u|^{p^*(x)-2}u\varphi dx - \int_\Omega \overline{\omega}\varphi dx, \quad \forall\, \varphi \in C_0^\infty(R^N).$$

So we derive that

$$-\Delta_{p(x)}u - \lambda|u|^{p^*(x)-2}u \in [\underline{f}(u), \overline{f}(u)]. \tag{3.14}$$

Now we applying Proposition 2.7 to prove that $\nu_j = 0$ in (3.6). Assume $\nu_j \neq 0$ for some $j \in J$. From (3.13), we have

$$I_\lambda(u_n) - \frac{1}{q_-}\langle \omega_n, u_n\rangle \geq (\frac{1}{p_+} - \frac{1}{q_-})\int_\Omega |\nabla u_n|^{p(x)}dx.$$

Since $I_\lambda(u_n) \to c$ and $m(u_n) \to 0$, using Proposition 2.7, taking $n \to \infty$, we obtain

$$\begin{aligned} c &\geq (\frac{1}{p_+} - \frac{1}{q_-})\int_\Omega |\nabla u|^{p(x)}dx + (\frac{1}{p_+} - \frac{1}{q_-})\sum_{j\in J}\mu_j \\ &\geq (\frac{1}{p_+} - \frac{1}{q_-})S^N \max\{\lambda^{1-\frac{N}{p_+}}, \lambda^{1-\frac{N}{p_-}}\}. \end{aligned}$$

Since $c < (\frac{1}{p_+} - \frac{1}{q_-})S^N \max\{\lambda^{1-\frac{N}{p_+}}, \lambda^{1-\frac{N}{p_-}}\}$, then $\nu_j = 0$ for all $j \in J$. Hence we have

$$\int_\Omega |u_n|^{p^*(x)}dx \to \int_\Omega |u|^{p^*(x)}dx. \tag{3.15}$$

So we can use [15, Lemma 2.1]. Set $v_n = u_n - u$ and we have

$$\int_\Omega |u_n|^{p^*(x)}dx = \int_\Omega |v_n|^{p^*(x)}dx + \int_\Omega |u|^{p^*(x)}dx + o(1), \tag{3.16}$$

$$\int_\Omega |\nabla u_n|^{p(x)}dx = \int_\Omega |\nabla v_n|^{p(x)}dx + \int_\Omega |\nabla u|^{p(x)}dx + o(1). \tag{3.17}$$

Thus, by (3.15) and (3.16), $u_n \to u$ strongly in $L^{p^*(x)}(\Omega)$. From (3.11), using (3.14) and (3.17), we obtain

$$\langle \omega_n, u_n\rangle = \int_\Omega |\nabla v_n|^{p(x)}dx + \int_\Omega |\nabla u|^{p(x)}dx - \lambda\int_\Omega |u_n|^{p^*(x)}dx - \int_\Omega \overline{\omega}_n u_n dx + o(1).$$

By (3.4) and (3.15), letting $n \to \infty$, we conclude that

$$\int_\Omega |\nabla v_n|^{p(x)} dx \to 0.$$

This fact and Proposition 2.2 imply that $u_n \to u$ strongly in $W_0^{1,p(x)}(\Omega)$. The proof is complete. $\square$

Lemma 3.4. *Suppose f satisfies* (F2), (F3). *Then, for every $\lambda > 0$, there are $\alpha, \rho > 0$, such that $I_\lambda(u) \geq \alpha$, $\|u\| = \rho$.*

Proof. By (F2) and (F3), we have

$$|f(t)| \leq \epsilon |t|^{p(x)-1} + C|t|^{q(x)} \leq \epsilon |t|^{p(x)-1} + C(\epsilon)|t|^{p^*(x)-1}.$$

Therefore,

$$\begin{aligned} I_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \lambda \int_\Omega \frac{1}{p^*(x)} |u|^{p^*(x)} dx - \int_\Omega F(u) dx \\ &\geq \frac{1}{p_+} \int_\Omega |\nabla u|^{p(x)} dx - \frac{\epsilon}{p_-} \int_\Omega |u|^{p(x)} dx - \frac{\lambda + C(\epsilon)}{p_-^*} \int_\Omega |u|^{p^*(x)} dx. \end{aligned} \tag{3.18}$$

we can take $\|u\| < 1$ sufficiently small such that $|u|_{p(x)} < 1$ and $|u|_{p^*(x)} < 1$. From (3.18), Propositions 2.1 and 2.4, and the definition of S, using the usual arguments, we obtain

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p_+} \|u\|^{p_+} - \frac{\epsilon}{p_-} |u|_{p(x)}^{p_-} - \frac{\lambda + C(\epsilon)}{p_-^*} |u|_{p^*(x)}^{p_-^*} \\ &\geq \frac{1}{2p_+} \|u\|^{p_+} - \frac{\lambda + C(\epsilon)}{p_-^*} S^{-p_-^*} \|u\|^{p_-^*} \\ &= (\frac{1}{2p_+} - \frac{\lambda + C(\epsilon)}{p_-^*} S^{-p_-^*} \|u\|^{p_-^* - p_+}) \|u\|^{p_+}. \end{aligned}$$

Considering

$$g(t) = \frac{1}{2p_+} - \frac{\lambda + C(\epsilon)}{p_-^*} S^{-p_-^*} \|t\|^{p_-^* - p_+},$$

since $p_+ < p_-^*$, we have $g(t) \to \frac{1}{2p_+}$ as $t \to 0$. Hence, there exists $\rho > 0$ such that $g(\rho) > 0$. So, we obtain α and $\rho > 0$, such that

$$I_\lambda(u) \geq \alpha, \quad \|u\| = \rho.$$

The proof is complete. $\square$

Next, we choose $\varphi(x) \in W_0^{1,p(x)}(\Omega)$, such that $\|\varphi\| = 1$.

Lemma 3.5. *Suppose f satisfies* (F4). *Then, there exists $\lambda_0 > 0$, $t_0 > 0$ such that $I_\lambda(t_0\varphi) < 0$, and for all $\lambda \in (0, \lambda_0)$,*

$$\sup_{t \geq 0} I_\lambda(t\varphi) < (\frac{1}{p_+} - \frac{1}{q_-}) S^N \max\{\lambda^{1-\frac{N}{p_+}}, \lambda^{1-\frac{N}{p_-}}\}.$$

Proof. By (F4), we have

$$f(u)u \geq \underline{f}(u)u \geq q_- F(u), \quad \forall u \neq 0.$$

This implies $F(tu) \geq t^{q_-} F(u)$, for all $t \geq 1$. Then, for any $t > 1$,

$$I_\lambda(t\varphi) \leq \frac{t^{p_+}}{p_-} - \frac{\lambda t^{p_+^*}}{p_+^*} \int_\Omega |\varphi|^{p^*(x)} dx - \int_\Omega F(t\varphi) dx \leq \frac{t_+^p}{p_-} - t^{q_-} \int_\Omega F(\varphi) dx = J_1(t\varphi).$$

Since $q_- > p_+$ and $F(\varphi) > 0$, there exists $t_0 > 0$ sufficiently large such that $I_\lambda(t_0\varphi) < 0$ and $\|t_0\varphi\| > \rho$ with ρ given by Lemma 3.4. If $0 \le t < 1$, then

$$I_\lambda(t\varphi) \le \frac{t^{p_-}}{p_-} - \int_\Omega F(t\varphi)dx = J_2(t\varphi).$$

Let $J(t\varphi) = \max\{J_1(t\varphi), J_2(t\varphi)\}$, so we have

$$\sup_{t\ge 0} I_\lambda(t\varphi) \le \sup_{t\ge 0} J(t\varphi).$$

Hence, we can find $\lambda_0 > 0$ such that

$$\sup_{t\ge 0} J_\lambda(t\varphi) < (\frac{1}{p_+} - \frac{1}{q_-})S^N \max\{\lambda^{1-\frac{N}{p_+}}, \lambda^{1-\frac{N}{p_-}}\}.$$

So, for all $\lambda \in (0, \lambda_0)$, we have

$$\sup_{t\ge 0} I_\lambda(t\varphi) < (\frac{1}{p_+} - \frac{1}{q_-})S^N \max\{\lambda^{1-\frac{N}{p_+}}, \lambda^{1-\frac{N}{p_-}}\}.$$

The proof is complete. □

Proof of Theorem 1.1. It is obvious that $I_\lambda(0) = 0$. By Lemmas 3.1, 3.3–3.5, according to Theorem 2.6, there exist $\lambda_0 > 0$, and for all $\lambda \in (0, \lambda_0)$, we can find an $u \in W_0^{1,p(x)}(\Omega)$ such that $I_\lambda(u) > 0$ and $0 \in \partial I_\lambda(u)$. Hence, u is a nontrivial solution of (1.1). The proof is complete. □

Proof of Theorem 1.2. In (1.2), $f(u) = bh(u-a)|u|^{q(x)-2}u$ has only one discontinuity point a, so by the consequence of Theorem 1.1, we obtain that an $u \in W_0^{1,p(x)}(\Omega)$ is a nontrivial nonnegative solution of (1.2). That is,

$$-\Delta_{p(x)}u - \lambda|u|^{p^*(x)-2}u \in \widehat{f}(u) \quad \text{a.e. in } \Omega \tag{3.19}$$

where $\widehat{f}(u)$ is the multivalued function given by

$$\widehat{f}(s) = \begin{cases} \{f(s)\} & s \ne a, \\ [0, bh(x)u^{q(x)-1}] & s = a. \end{cases} \tag{3.20}$$

If $V = \{x \in \Omega : u(x) = a\}$ exists, by (3.19) and (3.20), we have

$$-\Delta_{p(x)}u - \lambda|u|^{p^*(x)-2}u \in [0, bh(x)u^{q(x)-1}] \quad \text{a.e. in } V.$$

Using the Morrey-Stampacchia's theorem [23], we have $-\Delta_{p(x)}u = 0$ a.e. $x \in V$. So

$$-\lambda a^{p^*(x)-1} \ge 0 \quad \text{a.e. in } V.$$

This is a contradiction. Thus $|V| = 0$. The proof is complete. □

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