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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR M-POINT NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS ON THE HALF LINE

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Abstract. In this article we find sufficient conditions for existence and multiplicity of positive solutions for an m-point nonlinear fractional boundary-value problem on an infinite interval. Moreover, we prove that the set of positive solutions is compact. Nonexistence results for the boundary-value problem also are obtained.

1. Introduction

Fractional calculus has played a significant role in engineering, science, economy, and other fields. The monographs [3, 7, 6, 5] are commonly cited for the theory of fractional derivatives and integrals and applications to differential equations of fractional order. Recently, there have been some papers dealing with the existence and multiplicity of positive solutions of nonlinear boundary value problems of fractional order using the techniques of nonlinear analysis (fixed point theorem, Leray-Schauder theory, etc). See [4, 2, 9, 8, 11] for more details.

In this article we investigate existence and nonexistence results for a boundary-value problem of nonlinear fractional differential equation with m-point boundary conditions on an infinite interval of the form

$$D^{\alpha}_{0^+}u(t) + \lambda a(t) f(t, u(t)) = 0, \quad t \in (0, \infty),\ \alpha \in (2, 3), \tag{1.1}$$

$$u(0) + u'(0) = 0, \quad \lim_{t\to+\infty} D^{\alpha-1}_{0^+}u(t) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i), \tag{1.2}$$

$$0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty, \quad \beta_i \in \mathbb{R}^+ \cup \{0\}, \quad i = 1, 2, \dots, m-2 \tag{1.3}$$

where $D^{\alpha}_{0^+}$ is the fractional Riemann-Liouville derivative of order $\alpha > 0$ and λ is a positive parameter. We assume the following conditions:

(H1) $f \in C((0, \infty) \times [0, \infty), [0, \infty))$, $f(t, 0) \neq 0$ on any subinterval of $(0, +\infty)$, also when u is bounded $f(t, (1 + t^{\alpha-1})u)$ is bounded on $[0, +\infty)$.

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(H2) $a \in C((0,\infty),[0,\infty))$ and $a(t)$ is not identically zero on any interval of the form (t_0,∞). Also assume that
$$0 < \int_0^\infty a(s)ds < \infty.$$
(H3) $0 < \sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-2} < \Gamma(\alpha)$.

2. Preliminaries

In this section we introduce some fundamental tools of fractional calculus. We also remind the well known fixed point theorem due to Krasnosel'skii for operators acting on cones in Banach spaces.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \to \mathbb{R}$ is given by
$$I_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$$

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $u : (0,\infty) \to \mathbb{R}$ is defined by
$$D_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds$$
where $n = [\alpha]+1$.

Lemma 2.3 ([3])**.** *Let $u \in C(0,\infty) \cap L^1(0,\infty)$, $\beta \ge \alpha \ge 0$, then*
$$D_{0^+}^{\alpha}I_{0^+}^{\beta}u(t) = I_{0^+}^{\beta-\alpha}u(t)$$

Lemma 2.4 ([3])**.** *Let $\alpha > 0$ then*

(i) *If $\mu > -1$, $\mu \neq \alpha - i$ with $i = 1,2,\dots,[\alpha]+1$, $t > 0$ then*
$$D_{0^+}^{\alpha}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}.$$
(ii) *For $i = 1,2,\dots,[\alpha]+1$, we have $D_{0^+}^{\alpha}t^{\alpha-i} = 0$.*

(iii) *For every $t \in (0,\infty)$, $u \in L^1(0,\infty)$*
$$D_{0^+}^{\alpha}I_{0^+}^{\alpha}u(t) = u(t), \quad I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-i}, \quad c_i \in \mathbb{R},\ n = [\alpha]+1.$$
(iv) *$D_{0^+}^{\alpha}u(t) = 0$ if and only if $u(t) = \sum_{i=1}^{n} c_i t^{\alpha-i}$, $c_i \in \mathbb{R}$, $n = [\alpha]+1$.*

Lemma 2.5. *Let $h \in C[0,\infty)$ such that $0 < \int_0^{+\infty} h(s)ds < +\infty$, then the fractional boundary-value problem*
$$D_{0^+}^{\alpha}u(t) + h(t) = 0, \quad t \in (0,\infty),\ \alpha \in (2,3), \tag{2.1}$$
$$u(0) + u'(0) = 0, \quad \lim_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = \sum_{i=1}^{m-2}\beta_i u'(\xi_i) \tag{2.2}$$
has a unique solution
$$u(t) = \int_0^{+\infty} G(t,s)h(s)ds, \tag{2.3}$$

where

$$G(t,s) = H_1(t,s) + H_2(t,s) \tag{2.4}$$

with

$$H_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t < +\infty \\ t^{\alpha-1}, & 0 \le t \le s < +\infty \end{cases}, \tag{2.5}$$

$$H_2(t,s) = \frac{\sum_{i=1}^{m-2} \beta_i t^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2} (\alpha-1)\beta_i \xi_i^{\alpha-2}} \frac{\partial H_1(t,s)}{\partial t}\Big|_{t=\xi_i}. \tag{2.6}$$

The function $G(t,s)$ is called Green's function of boundary-value problem (2.1)-(2.2).

Proof. By Lemmas 2.3 and 2.4 and considering (2.1), we have

$$u(t) = -c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - c_3 t^{\alpha-3} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.$$

Then

$$u'(t) = -(\alpha-1)c_1 t^{\alpha-2} - (\alpha-2)c_2 t^{\alpha-3} - (\alpha-3)c_3 t^{\alpha-4} - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds.$$

Now by imposing the boundary condition $u(0) + u'(0) = 0$ we conclude that $c_2 = c_3 = 0$, also using boundary condition

$$\lim_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i)$$

we have

$$c_1 = \frac{1}{\Gamma(\alpha) - \sum_{i=1}^{m-2}(\alpha-1)\beta_i \xi_i^{\alpha-2}} \Big[\sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - \int_0^{+\infty} h(s)ds \Big].$$

Thus

$$\begin{aligned}
u(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2}(\alpha-1)\beta_i \xi_i^{\alpha-2}} \int_0^{+\infty} h(s)ds \\
&\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2}(\alpha-1)\beta_i \xi_i^{\alpha-2}} \Big[\sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds \Big] \\
&\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\
&= \int_0^{+\infty} H_1(t,s)h(s)ds + \frac{\sum_{i=1}^{m-2}(\alpha-1)\beta_i \xi_i^{\alpha-2}}{\Gamma(\alpha) - \sum_{i=1}^{m-2}(\alpha-1)\beta_i \xi_i^{\alpha-2}} \int_0^{+\infty} \frac{t^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\
&\quad - \frac{\sum_{i=1}^{m-2} \beta_i t^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2}(\alpha-1)\beta_i \xi_i^{\alpha-2}} \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds \\
&= \int_0^{+\infty} H_1(t,s)h(s)ds \\
&\quad + \frac{\sum_{i=1}^{m-2} \beta_i t^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2}(\alpha-1)\beta_i \xi_i^{\alpha-2}} \int_0^{\xi_i} \frac{\xi_i^{\alpha-2} - (\xi_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}}{\Gamma(\alpha)-\sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-2}}\int_{\xi_i}^{+\infty}\frac{\xi_i^{\alpha-2}}{\Gamma(\alpha-1)}h(s)ds\\
&=\int_0^{+\infty}H_1(t,s)h(s)ds\\
&\quad+\frac{\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}}{\Gamma(\alpha)-\sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-2}}\int_0^{+\infty}\frac{\partial H_1(t,s)}{\partial t}\Big|_{t=\xi_i}h(s)ds\\
&=\int_0^{+\infty}H_1(t,s)h(s)ds+\int_0^{+\infty}H_2(t,s)h(s)ds\\
&=\int_0^{+\infty}G(t,s)h(s)ds
\end{aligned}
$$

where $G(t,s)$ is Green's function defined by (2.4). Now by uniqueness of constants c_1, c_2, c_3 we conclude that (2.3) is the unique solution of boundary value problem (2.1)-(2.2). This completes the proof. □

Lemma 2.6. *The function $H_1(t,s)$ defined by (2.5) has the following properties:*

- (i) *$H_1(t,s)$ is a nonnegative continuous function for $t,s\in[0,+\infty)$;*
- (ii) *$H_1(t,s)$ is increasing function with respect to the first variable;*
- (iii) *$H_1(t,s)$ is a concave function with respect to the first variable, for every $0<s<t<+\infty$.*

Proof. Using (2.5) it is easy to see that property (i) obviously holds. Now we show that (ii) holds. Considering (i) we know that $H_1(t,s)\in C([0,\infty)\times[0,\infty),[0,\infty))$, hence

$$
\frac{\partial H_1(t,s)}{\partial t}=\frac{(\alpha-1)}{\Gamma(\alpha)}\begin{cases} t^{\alpha-2}-(t-s)^{\alpha-2}, & 0\le s\le t<+\infty\\ t^{\alpha-2}, & 0\le t\le s<+\infty;\end{cases}
$$

thus $H_1(t,s)$ is an increasing function with respect to first variable.

To prove (iii) we note that

$$
\frac{\partial^2 H_1(t,s)}{\partial t^2}=\frac{1}{\Gamma(\alpha-2)}\begin{cases} t^{\alpha-3}-(t-s)^{\alpha-3}, & 0\le s\le t<+\infty\\ t^{\alpha-3}, & 0\le t\le s<+\infty.\end{cases}
$$

On the other hand $\alpha\in(2,3)$, thus for $0<s<t<+\infty$,

$$
\frac{\partial^2 H_1(t,s)}{\partial t^2}<0
$$

So $H_1(t,s)$ is a concave function with respect to first variable, for $0<s<t<+\infty$. This completes the proof. □

Remark 2.7. According to definition of $H_1(t,s)$ in (2.5) we have for $t,s\in[0,+\infty)$,

$$
\frac{H_1(t,s)}{1+t^{\alpha-1}}\le\frac{1}{\Gamma(\alpha)},\quad \frac{G(t,s)}{1+t^{\alpha-1}}\le L,
$$
$$
L=\frac{1}{\Gamma(\alpha)}\Big(1+\frac{\sum_{i=1}^{m-2}\beta_i\xi_{m-2}^{\alpha-1}}{\Gamma(\alpha)-\sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-2}}\Big).
$$

Lemma 2.8. *There exist positive constant γ_1 such that for every $k>1$,*

$$
\min_{1/k\le t\le k}\frac{H_1(t,s)}{1+t^{\alpha-1}}\ge\gamma_1\sup_{0\le t<+\infty}\frac{H_1(t,s)}{1+t^{\alpha-1}},
$$

where $H_1(t,s)$ *is defined by* (2.5).

Proof. Using (2.5), we have

$$\frac{H_1(t,s)}{1+t^{\alpha-1}}=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{1+t^{\alpha-1}}, & 0\le s\le t<+\infty\\ \frac{t^{\alpha-1}}{1+t^{\alpha-1}}, & 0\le t\le s<+\infty\,.\end{cases}$$

Now let

$$h_1(t,s)=\frac{1}{\Gamma(\alpha)}\frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{1+t^{\alpha-1}},\quad s\le t$$
$$h_2(t,s)=\frac{1}{\Gamma(\alpha)}\frac{t^{\alpha-1}}{1+t^{\alpha-1}},\quad t\le s.$$

First of all we must note that, h_1 is decreasing and h_2 is increasing with respect to t, respectively, also h_1 is increasing with respect to s. So by a direct computation, we conclude that

$$\min_{1/k\le t\le k}h_1(t,s)\ge\frac{(k^{\alpha-1}-(k-s)^{\alpha-1})}{\Gamma(\alpha)(1+k^{\alpha-1})}\ge h_1(k)=\frac{k^{2(\alpha-1)}-(k^2-1)^{\alpha-1}}{\Gamma(\alpha)k^{\alpha-1}(1+k^{\alpha-1})},$$
$$\sup_{0\le t<+\infty}h_1(t,s)\le\frac{1}{\Gamma(\alpha)}$$
$$\min_{1/k\le t\le k}h_2(t,s)\ge h_2(1/k)=\frac{1}{\Gamma(\alpha)(1+k^{\alpha-1})},$$
$$\sup_{0\le t<+\infty}h_2(t,s)=\frac{1}{\Gamma(\alpha)}.$$

Now defining

$$m_1=\min\Big\{\frac{k^{2(\alpha-1)}-(k^2-1)^{\alpha-1}}{\Gamma(\alpha)k^{\alpha-1}(1+k^{\alpha-1})},\frac{1}{\Gamma(\alpha)(1+k^{\alpha-1})}\Big\},\quad M_1=\frac{1}{\Gamma(\alpha)},$$

and setting

$$\gamma_1=\frac{m_1}{M_1}=\min\Big\{\frac{k^{2(\alpha-1)}-(k^2-1)^{\alpha-1}}{k^{\alpha-1}(1+k^{\alpha-1})},\frac{1}{(1+k^{\alpha-1})}\Big\}\tag{2.7}$$

we conclude that

$$\min_{1/k\le t\le k}\frac{H_1(t,s)}{1+t^{\alpha-1}}\ge\gamma_1\sup_{0\le t<+\infty}\frac{H_1(t,s)}{1+t^{\alpha-1}}$$

This completes the proof. $\square$

Lemma 2.9. *For* $H_2(t,s)$, *defined by* (2.6) *there exist positive constant* γ_2 *such that*

$$\min_{1/k\le t\le k}\frac{H_2(t,s)}{1+t^{\alpha-1}}\ge\gamma_2\sup_{0\le t<+\infty}\frac{H_2(t,s)}{1+t^{\alpha-1}},k>1\,.$$

Proof. Considering $H_2(t,s)$ in (2.6) we have

$$\min_{1/k\le t\le k}\frac{H_2(t,s)}{1+t^{\alpha-1}}=\frac{1}{1+k^{\alpha-1}}\frac{\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)-\sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-2}}\frac{\partial H_1(t,s)}{\partial t}\Big|_{t=\xi_i}=m_2,$$
$$\sup_{0\le t<+\infty}\frac{H_2(t,s)}{1+t^{\alpha-1}}=\frac{\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)-\sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-2}}\frac{\partial H_1(t,s)}{\partial t}\Big|_{t=\xi_i}=M_2.$$

Now setting

$$\gamma_2 = \frac{m_2}{M_2} = \frac{1}{1+k^{\alpha-1}},$$

we conclude that

$$\min_{1/k\le t\le k} \frac{H_2(t,s)}{1+t^{\alpha-1}} \ge \gamma_2 \sup_{0\le t<+\infty} \frac{H_2(t,s)}{1+t^{\alpha-1}}.$$

The proof is complete. □

Lemma 2.10. *Let $k>1$ be fixed and $G(t,s)$ be defined by* (2.4)-(2.6). *Then*

$$\min_{1/k\le t\le k} \frac{G(t,s)}{1+t^{\alpha-1}} \ge \lambda(k) \sup_{0\le t<+\infty} \frac{G(t,s)}{1+t^{\alpha-1}},$$
$$\lambda(k) = \min\{\gamma_1,\gamma_2\} = \gamma_1.$$

Definition 2.11. We introduce the Banach space

$$B = \{u \in C[0,+\infty) : \|u\| < +\infty\}$$

which is equipped with the norm

$$\|u\| = \sup_{t\in[0,+\infty)} \frac{|\ u(t)\ |}{1+t^{\alpha-1}}.$$

Also we define the cone $P \subset B$ as follows

$$P = \{u \in B : u(t) \ge 0,\ \min_{t\in[\frac{1}{k},k]} \frac{u(t)}{1+t^{\alpha-1}} \ge \lambda(k)\|u\|\}.$$

Lemma 2.12. *Let conditions* (H1)–(H3) *be satisfied and define the Hammerstein integral operator $T : P \to B$ by*

$$Tu(t) = \lambda \int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds. \tag{2.8}$$

Then $TP \subset P$.

Proof. Let $u \in P$. Considering conditions (H1), (H2) and Lemma 2.6 it is clear that

$$Tu(t) = \lambda \int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds \ge 0.$$

Also we have

$$\begin{aligned}
\min_{1/k\le t\le k} \frac{Tu(t)}{1+t^{\alpha-1}} &= \min_{1/k\le t\le k} \frac{\lambda\int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds}{1+t^{\alpha-1}} \\
&\ge \lambda \int_0^{+\infty} \min_{1/k\le t\le k} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f(s,u(s))ds \\
&\ge \lambda \int_0^{+\infty} \lambda(k) \sup_{0\le t<+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f(s,u(s))ds \\
&\ge \lambda\lambda(k) \sup_{0\le t<+\infty} \frac{\int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds}{1+t^{\alpha-1}} \\
&= \lambda(k)\|Tu\|.
\end{aligned}$$

This shows that $TP \subset P$. □

Definition 2.13 ([4]). Let

$$V = \{u \in B : \|u\| < l,\, l > 0\}, \quad W = \{\frac{u(t)}{1+t^{\alpha-1}} : u \in V\}.$$

The set W is called equiconvergent at infinity if for each $\epsilon > 0$ there exists $\mu(\epsilon) > 0$, such that for all $u \in W$ and all $t_1, t_2 \geq \mu$, we have

$$|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}| < \epsilon.$$

Lemma 2.14 ([4]). *Assume*

$$V = \{u \in B : \|u\| < l,\ l > 0\}, W = \{\frac{u(t)}{1+t^{\alpha-1}} | u \in V\}.$$

If V is equicontinuous on any compact interval of $[0, +\infty)$ and equiconvergent at infinity, then V is relatively compact on B.

Lemma 2.15. *If conditions* (H1)–(H3) *hold, then integral operator $T : P \to P$ is completely continuous.*

Proof. First we prove that the operator T is uniformly bounded on P. Considering real Banach space B, we choose a positive constant r_0 such that for every $u \in P$, $\|u\| < r_0$. Let

$$B_{r_0} = \sup\{f(t, (1 + t^{\alpha-1})u) : (t, u) \in [0, +\infty) \times [0, r_0]\}$$

and Ω be a bounded subset of P. Thus there exist a positive constant r such that

$$\|u\| \leq r.$$

Using Definition 2.11, we have

$$\|Tu\| = \lambda \sup_{t\in[0,+\infty)} \frac{\int_0^{+\infty} G(t, s)a(s)f(s, u(s))ds}{1 + t^{\alpha-1}} \leq \lambda L B_r \int_0^{+\infty} a(s)ds < +\infty.$$

Thus $T\Omega$ is bounded. Now we show that operator T is continuous. We consider $\{u_n\}_{n=1}^{\infty} \subset P$, such that $u_n \to u$ as $n \to \infty$, so by the Lebesgue dominated convergence theorem we find that

$$\int_0^{+\infty} a(s)f(s, u_n(s))ds \to \int_0^{+\infty} a(s)f(s, u(s))ds$$

as $n \to \infty$. Hence by (2.8) we have

$$\|Tu_n - Tu\| \leq L\lambda \Big| \int_0^{+\infty} a(s)f(s, u_n(s))ds - \int_0^{+\infty} a(s)f(s, u(s))ds \Big| \to 0,$$

as $n \to \infty$. Hence T is a continuous operator. Now we show that operator $T : P \to P$ is an equiconvergent operator at infinity. For each $u \in \Omega$, we have

$$\int_0^{+\infty} a(s)f(s, u(s))ds \leq B_r \int_0^{+\infty} a(s)ds < +\infty.$$

Since

$$\lim_{t\to+\infty} \frac{1}{1 + t^{\alpha-1}} \int_0^{+\infty} H_1(t, s)a(s)f(s, u(s))ds = 0$$

and for $i = 1, 2, \dots, m-2$, $\xi_i < \infty$, also by condition (H2), we conclude that

$$\lim_{t\to+\infty} |\frac{Tu(t)}{1 + t^{\alpha-1}}|$$

$$
\begin{aligned}
&= \lim_{t\to+\infty} \frac{1}{1+t^{\alpha-1}} \lambda \int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds \\
&= \lim_{t\to+\infty} \frac{1}{1+t^{\alpha-1}} \lambda \int_0^{+\infty} H_1(t,s)a(s)f(s,u(s))ds \\
&\quad + \lim_{t\to+\infty} \frac{\lambda t^{\alpha-1}}{1+t^{\alpha-1}} \frac{\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)-\sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-2}} \\
&\quad \times \int_0^{+\infty} \frac{\partial H_1(t,s)}{\partial t}\big|_{t=\xi_i} a(s)f(s,u(s)ds) \\
&= \frac{\lambda\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)-\sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-2}} \int_0^{+\infty} \frac{\partial H_1(t,s)}{\partial t}\big|_{t=\xi_i} a(s)f(s,u(s))ds.
\end{aligned}
$$

Then

$$
\lim_{t\to+\infty} \big|\frac{Tu(t)}{1+t^{\alpha-1}}\big| < +\infty.
$$

Thus $T\Omega$ is equiconvergent at infinity.

Finally we prove that T is an equicontinuous operator. For every $s\in(0,+\infty)$, let $t_1,t_2\in[0,s]$, with $t_1<t_2$. Then we have

$$
\begin{aligned}
\big|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\big|
&\le \lambda B_r \int_0^{+\infty} \big|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\big| a(s)ds \\
&\le \lambda B_r \int_0^{+\infty} \big|\frac{H_1(t_2,s)}{1+t_2^{\alpha-1}} - \frac{H_1(t_1,s)}{1+t_1^{\alpha-1}}\big| a(s)ds \\
&\quad + \lambda B_r \int_0^{+\infty} \big|\frac{H_2(t_2,s)}{1+t_2^{\alpha-1}} - \frac{H_2(t_1,s)}{1+t_1^{\alpha-1}}\big| a(s)ds \\
&\le \lambda B_r \int_0^{+\infty} \big|\frac{H_1(t_2,s)}{1+t_1^{\alpha-1}} - \frac{H_1(t_1,s)}{1+t_1^{\alpha-1}}\big| a(s)ds \\
&\quad + \lambda B_r \int_0^{+\infty} \big|\frac{H_1(t_2,s)}{1+t_2^{\alpha-1}} - \frac{H_1(t_2,s)}{1+t_1^{\alpha-1}}\big| a(s)ds \\
&\quad + \lambda B_r \frac{\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)-\sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-2}} \\
&\quad \times \big|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\big| \int_0^{+\infty} \frac{\partial H_1(t,s)}{\partial t}\big|_{t=\xi_i} a(s)ds.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\int_0^{+\infty} \big|\frac{H_1(t_2,s)}{1+t_1^{\alpha-1}} - \frac{H_1(t_1,s)}{1+t_1^{\alpha-1}}\big| a(s)ds \\
&\le \int_0^{t_1} \big|\frac{H_1(t_2,s)}{1+t_1^{\alpha-1}} - \frac{H_1(t_1,s)}{1+t_1^{\alpha-1}}\big| a(s)ds \\
&\quad + \int_{t_1}^{t_2} \big|\frac{H_1(t_2,s)}{1+t_1^{\alpha-1}} - \frac{H_1(t_1,s)}{1+t_1^{\alpha-1}}\big| a(s)ds \\
&\quad + \int_{t_2}^{+\infty} \big|\frac{H_1(t_2,s)}{1+t_1^{\alpha-1}} - \frac{H_1(t_1,s)}{1+t_1^{\alpha-1}}\big| a(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{t_1} \frac{|(t_2^{\alpha-1} - t_1^{\alpha-1}) + (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}|}{1 + t_1^{\alpha-1}} a(s)ds \\
&\quad + \int_{t_1}^{t_2} \frac{|(t_2^{\alpha-1} - t_1^{\alpha-1}) - (t_2 - s)^{\alpha-1}|}{1 + t_1^{\alpha-1}} a(s)ds + \int_{t_2}^{+\infty} \frac{|(t_2^{\alpha-1} - t_1^{\alpha-1})|}{1 + t_1^{\alpha-1}} a(s)ds.
\end{aligned}
$$

Thus when $t_1 \to t_2$, we conclude that

$$
\int_0^{+\infty} \Big|\frac{H_1(t_2, s)}{1 + t_1^{\alpha-1}} - \frac{H_1(t_1, s)}{1 + t_1^{\alpha-1}}\Big| a(s)ds \to 0 \tag{2.9}
$$

Similar to (2.9), when $t_1 \to t_2$, we have

$$
\int_0^{+\infty} \Big|\frac{H_1(t_2, s)}{1 + t_2^{\alpha-1}} - \frac{H_1(t_2, s)}{1 + t_1^{\alpha-1}}\Big| a(s)ds \to 0 \tag{2.10}
$$

From (2.9) and(2.10) when $t_1 \to t_2$, we obtain that

$$
\Big|\frac{Tu(t_2)}{1 + t_2^{\alpha-1}} - \frac{Tu(t_1)}{1 + t_1^{\alpha-1}}\Big| \to 0.
$$

Thus $T\Omega$ is equicontinuous on $(0, +\infty)$. Using Lemma 2.14 we attain that operator $T : P \to P$ is completely continuous. This complete the proof. □

Theorem 2.16 ([8]). *Let X be a real Banach space and $P \subset X$ be a cone in X . Assume Ω_1, Ω_2 are two open bounded subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that*

- (i) *$\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$, $u \in P \cap \partial\Omega_2$, or*
- (ii) *$\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_2$ and $\|Tu\| \ge \|u\|$, $u \in P \cap \partial\Omega_1$.*

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Main Results

We introduce the following notation:

$$
\begin{gathered}
f_0 = \lim_{u\to 0^+} \min \frac{(1 + t^{\alpha-1})f(t, u)}{u}, \quad f_\infty = \lim_{u\to +\infty} \min \frac{(1 + t^{\alpha-1})f(t, u)}{u}, \quad t \in [\frac{1}{k}, k] \\
f^0 = \limsup_{u\to 0^+} \frac{(1 + t^{\alpha-1})f(t, u)}{u}, \quad f^\infty = \limsup_{u\to +\infty} \frac{(1 + t^{\alpha-1})f(t, u)}{u}, \quad t \in (0, \infty) \\
A = \Big(L \int_0^{+\infty} a(s)ds\Big)^{-1}, \quad B = \Big(\frac{\lambda^2(k)}{k^{\alpha-1}} \int_{1/k}^{k} a(s)ds\Big)^{-1}.
\end{gathered}
$$

The following theorem rely on Theorem 2.16 which has two possibilities that may occur.

Theorem 3.1. *Let conditions* (H1)–(H3) *hold. Then* (1.1)-(1.2) *has at least one positive solution on P in each one of the two cases:*

- (C1) *For every $\lambda \in (\frac{B}{f_0}, \frac{A}{f^\infty})$ such that $f_0, f^\infty \in (0, \infty)$ with $\lambda(k)f_0 > f^\infty$, or*
- (C2) *For every $\lambda \in (\frac{B}{f_\infty}, \frac{A}{f^0})$ such that $f_\infty, f^0 \in (0, \infty)$ with $\lambda(k)f_\infty > f^0$.*

Proof. Let

$$
\Omega_i = \{u \in B : \|u\| < R_i\}, \; i = 1, 2, \; R_1 < R_2.
$$

Then Ω_1, Ω_2 are two open bounded subset of B such that $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$.

Case1: Let $f_0, f^\infty \in (0,\infty)$ and $\lambda(k)f_0 > f^\infty$, also $\lambda \in (\frac{B}{f_0}, \frac{A}{f^\infty})$. Let $\epsilon > 0$ be chosen such that

$$\frac{B}{f_0 - \epsilon} < \lambda < \frac{A}{f^\infty + \epsilon} \tag{3.1}$$

Since $f_0 \in (0,\infty)$, thus there exist a positive constant R_1 such that for every $t \in [1/k, k]$ and $u \in [0, R_1]$,

$$f(t,u) = f(t, \frac{(1+t^{\alpha-1})u}{1+t^{\alpha-1}}) \geq (f_0 - \epsilon)\frac{u}{1+t^{\alpha-1}}.$$

So if $u \in P$ with $\|u\| = R_1$, then

$$f(t,u) \geq (f_0 - \epsilon)\frac{u}{1+t^{\alpha-1}} \geq \lambda(k)(f_0 - \epsilon)\|u\|, \quad t \in [1/k, k]$$

hence from (3.1) we have

$$\begin{aligned}
Tu(t) &= \lambda \int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds \\
&\geq \lambda\lambda(k)(f_0 - \epsilon)\|u\| \int_0^{+\infty} G(t,s)a(s)ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\|Tu\| &\geq \lambda\lambda(k)(f_0 - \epsilon)\|u\| \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}}a(s)ds \\
&\geq \lambda\lambda(k)(f_0 - \epsilon)\|u\| \int_{1/k}^{k} \frac{H_1(t,s)}{1+t^{\alpha-1}}a(s)ds \\
&\geq \lambda(f_0 - \epsilon)\|u\| \frac{\lambda^2(k)}{k^{\alpha-1}} \int_{1/k}^{k} a(s)ds \\
&= \lambda(f_0 - \epsilon)B^{-1}\|u\| > \|u\|.
\end{aligned}$$

Therefore,

$$\|Tu\| \geq \|u\| \quad \forall u \in P \cap \partial\Omega_1. \tag{3.2}$$

On the other hand, since $f^\infty \in (0,\infty)$, there exist a positive constant R such that for all $u \geq R$, we have

$$f(t,u) = f(t, \frac{(1+t^{\alpha-1})u}{1+t^{\alpha-1}}) \leq (f^\infty + \epsilon)\frac{u}{1+t^{\alpha-1}} \leq (f^\infty + \epsilon)\|u\|.$$

Let $R_2 = \max\{1 + R_1, R\lambda^{-1}(k)\}$ and $u \in P \cap \partial\Omega_2$. Using (3.1) we have

$$\begin{aligned}
Tu(t) &= \lambda \int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds \\
&\leq \lambda(f^\infty + \epsilon)\|u\| \int_0^{+\infty} G(t,s)a(s)ds.
\end{aligned}$$

So

$$\begin{aligned}
\|Tu\| &\leq \lambda(f^\infty + \epsilon)\|u\| \int_0^{+\infty} \sup_{t \in [0,+\infty)} \frac{G(t,s)}{1+t^{\alpha-1}}a(s)ds \\
&\leq \lambda(f^\infty + \epsilon)\|u\| L \int_0^{+\infty} a(s)ds \\
&\leq \lambda(f^\infty + \epsilon)A^{-1}\|u\| \leq \|u\|.
\end{aligned}$$

Thus we find that
$$\|Tu\| \le \|u\| \quad \forall u \in P \cap \partial\Omega_2. \tag{3.3}$$
Hence, using the Theorem 2.16 and (3.2), (3.3) we conclude that the boundary value problem (1.1)-(1.2) has at least one positive solution in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Case 2: Let $f_\infty, f^0 \in (0, \infty)$, $\lambda(k) f_\infty > f^0$ and $\lambda \in (\frac{B}{f_\infty}, \frac{A}{f^0})$. Similar to the case1, let $\epsilon > 0$ be chosen such that
$$\frac{B}{f_\infty - \epsilon} < \lambda < \frac{A}{f^0 + \epsilon}. \tag{3.4}$$
We can choose positive constants $R_2 > R_1$ such that
$$\|Tu\| \ge \|u\| \quad \forall u \in P \cap \partial\Omega_1, \tag{3.5}$$
$$\|Tu\| \le \|u\| \quad \forall u \in P \cap \partial\Omega_2. \tag{3.6}$$
Considering Theorem 2.16 and (3.5),(3.6) we conclude that the boundary value problem (1.1)-(1.2) has at least one positive solution in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. □

To prove multiplicity of positive solutions for (1.1)-(1.2), we need following condition.

(H4) Assume that function $f(t, u)$ is nondecreasing with respect to the second variable; i.e., for all $u_1, u_2 \in B$, if $u_1 \le u_2$ then $f(t, u_1) \le f(t, u_2)$.

Theorem 3.2. *Let conditions* (H1)–(H4) *hold. Assume that there exist positive constants $R_2 > R_1$, such that*
$$\frac{BR_1}{\min_{t\in[1/k,k]} f(t, \lambda(k)R_1)} \le \lambda \le \frac{AR_2}{\sup_{t\in[0,+\infty)} f(t, R_2)}. \tag{3.7}$$
Then (1.1)-(1.2) *has at least two positive solutions v_1, v_2 such that*
$$R_1 \le \|v_1\| \le R_2, \quad \lim_{n\to\infty} T^n u_0 = v_1, \quad u_0 = R_2, \quad t \in [0, +\infty),$$
$$R_1 \le \|v_2\| \le R_2, \quad \lim_{n\to\infty} T^n w_0 = v_2, \quad w_0 = R_1, \quad t \in [0, +\infty).$$

Proof. We define
$$P_{[R_1,R_2]} = \{u \in P : R_1 \le \|u\| \le R_2\}.$$
First we prove that $TP_{[R_1,R_2]} \subset P_{[R_1,R_2]}$. Let $u \in P_{[R_1,R_2]}$, thus obviously we have
$$\lambda(k)R_1 \le \lambda(k)\|u\| \le \frac{u(t)}{1 + t^{\alpha-1}} \le u(t) \le \|u\| \le R_2, \quad t \in [1/k, k]. \tag{3.8}$$
Using (H4), (3.7) and (3.8), we have
$$Tu(t) = \lambda \int_0^{+\infty} G(t, s)a(s)f(s, u(s))ds \le \lambda \int_0^{+\infty} G(t, s)a(s)f(s, R_2)ds.$$
Hence
$$\begin{aligned}\|Tu\| &\le \lambda \int_0^{+\infty} \sup_{t\in[0,+\infty)} \frac{G(t, s)}{1 + t^{\alpha-1}} a(s)f(s, R_2)ds \\ &\le \lambda \sup_{t\in[0,+\infty)} f(t, R_2)A^{-1} \le R_2.\end{aligned}$$
Also considering (3.7) and (3.8), we have
$$Tu(t) \ge \lambda \int_0^{+\infty} G(t, s)a(s)f(s, \lambda(k)R_1)ds.$$

Thus

$$\begin{aligned}\|Tu\| &\geq \lambda \frac{\lambda^2(k)}{k^{\alpha-1}} \int_{1/k}^{k} a(s)ds \min_{t\in[1/k,k]} f(t,\lambda(k)R_1) \\ &= \lambda \min_{t\in[1/k,k]} f(t,\lambda(k)R_1)B^{-1} \geq R_1.\end{aligned}$$

This implies $TP_{[R_1,R_2]} \subset P_{[R_1,R_2]}$. For every $t \in (0,+\infty)$ and $u_0 = R_2$, clearly $u_0 \in P_{[R_1,R_2]}$. Now we consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ in $P_{[R_1,R_2]}$ and define

$$u_n = Tu_{n-1} = T^n u_0, \quad i = 1,2,3,\dots. \tag{3.9}$$

Since, T is completely continuous, there exist a subsequence $\{u_{nk}\}$ of the sequence $\{u_n\}_{n\in\mathbb{N}}$ such that it converges uniformly to $v_1 \in B$. On the other hand considering the condition (H4), we can see that the operator $T : P_{[R_1,R_2]} \to P_{[R_1,R_2]}$, is nondecreasing. Since for every $t \in (0,+\infty)$

$$0 \leq u_1(t) \leq \|u_1\| \leq R_2 = u_0(t).$$

Thus $Tu_1 \leq Tu_0$. Considering (3.9) we conclude that $u_2 \leq u_1$. Similarly by induction we deduce that $u_{n+1} \leq u_n$. Hence $\{u_n\}_{n\in\mathbb{N}}$ is a decreasing sequence, such that has a subsequence $\{u_{nk}\}$ converges to v_1. Thus $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly to v_1. Letting $n \to +\infty$ in (3.9) yields

$$Tv_1 = v_1. \tag{3.10}$$

Let $w_0 = R_1$ for every $t \in (0,+\infty)$. So $w_0 \in P_{[R_1,R_2]}$. Now consider the sequence $\{w_n\}_{n\in\mathbb{N}}$ given by

$$w_n = Tw_{n-1}, \quad n = 1,2,3,\dots \tag{3.11}$$

From (3.11) we have $\{w_n\}_{n\in\mathbb{N}} \subset P_{[R_1,R_2]}$. Moreover, using definition (2.8), we conclude that

$$\begin{aligned}w_1(t) = Tw_0(t) &= \lambda \int_0^{+\infty} G(t,s)a(s)f(s,w_0(s))ds \\ &\geq \lambda \int_0^{+\infty} G(t,s)a(s)f(s,\lambda(k)R_1)ds \\ &\geq R_1 = w_0(t) \quad t \in (0,+\infty).\end{aligned}$$

Thus using the same argument as above, we deduce that $\{w_n\}_{n\in\mathbb{N}}$ is a increasing sequence with subsequence $\{w_{nk}\}$ such that $\{w_{nk}\}$ converges uniformly to $v_2 \in P_{[R_1,R_2]}$. Thus $\{w_n\}_{n\in\mathbb{N}}$ converges uniformly to $v_2 \in P_{[R_1,R_2]}$. Letting $n \to +\infty$, from (3.11) we find that

$$Tv_2 = v_2. \tag{3.12}$$

Finally from (3.10) and (3.12) we conclude that the boundary-value problem (1.1)-(1.2) has at least two positive solutions v_1, v_2 in P which completes the proof. $\square$

We conclude this article with two nonexistence results stated in the following theorems. Moreover, we show the compactness of the solutions set.

Theorem 3.3. *Let conditions* (H1)–(H3) *hold. If $f^0, f^\infty < \infty$, then there exist a positive constant λ_0, such that for every $0 < \lambda < \lambda_0$, the boundary value problem* (1.1)-(1.2) *has no positive solution.*

Proof. Since $f^0, f^\infty < \infty$, for every $t \in (0,+\infty)$, there exist positive constants c_1, c_2, r_1, r_2 with $r_1 < r_2$ such that

$$f(t,u) \le c_1 \frac{u}{1+t^{\alpha-1}}, \quad u \in [0, r_1]$$
$$f(t,u) \le c_2 \frac{u}{1+t^{\alpha-1}}, \quad u \in [r_2, +\infty).$$

Let

$$C = \max\big\{c_1, c_2, \sup_{r_1 \le u \le r_2} \frac{(1+t^{\alpha-1})f(t,u)}{u}\big\}.$$

Thus we have

$$f(t,u) \le C\frac{u}{1+t^{\alpha-1}}, \quad u \in [0,+\infty),\ t \in (0,+\infty).$$

Assume $w(t)$ is a positive solution of the boundary value problem (1.1)-(1.2). We will show that this leads to a contradiction for every $0 < \lambda < \lambda_0$ with $\lambda_0 = \frac{A}{C}$.

$$w(t) = Tw(t) = \lambda \int_0^{+\infty} G(t,s)a(s)f(s,w(s))ds \le \lambda C\|w\| \int_0^{+\infty} G(t,s)a(s)ds.$$

Hence

$$\|w\| \le \lambda C\|w\| \int_0^{+\infty} \sup_{t\in[0,+\infty)} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)ds = \frac{\lambda C}{A}\|w\| < \|w\|,$$

which is a contradiction. Therefore, (1.1)-(1.2) has no positive solution. □

Theorem 3.4. *Assume that conditions*(H1)–(H3) *hold. If $f_0, f_\infty > 0$, then there exist a positive constant λ_0, such that for every $\lambda > \lambda_0$, the boundary value problem* (1.1)-(1.2) *has no positive solution.*

Proof. Since $f_0, f_\infty > 0$, we conclude that for all $t \in [1/k, k]$, there exist positive constants m_1, m_2, r_1, r_2 with $r_1 < r_2$ such that

$$f(t,u) \ge m_1 \frac{u}{1+t^{\alpha-1}}, \quad u \in [0, r_1]$$
$$f(t,u) \ge m_2 \frac{u}{1+t^{\alpha-1}}, \quad u \in [r_2, +\infty).$$

Assume that

$$m = \min\big\{m_1, m_2, \min_{r_1 \le u \le r_2} \frac{(1+t^{\alpha-1})f(t,u)}{u}\big\}.$$

Hence we have

$$f(t,u) \ge m\frac{u}{1+t^{\alpha-1}} \ge m\lambda(k)\|u\|, \quad u \in [0,+\infty),\ t \in [1/k, k].$$

Let $w(t)$ be a positive solution of (1.1)-(1.2). We will show that this leads to a contradiction for every $\lambda > \lambda_0$, with $\lambda_0 = B/m$.

$$w(t) = Tw(t) = \lambda \int_0^{+\infty} G(t,s)a(s)f(s,w(s))ds \ge m\lambda\lambda(k)\|w\| \int_0^{+\infty} G(t,s)a(s)ds.$$

So

$$\|w\| \ge m\lambda \frac{\lambda^2(k)}{k^{\alpha-1}}\|w\| \int_{1/k}^{k} a(s)ds = \frac{\lambda m}{B}\|w\| > \|w\|,$$

which is a contradiction. Therefore (1.1)-(1.2) has no positive solution. This completes the proof. □

Theorem 3.5. *Assume conditions* (H1)–(H3) *hold and that*

$$f_0, f^\infty \in (0,+\infty), \quad f_0\lambda(k) > f^\infty, \quad \lambda \in (\frac{B}{f_0}, \frac{A}{f^\infty}). \tag{3.13}$$

Then the set of positive solutions of (1.1)-(1.2) *is nonempty and compact.*

Proof. Let $S = \{u \in P : u = Tu\}$. Theorem 3.1 implies that S is nonempty. It is sufficient to show that S is compact in B. First of all we claim that S is closed in B. Let $\{u_n\}_{n\in\mathbb{N}}$ be sequence in S, such that $\lim_{n\to\infty} \|u_n - u\| = 0$. Thus for every $t \in (0,+\infty)$, we have

$$\begin{aligned}
&\big|u(t) - \lambda\int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds\big| \\
&\leq |u_n - u| + \big|u_n(t) - \lambda\int_0^{+\infty} G(t,s)a(s)f(s,u_n(s))ds\big| \\
&\quad + \lambda\int_0^{+\infty} G(t,s)a(s)|f(s,u(s)) - f(s,u_n(s))|ds.
\end{aligned}$$

Let $n \to \infty$, using the continuity of f and by dominated convergence theorem, we deduce that for all $t \in (0,+\infty)$

$$u(t) = \lambda\int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds.$$

Thus $u \in S$ and S is closed in B.

It remains to check that S is relatively compact in B. Let (3.13) hold. Choosing $\epsilon > 0$ such that

$$\lambda \in (\frac{B}{f_0 - \epsilon}, \frac{A}{f^\infty + \epsilon}),$$

we find that there exists a positive constant R such that for every $u \in [R,+\infty)$,

$$f(t,u) \leq (f^\infty + \epsilon)\frac{u}{1 + t^{\alpha-1}} \leq (f^\infty + \epsilon)\|u\|.$$

Hence for $t \in (0,+\infty)$, we have

$$\begin{gathered}
f(t,u) \leq (f^\infty + \epsilon)\|u\| + \gamma, \\
\gamma = \max\{f(t,u) : t \in [1/k, k],\, u \in [0,R]\}.
\end{gathered}$$

Thus for every $u \in S$ and $t \in (0,+\infty)$, we have

$$\begin{aligned}
u(t) &= \lambda\int_0^{+\infty} G(t,s)a(s)f(s,u(s))ds \\
&\leq \lambda\left[(f^\infty + \epsilon)\|u\| + \gamma\right]\int_0^{+\infty} G(t,s)a(s)ds.
\end{aligned}$$

Then

$$\|u\| \leq \lambda(\frac{(f^\infty + \epsilon)\|u\| + \gamma}{A}).$$

Therefore, S is bounded in B. Now by compactness of the operator $T : P \to P$ we deduce that $S = TS$ is relatively compact, which completes the proof. $\square$

References

[1] R. I. Avery, A. C. Peterson; Three positive fixed points of nonlinear operators on ordered banach spaces, *Comput. math. Appl.* **42** (2001), 313-322.

[2] R. P. Agrawal, D. O'Regon; Positive solutions for Dirichlet problem of singular nonlinear fractional differential equations, *J. Math Anal Appl.* **371** (2010), 57-68.

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; Theory and Applications of fractional Differential Equations, *North-Holland mathematics studies, Elsvier science.* **204** (2006).

[4] S. Liang, J. Zhang; Existence of three positive solutions for m-point boundary value problem of some nonlinear fractional differential equations on an infinite interval, *Computers and Mathematics with Applications,* **61** (2011), 3343-3354.

[5] K. S. Miller, B. Ross; An Introduction to fractional calculus and fractioal differential equation, *John Wiley*, New York, (1993).

[6] K. B. Oldham, J. Spanier; The fractional calculus, *Academic Press, New York*, (1974).

[7] I. Poudlobny; Fractional Differential Equations, *Mathematics in Science and Applications, Academic Press.* **19** (1999).

[8] L. X. Troung, P. D. Phung; Existence of positive solutions for a multi point fourth-order boundary value problem, *Electronic Journal of Differential equations*, **2011** (2011) no. 129, 1-10.

[9] S. Zhang, G. Han; The existence of a positive solution for a nonlinear fractional differential equation, *J. Math Anal. Appl.* **252** (2000), 804-812.

[10] S. Zhang; Existence of positive solutions for some class of fractional differential equations, *J. Math Anal Aappl.* **278** (2003), 136-148.

[11] X. Zhao, W. Ge; Unbounded solutions for a fractional boundary value problem on the infinite interval, *Acta Appl Math,* **109** (2010), 495-505.

[12] Y. Zhao, S. Sun, Z. Han, Q. Li; The existence of multiple positive solutions for boundary value problem of nonlinear fractional differential equations, *Commun Nonlinear Sci Numer Simulat.* **16** (2011), 2086-2097.

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