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MULTIPLICITY OF POSITIVE SOLUTIONS FOR A GRADIENT SYSTEM WITH AN EXPONENTIAL NONLINEARITY

NASREDDINE MEGREZ, K. SREENADH, BRAHIM KHALDI

Abstract. In this article, we consider the problem

$$\begin{gathered}
-\Delta u = \lambda u^q + f_1(u,v) \quad \text{in } \Omega \\
-\Delta v = \lambda v^q + f_2(u,v) \quad \text{in } \Omega \\
u, v > 0 \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $0 < q < 1$, and $\lambda > 0$. We show that there exists a real number Λ such that the above problem admits at least two solutions for $\lambda \in (0, \Lambda)$, and no solution for $\lambda > \Lambda$.

1. Introduction

In this article, we study the existence of multiple solutions of the system of partial differential equations

$$\begin{gathered}
-\Delta u = \lambda u^q + f_1(u,v) \quad \text{in } \Omega \\
-\Delta v = \lambda v^q + f_2(u,v) \quad \text{in } \Omega \\
u, v > 0 \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $0 < q < 1$, $\lambda > 0$, and f_i, $i = 1, 2$ satisfy the following conditions:

- (H1) $f_i \in C^1(\mathbb{R}^2)$, $f_i(t,s) > 0$ for $t > 0$ and $s > 0$; $f_i(t,s) = 0$ if $t \leq 0$ or $s \leq 0$.
- (H2) The maps $t \mapsto f_i(t,.)$ and $s \mapsto f_i(.,s)$ are nondecreasing for all $t > 0$, $s > 0$.
- (H3) For all $\epsilon > 0$,

$$\lim_{t^2+s^2\to\infty} f_i(t,s)e^{-(1-\epsilon)(t^2+s^2)} = \infty, \quad \lim_{t^2+s^2\to\infty} f_i(t,s)e^{-(1+\epsilon)(t^2+s^2)} = 0.$$

- (H4) There exists $\overline{\lambda}$ such that

$$\lambda t^q + f_1(t,s) > \lambda_1 t \quad \text{and} \quad \lambda s^q + f_2(t,s) > \lambda_1 s$$

for all $\lambda > \overline{\lambda}$ and $s, t > 0$, where λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$.

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(H5) Let $F(t,s)$ be a C^2 function such that

$$\frac{\partial F}{\partial t} = f_1(t,s), \quad \frac{\partial F}{\partial s} = f_2(t,s),$$

$$\lim_{(t,s)\to(0,0)} \frac{F(t,s)}{t^k + s^k} = 0 \quad \text{for some } k > 1,$$

$$\lim_{t^2+s^2\to\infty} \frac{F(t,s)}{e^{(1+\epsilon)(t^2+s^2)}} = 0, \quad \forall \epsilon > 0.$$

(H6) There exists a constant $\kappa \geq 0$ such that

$$\lim_{|t|,|s|\to+\infty} \frac{F(t,s)}{f_1(t,s) + f_2(t,s)} = \kappa$$

(H7) For every $\epsilon > 0$,

$$\lim_{t^2+s^2\to\infty} \frac{\partial f_i(t,s)}{\partial t} e^{-(1-\epsilon)(t^2+s^2)} = \infty, \quad \lim_{t^2+s^2\to\infty} \frac{\partial f_i(t,s)}{\partial s} e^{-(1+\epsilon)(t^2+s^2)} = 0.$$

As examples of a function satisfying the above assumptions, we have

$$F(t,s) = \begin{cases} (t^2+s^2)e^{t^2+s^2} & \text{if } t > 0, \ s > 0 \\ 0 & \text{otherwise.} \end{cases}$$

So

$$f_1(t,s) = \begin{cases} 2t(t^2+s^2+1)e^{t^2+s^2} & \text{if } t > 0, \ s > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$f_2(t,s) = \begin{cases} 2s(t^2+s^2+1)e^{t^2+s^2} & \text{if } t > 0, \ s > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Starting from the work of Adimurthi [1], there are many results in the scalar case for problems involving exponential growth, for example [8], [12]. Systems involving exponential nonlinearities in two dimension have also been studied in [9]. Recently, lot of interest has been shown for studying the multiplicity of positive solutions with nonlinearities of sublinear growth at origin. In this direction we mention the works of Ambrosetti-Brezis-Cerami [3] for higher dimensions, and Prashanth-Sreenadh [21] in the case of $\mathbb{R}^2$.

Our aim in this article is to generalize the result in [21] to the case of systems. One of the motivations of this work is that parameter dependent systems with exponential nonlinearities have been recently shown to be very involved in relativistic Abelian Chern-Simons model with two Higgs particles and two gauge fields, see [7, 16, 17, 18]. Chern-Simons theories are applied in condensed matter physics, anyon physics, superconductivity, quantum mechanics, and electro magnetic spin density, to mention a few. This system may also be applied to heat transfer modeling in a nuclear fuel rod where the nonlinearities f_1 and f_2 represent the energy production.

We shall find the weak solutions of the system (P_λ) in the space

$$\mathcal{H} := H_0^1(\Omega) \times H_0^1(\Omega)$$

endowed with the norm

$$\|(u,v)\|_{\mathcal{H}} := \Big[\int_\Omega \Big(|\nabla u|^2 + |\nabla v|^2 \Big) dx \Big]^{1/2}.$$

Throughout this paper, we denote by $\|\cdot\|_{1,2}$ the norm of Sobolev space $H_0^1(\Omega)$. To handle exponential nonlinearity for dimension $N = 2$, the Moser-Trudinger inequality [19, 22] plays the same role as the Sobolev imbedding Theorem for the case of polynomial nonlinearity in dimension $N \geq 3$. In this paper, we will use the following adapted version of Moser-Trudinger inequality for the pair (u, v) [15]:

Lemma 1.1. *Let $(u, v) \in \mathcal{H}$, then $\int_\Omega e^{\gamma(u^2+v^2)}dx < +\infty$ for any $\gamma > 0$. Moreover, there exists a constant $C = C(\Omega)$ such that*

$$\sup_{\|(u,v)\|_{\mathcal{H}}=1} \int_\Omega e^{\gamma(u^2+v^2)}dx \leq C, \quad \textit{provided } \gamma \leq 4\pi. \tag{1.2}$$

Proof. Let $r_1 = \|u\|_{1,2}^2$ and $r_2 = \|v\|_{1,2}^2$, be such that $r_1 + r_2 = 1$. Then, we have

$$\int_\Omega e^{\gamma \frac{u^2}{r_1}}\,dx < C, \quad \int_\Omega e^{\gamma \frac{v^2}{r_2}}\,dx < C, \quad \text{for all } \gamma \leq 4\pi.$$

Hence, using Hölder inequality, (1.2) can be obtained as follows

$$\int_\Omega e^{\gamma(u^2+v^2)}\,dx \leq (\int_\Omega e^{\gamma \frac{u^2}{r_1}}\,dx)^{r_1}(\int_\Omega e^{\gamma \frac{v^2}{r_2}}\,dx)^{r_2} \leq C^{r_1+r_2} = C.$$

□

It follows from the above inequality that the imbedding

$$(u, v) \in \mathcal{H} \mapsto e^{(|u|^\alpha+|v|^\alpha)} \in L^1(\Omega)$$

is compact for $\alpha < 2$. Also, it can be shown using a class of functions called the Moser functions, that the above imbedding is not compact for $\alpha = 2$.

Weak solutions of (1.1) are the functions $u, v \in H_0^1(\Omega)$ such that

$$\begin{gathered}
\int_\Omega \nabla u.\nabla\phi\,dx = \lambda\int_\Omega u^q\phi\,dx + \int_\Omega f_1(u,v)\phi\,dx,\\
\int_\Omega \nabla v.\nabla\psi\,dx = \lambda\int_\Omega v^q\psi\,dx + \int_\Omega f_2(u,v)\psi\,dx,
\end{gathered}$$

for all $\phi, \psi \in H_0^1(\Omega)$.

The main results of this article are given in the following theorem.

Theorem 1.2. *There exists $\Lambda > 0$ such that* (1.1) *admits at least two solutions for all $\lambda \in (0, \Lambda)$, and no solution for $\lambda > \Lambda$.*

Let us write $H(u, v)$ as

$$H(u, v) = \frac{\lambda}{q+1}\Big(|u|^{q+1} + |v|^{q+1}\Big) + F(u, v),$$

and let

$$h_1(u, v) = \frac{\partial H}{\partial u} = \lambda u^q + f_1(u, v), \quad h_2(u, v) = \frac{\partial H}{\partial v} = \lambda v^q + f_2(u, v).$$

Using (H6), for R sufficiently large

$$H(u, v) \leq C(h_1(u, v) + h_2(u, v)), \quad \text{for } |u| > R, \text{ and } |v| > R. \tag{1.3}$$

The functional associated with (1.1) is given by

$$E(u, v) = \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 + |\nabla v|^2\Big)dx - \int_\Omega H(u, v)\,dx.$$

It is well defined on $\mathcal{H}$ and $C^1(\mathcal{H},\mathbb{R})$. Also, for all $(\phi,\psi)\in\mathcal{H}$, we have

$$E'(u,v)(\phi,\psi)=\int_\Omega \nabla u.\nabla\phi\,dx+\int_\Omega \nabla v.\nabla\psi\,dx-\int_\Omega h_1(u,v)\phi\,dx-\int_\Omega h_2(u,v)\psi\,dx,$$

Our approach is to construct an H^1 local minimum (u_λ,v_λ) for E for λ in the largest interval of existence $(0,\Lambda)$, and then use the generalized mountain-pass Theorem of Ghoussoub-Priess [10] about (u_λ,v_λ) to obtain a second solution.

2. Existence of a minimal solution and a local minimum for E

In this section, we prove the existence of a minimal solution (u_λ,v_λ) of (1.1), and then we show that this minimal solution is a local minimum for E. A solution (u_λ,v_λ) is said to be minimal if any other solution (u,v) of (1.1) satisfies $u\ge u_\lambda$ and $v\ge v_\lambda$ in Ω.

Lemma 2.1. *There exists $\lambda_0>0$ such that* (1.1) *admits a solution for all $\lambda\in(0,\lambda_0)$.*

Proof. From assumption (H5), for any $\epsilon>0$, we obtain $C>0$, such that

$$|F(u,v)|\le C\Big(|u|^{k+1}+|v|^{k+1}\Big)e^{(1+\epsilon)(u^2+v^2)}$$

For $\|(u,v)\|_{\mathcal{H}}=\rho$ such that $\rho^2\le\frac{2\pi}{1+\epsilon}$, and by Hölder's inequality and the Moser-Trudinger inequality (1.2), we obtain

$$\begin{aligned}
&|\int_\Omega F(u,v)dx|\\
&\le C\Big(\int_\Omega \big(|u|^{(k+1)}+|v|^{(k+1)}\big)e^{(1+\epsilon)(u^2+v^2)}dx\Big)\\
&\le C\Big(\int_\Omega \big(|u|^{(k+1)}+|v|^{(k+1)}\big)\\
&\quad\times\exp\Big((1+\epsilon)(\frac{u^2}{\|u\|_{1,2}^2+\|v\|_{1,2}^2}+\frac{v^2}{\|u\|_{1,2}^2+\|v\|_{1,2}^2})(\|u\|_{1,2}^2+\|v\|_{1,2}^2)\Big)dx\Big)\\
&\le C\Big(\Big(\int_\Omega |u|^{2(k+1)}dx\Big)^{1/2}+\Big(\int_\Omega |v|^{2(k+1)}dx\Big)^{1/2}\Big)\\
&\le C\Big(\|u\|_{1,2}^{k+1}+\|v\|_{1,2}^{k+1}\Big),
\end{aligned}$$

where C is a generic constant. Therefore, for $\|(u,v)\|_{\mathcal{H}}=\rho$, we have

$$\begin{aligned}
E(u,v)&\ge\frac12\|(u,v)\|_{\mathcal{H}}^2-C\Big(\|u\|_{1,2}^{k+1}+\|v\|_{1,2}^{k+1}\Big)-\frac{\lambda}{q+1}\Big(\|u\|_{L^{q+1}}^{q+1}+\|v\|_{L^{q+1}}^{q+1}\Big)\\
&\ge\frac12\|(u,v)\|_{\mathcal{H}}^2-C\Big(\|u\|_{1,2}^{k+1}+\|v\|_{1,2}^{k+1}\Big)-\frac{\lambda}{q+1}\Big(C_1\|u\|_{1,2}^{q+1}+C_2\|v\|_{1,2}^{q+1}\Big)\\
&\ge\frac12\|(u,v)\|_{\mathcal{H}}^2-C\Big(\|(u,v)\|_{\mathcal{H}}^{k+1}+\|(u,v)\|_{\mathcal{H}}^{k+1}\Big)\\
&\quad-\frac{\lambda}{q+1}\Big(C_1\|(u,v)\|_{\mathcal{H}}^{q+1}+C_2\|(u,v)\|_{\mathcal{H}}^{q+1}\Big)\\
&\ge\frac12\rho^2-2C\rho^{k+1}-\tilde{C}\lambda\rho^{q+1}.
\end{aligned}$$

Now, we may fix $\rho,\lambda_0>0$ small enough such that $E(u,v)>0$ for all $\lambda\in(0,\lambda_0)$. We note that $E(tu,tv)<0$ for $t>0$ small enough. So, $\inf_{\|(u,v)\|_{\mathcal{H}}\le\rho}E(u,v)<0$

and if this infimum is achieved at some (u_λ, v_λ), then (u_λ, v_λ) becomes a solution of (1.1). Let $\{(u_n, v_n)\} \subset \{\|(u_n, v_n)\|_{\mathcal{H}} \le \rho\}$ be a minimizing sequence and let $(u_n, v_n) \rightharpoonup (u_\lambda, v_\lambda)$ in $\mathcal{H}$. Clearly,

$$\|(u_\lambda, v_\lambda)\|_{\mathcal{H}} \le \liminf_{n\to\infty} \|(u_n, v_n)\|_{\mathcal{H}}.$$

Now, we can choose $\rho < \pi$ so that $\{F(u_n, v_n)\}$ is bounded in $L^r(\Omega)$ for some $r > 1$. Using this fact and Holders inequality, it is not difficult to show that $\{F(u_n, v_n)\}$ is equi-integrable family in $L^1(\Omega)$ and $\lim_{|A|\to 0} \int_A |F(u_n, v_n)|dx = 0$. Therefore, by Vitali's convergence Theorem, we obtain

$$\int_\Omega F(u_n, v_n)dx \to \int_\Omega F(u_\lambda, v_\lambda)dx.$$

Hence, (u_λ, v_λ) is a minimizer of $E(u, v)$. By the maximum principle, we obtain $u_\lambda, v_\lambda > 0$ in Ω. □

Lemma 2.2. *Let* $\Lambda := \sup\{\lambda :$ (1.1) *admits a solution*$\}$. *Then* $0 < \Lambda < \infty$.

Proof. By Lemma 2.1, it is clear that $\Lambda > 0$. Suppose $\Lambda = \infty$. Then there exists a sequence $\lambda_n \to \infty$ such that (1.1) with $\lambda = \lambda_n$ has a solution $(u_{\lambda_n}, v_{\lambda_n})$. Hence

$$\lambda_1 \int_\Omega u_{\lambda_n} \phi_1\, dx = \int_\Omega \nabla u_{\lambda_n} \nabla \phi_1\, dx = \int_\Omega \Big(\lambda_n u_{\lambda_n}^q + f_1(u_{\lambda_n}, v_{\lambda_n})\Big)\phi_1\, dx. \tag{2.1}$$

where ϕ_1 is the eigenfunction associated with the first eigenvalue λ_1 of $-\Delta$ on $H_0^1(\Omega)$.

Now, we choose $\lambda_n > \overline{\lambda}$. By (H4) we have

$$\lambda_n t^q + f_1(t, s) > \lambda_1 t, \quad \lambda_n s^q + f_2(t, s) > \lambda_1 s. \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$\lambda_1 \int_\Omega u_{\lambda_n} \phi_1 dx > \lambda_1 \int_\Omega u_{\lambda_n} \phi_1 dx \tag{2.3}$$

which is absurd. Hence, Λ is finite. □

Lemma 2.3. *For all* $\lambda \in (0, \Lambda)$, (1.1) *admits a solution.*

Proof. Suppose $\lambda' < \lambda < \lambda'' < \Lambda$ and (1.1) with $\lambda = \lambda'$, and with $\lambda = \lambda''$ admit solutions $(u_{\lambda'}, v_{\lambda'})$, $(u_{\lambda''}, v_{\lambda''})$ respectively. Then $(u_{\lambda'}, v_{\lambda'})$ is a subsolution of (1.1), and $(u_{\lambda''}, v_{\lambda''})$ is a supersolution of (1.1), and hence, by the monotone iterative procedure, there exists a solution of (1.1). □

We recall the following well known comparison Theorem, whose proof can be found in [20].

Lemma 2.4. *Let* $f : [0, \infty) \to [0, \infty)$ *be such that* $f(t)/t$ *is non-increasing for* $t > 0$. *Let* $v, w \in W_0^{1,2}(\Omega)$ *be weak sub and super solutions (respectively) of*

$$\begin{gathered} -\Delta u = f(u), \quad u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Then $w \ge v$ *a.e. in* Ω.

Lemma 2.5. *For all* $\lambda \in (0, \Lambda)$, (1.1) *admits a minimal solution* (u_λ, v_λ).

Proof. Let v_0 be the unique solution of the problem

$$\begin{gathered} -\Delta u = \lambda u^q, \quad u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.4}$$

Then, (v_0, v_0) is a subsolution of (1.1). Now, let (u, v) be a solution of (1.1). Then, u and v are supersolutions of (2.4), and by the above weak comparison Theorem, $u \geq v_0$, and $v \geq v_0$. By the monotone iteration procedure with $\underline{U} = (v_0, v_0)$, and $\overline{U} = (u, v)$ as sub and super-solutions of (1.1), we obtain a solution (u_λ, v_λ). It is easy to see that (u_λ, v_λ) is the minimal solution. □

Lemma 2.6. *(u_λ, v_λ) is a local minimum of E in $\mathcal{H}$.*

Proof. We use Perron's method as in [11] and [13]. Arguing by contradiction, let us suppose that there exists $\lambda \in (0, \Lambda)$ and a sequence (u_n, v_n) such that $(u_n, v_n) \to (u_\lambda, v_\lambda)$ strongly in $\mathcal{H}$ and $E(u_n, v_n) < E(u_\lambda, v_\lambda)$.

Let $\lambda < \lambda_0 < \Lambda$ and let $(u_{\lambda_0}, v_{\lambda_0})$ be the minimal solution of (1.1) with $\lambda = \lambda_0$. Let $\overline{U} = (\overline{u}, \overline{v}) = (u_{\lambda_0}, v_{\lambda_0})$, and let $\underline{U} = (\underline{u}, \underline{v}) = (v_0, v_0)$, where v_0 is the unique solution of (2.4). Consider the following cut-off functions:

$$y_{1,n}(x) = \begin{cases} \underline{u}(x) & \text{if } u_n(x) \leq \underline{u}(x) \\ u_n(x) & \text{if } \underline{u}(x) \leq u_n(x) \leq \overline{u}(x) \\ \overline{u}(x) & \text{if } u_n(x) \geq \overline{u}(x) \end{cases}$$

$$y_{2,n}(x) = \begin{cases} \underline{v}(x) & \text{if } v_n(x) \leq \underline{v}(x) \\ v_n(x) & \text{if } \underline{v}(x) \leq v_n(x) \leq \overline{v}(x) \\ \overline{v}(x) & \text{if } v_n(x) \geq \overline{v}(x). \end{cases}$$

Also define

$$\begin{gathered} W_n = (w_{1,n}, w_{2,n}) := ((u_n - \overline{u})^+, (v_n - \overline{v})^+), \\ Z_n = (z_{1,n}, z_{2,n}) := ((u_n - \underline{u})^-, (v_n - \underline{v})^-). \end{gathered}$$

We also define the following subsets:

$$\begin{gathered} S_{1,n} = \{x \in \Omega : u_n(x) < \underline{u}(x) \text{ and } v_n(x) < \underline{v}(x)\}, \\ S_{2,n} = \{x \in \Omega : u_n(x) < \underline{u}(x) \text{ and } \underline{v}(x) \leq v_n(x) \leq \overline{v}(x)\}, \\ S_{3,n} = \{x \in \Omega : u_n(x) < \underline{u}(x) \text{ and } v_n(x) > \overline{v}(x)\}, \\ S_{4,n} = \{x \in \Omega : \underline{u}(x) \leq u_n(x) \leq \overline{u}(x) \text{ and } v_n(x) < \underline{v}(x)\}, \\ S_{5,n} = \{x \in \Omega : \underline{u}(x) \leq u_n(x) \leq \overline{u}(x) \text{ and } v_n(x) > \overline{v}(x)\}, \\ S_{6,n} = \{x \in \Omega : u_n(x) > \underline{u}(x) \text{ and } v_n(x) < \underline{v}(x)\}, \\ S_{7,n} = \{x \in \Omega : u_n(x) > \underline{u}(x) \text{ and } \underline{v}(x) \leq v_n(x) \leq \overline{v}(x)\}, \\ S_{8,n} = \{x \in \Omega : u_n(x) > \underline{u}(x) \text{ and } v_n(x) > \overline{v}(x)\}. \end{gathered}$$

Then,

$$\begin{gathered} (u_n, v_n) = (y_{1,n}, y_{2,n}) - (z_{1,n}, z_{2,n}) + (w_{1,n}, w_{2,n}), \\ (y_{1,n}, y_{2,n}) \in M := \big\{(u, v) \in \mathcal{H} : \underline{u} \leq u \leq \overline{u} \text{ and } \underline{v} \leq v \leq \overline{v}\big\}, \\ E(u_n, v_n) = E(y_{1,n}, y_{2,n}) + \sum_{i=1}^{8} A_{i,n}, \end{gathered} \tag{2.5}$$

where

$$\begin{aligned}
A_{1,n} &= \frac{1}{2}\int_{S_{1,n}}\Big(|\nabla u_n|^2 - |\nabla \underline{u}|^2\Big)dx + \frac{1}{2}\int_{S_{1,n}}\Big(|\nabla v_n|^2 - |\nabla \underline{v}|^2\Big)dx \\
&\quad - \int_{S_{1,n}}\Big(H(u_n,v_n) - H(\underline{u},\underline{v})\Big)dx,\\
A_{2,n} &= \frac{1}{2}\int_{S_{2,n}}\Big(|\nabla u_n|^2 - |\nabla \underline{u}|^2\Big)dx - \int_{S_{2,n}}\Big(H(u_n,v_n) - H(\underline{u},v_n)\Big)dx,\\
A_{3,n} &= \frac{1}{2}\int_{S_{3,n}}\Big(|\nabla u_n|^2 - |\nabla \underline{u}|^2\Big)dx + \frac{1}{2}\int_{S_{3,n}}\Big(|\nabla v_n|^2 - |\nabla \overline{v}|^2\Big)dx \\
&\quad - \int_{S_{3,n}}\Big(H(u_n,v_n) - H(\underline{u},\overline{v})\Big)dx,\\
A_{4,n} &= \frac{1}{2}\int_{S_{4,n}}\Big(|\nabla v_n|^2 - |\nabla \underline{v}|^2\Big)dx - \int_{S_{4,n}}\Big(H(u_n,v_n) - H(u_n,\underline{v})\Big)dx,\\
A_{5,n} &= \frac{1}{2}\int_{S_{5,n}}\Big(|\nabla v_n|^2 - |\nabla \overline{v}|^2\Big)dx - \int_{S_{5,n}}\Big(H(u_n,v_n) - H(u_n,\overline{v})\Big)dx,\\
A_{6,n} &= \frac{1}{2}\int_{S_{6,n}}\Big(|\nabla u_n|^2 - |\nabla \overline{u}|^2\Big)dx + \frac{1}{2}\int_{S_{6,n}}\Big(|\nabla v_n|^2 - |\nabla \underline{v}|^2\Big)dx \\
&\quad - \int_{S_{6,n}}\Big(H(u_n,v_n) - H(\overline{u},\underline{v})\Big)dx,\\
A_{7,n} &= \frac{1}{2}\int_{S_{7,n}}\Big(|\nabla u_n|^2 - |\nabla \overline{u}|^2\Big)dx - \int_{S_{7,n}}\Big(H(u_n,v_n) - H(\overline{u},v_n)\Big)dx,\\
A_{8,n} &= \frac{1}{2}\int_{S_{8,n}}\Big(|\nabla u_n|^2 - |\nabla \overline{u}|^2\Big)dx + \frac{1}{2}\int_{S_{8,n}}\Big(|\nabla v_n|^2 - |\nabla \overline{v}|^2\Big)dx \\
&\quad - \int_{S_{8,n}}\Big(H(u_n,v_n) - H(\overline{u},\overline{v})\Big)dx.
\end{aligned}$$

Following Perron's method as in the proof of [2, Proposition 2.2], one can states that $E(u_\lambda,v_\lambda) = \inf_M E(u,v)$, and then concludes that

$$E(u_n,v_n) \ge E(u_\lambda,v_\lambda) + \sum_{i=1}^{8} A_{i,n}.$$

Now, since $(u_n,v_n) \to (u_\lambda,v_\lambda)$ strongly in $\mathcal{H}$, $\underline{u} < u_\lambda < \overline{u}$ and $\underline{v} < v_\lambda < \overline{v}$ in $\overline{\Omega}$, we have $\operatorname{meas}(S_{i,n})_{i=1-8} \to 0$ as $n \to \infty$. Therefore,

$$\|W_n\|_{\mathcal{H}},\ \|Z_n\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty.$$

Using (2.5), mean value Theorem, and (H2), we obtain for some $0 < \theta < 1$:

$$\begin{aligned}
\sum_{i=1}^{8} A_{i,n} &\ge \frac{1}{2}\Big(\|W_n\|_{\mathcal{H}}^2 + \|Z_n\|_{\mathcal{H}}^2\Big) - \int_\Omega \nabla\underline{u}\nabla z_{1,n}dx \\
&\quad + \int_{S_{1,n}\cup S_{2,n}\cup S_{3,n}} h_1(\underline{u}-\theta z_{1,n},\underline{v}-\theta z_{2,n})z_{1,n}dx - \int_\Omega \nabla\underline{v}\nabla z_{2,n}dx \\
&\quad + \int_{S_{1,n}\cup S_{4,n}\cup S_{6,n}} h_2(\underline{u}-\theta z_{1,n},\underline{v}-\theta z_{2,n})z_{2,n}dx + \int_\Omega \nabla\overline{u}\nabla w_{1,n}dx
\end{aligned}$$

$$
\begin{aligned}
&- \int_{S_{6,n}\cup S_{7,n}\cup S_{8,n}} h_1(\overline{u}+\theta w_{1,n},\overline{v}+\theta w_{2,n})w_{1,n}dx + \int_\Omega \nabla\overline{v}\nabla w_{2,n}dx \\
&- \int_{S_{3,n}\cup S_{5,n}\cup S_{8,n}} h_2(\overline{u}+\theta w_{1,n},\overline{v}+\theta w_{2,n})w_{2,n}dx.
\end{aligned}
$$

Since $(\overline{u},\overline{v})$(resp. $(\underline{u},\underline{v})$) is a supersolution (resp. subsolution) of (1.1),

$$
\begin{aligned}
\sum_{i=1}^{8} A_{i,n} &\geq \frac{1}{2}\Big(\|W_n\|_{\mathcal{H}}^2 + \|Z_n\|_{\mathcal{H}}^2\Big) \\
&\quad + \int_{S_{1,n}\cup S_{2,n}\cup S_{3,n}} \Big(h_1(\underline{u}-\theta z_{1,n},\underline{v}-\theta z_{2,n}) - h_1(\underline{u},\underline{v})\Big)z_{1,n}dx \\
&\quad + \int_{S_{1,n}\cup S_{4,n}\cup S_{6,n}} \Big(h_2(\underline{u}-\theta z_{1,n},\underline{v}-\theta z_{2,n}) - h_2(\underline{u},\underline{v})\Big)z_{2,n}dx \\
&\quad - \int_{S_{6,n}\cup S_{7,n}\cup S_{8,n}} \Big(h_1(\overline{u}+\theta w_{1,n},\overline{v}+\theta w_{2,n}) - h_1(\overline{u},\overline{v})\Big)w_{1,n}dx \\
&\quad - \int_{S_{3,n}\cup S_{5,n}\cup S_{8,n}} \Big(h_2(\overline{u}+\theta w_{1,n},\overline{v}+\theta w_{2,n}) - h_2(\overline{u},\overline{v})\Big)w_{2,n}dx \\
&\geq \frac{1}{2}\Big(\|W_n\|_{\mathcal{H}}^2 + \|Z_n\|_{\mathcal{H}}^2\Big) - \int_{S_{1,n}\cup S_{2,n}\cup S_{3,n}} \Big(\frac{\partial h_1}{\partial t}(\underline{u}-\theta' z_{1,n},\underline{v}-\theta' z_{2,n}) \\
&\quad + \frac{\partial h_1}{\partial s}(\underline{u}-\theta' z_{1,n},\underline{v}-\theta' z_{2,n})\Big)z_{1,n}^2dx \\
&\quad - \int_{S_{1,n}\cup S_{4,n}\cup S_{6,n}} \Big(\frac{\partial h_2}{\partial t}(\underline{u}-\theta' z_{1,n},\underline{v}-\theta' z_{2,n}) \\
&\quad - \frac{\partial h_2}{\partial s}(\underline{u}-\theta' z_{1,n},\underline{v}-\theta' z_{2,n})\Big)z_{2,n}^2dx \\
&\quad - \int_{S_{6,n}\cup S_{7,n}\cup S_{8,n}} \Big(\frac{\partial h_1}{\partial t}(\overline{u}+\theta' w_{1,n},\overline{v}+\theta' w_{2,n}) \\
&\quad + \frac{\partial h_1}{\partial t}(\overline{u}+\theta' w_{1,n},\overline{v}+\theta' w_{2,n})\Big)w_{1,n}^2dx \\
&\quad - \int_{S_{3,n}\cup S_{5,n}\cup S_{8,n}} \Big(\frac{\partial h_2}{\partial t}(\overline{u}+\theta' w_{1,n},\overline{v}+\theta' w_{2,n}) \\
&\quad + \frac{\partial h_2}{\partial s}(\overline{u}+\theta' w_{1,n},\overline{v}+\theta' w_{2,n})\Big)w_{2,n}^2dx.
\end{aligned}
$$

It follows from (H7), (1.2), Hölder's and Sobolev's inequalities that for n sufficiently large,

$$
\begin{aligned}
\sum_{i=1}^{8} A_{i,n} &\geq \frac{1}{2}\Big(\|W_n\|_{\mathcal{H}}^2 + \|Z_n\|_{\mathcal{H}}^2\Big) \\
&\quad - C_1 \int_{S_{1,n}\cup S_{2,n}\cup S_{3,n}} e^{(1+\varepsilon)(\underline{u}^2+\underline{v}^2)} z_{1,n}^2dx \\
&\quad - C_2 \int_{S_{1,n}\cup S_{4,n}\cup S_{6,n}} e^{(1+\varepsilon)(\underline{u}^2+\underline{v}^2)} z_{2,n}^2dx \\
&\quad - C_3 \int_{S_{6,n}\cup S_{7,n}\cup S_{8,n}} e^{(1+\varepsilon)((\overline{u}+w_{1,n})^2+(\overline{v}+w_{2,n})^2)} w_{1,n}^2dx
\end{aligned}
$$

$$
\begin{aligned}
&- C_4 \int_{S_{3,n}\cup S_{5,n}\cup S_{8,n}} e^{(1+\varepsilon)((\overline{u}+w_{1,n})^2+(\overline{v}+w_{2,n})^2)} w_{2,n}^2 dx \\
&\geq \frac{1}{2}\Big(\|W_n\|_{\mathcal{H}}^2 + \|Z_n\|_{\mathcal{H}}^2\Big) - o(1)\Big(\|W_n\|_{\mathcal{H}}^2 + \|Z_n\|_{\mathcal{H}}^2\Big).
\end{aligned}
$$

Hence $E(u_n, v_n) \geq E(u_\lambda, v_\lambda)$ which is a contradiction. □

3. Existence of a second solution

Throughout this section, we fix $\lambda \in (0, \Lambda)$ and we denote by (u_λ, v_λ) the local minimum of E obtained in the previous section as the minimal solution of (1.1). Using min-max methods and Mountain pass lemma around a closed set, we prove the existence of a second solution $(\overline{u}_\lambda, \overline{v}_\lambda)$ of (1.1) such that $\overline{u}_\lambda \geq u_\lambda$ and $\overline{v}_\lambda \geq v_\lambda$ in Ω. Let $T = \{(u, v) : u \geq u_\lambda, v \geq v_\lambda \text{ a.e. in } \Omega\}$.

We note that $\lim_{t\to+\infty} E(u_\lambda + tu, v_\lambda + tv) = -\infty$ for any $(u, v) \in \mathcal{H}\setminus\{0\}$. Hence, we may fix $(\tilde{u}, \tilde{v}) \in \mathcal{H}\setminus\{0\}$ such that $E(u_\lambda + \tilde{u}, v_\lambda + \tilde{v}) < 0$. We define the mountain pass level

$$
\rho_0 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} E(\gamma(t)), \tag{3.1}
$$

where $\Gamma = \{\gamma : [0, 1] \to \mathcal{H} : \gamma \in C,\ \gamma(0) = (u_\lambda, v_\lambda),\ \text{and } \gamma(1) = (u_\lambda + \tilde{u}, v_\lambda + \tilde{v})\}$. It follows that $\rho_0 \geq E(u_\lambda, v_\lambda)$. If $\rho_0 = E(u_\lambda, v_\lambda)$, we obtain that $\inf\{E(u, v) : \|(u, v) - (u_\lambda, v_\lambda)\|_{\mathcal{H}} = R\} = E(u_\lambda, v_\lambda)$ for all $R \in (0, R_0)$ for some R_0 small.

We now let $\mathcal{F} = T$ if $\rho_0 > E(u_\lambda, v_\lambda)$, and $\mathcal{F} = T \cap \{\|(u - u_\lambda, v - v_\lambda)\|_{\mathcal{H}} = \frac{R_0}{2}\}$ if $\rho_0 = E(u_\lambda, v_\lambda)$. We have the following upper bound on ρ_0.

Lemma 3.1. *With ρ_0 defined as in (3.1), we have $\rho_0 < E(u_\lambda, v_\lambda) + 2\pi$.*

Proof. Without loss of generality, we assume that $0 \in \Omega$. Define the sequence

$$
\tilde{\psi}_n(x) = \begin{cases} \frac{1}{2\sqrt{\pi}}(\log n)^{1/2} & \text{if } 0 \leq |x| \leq \frac{1}{n} \\ \frac{1}{2\sqrt{\pi}} \frac{\log(1/|x|)}{(\log n)^{1/2}} & \text{if } \frac{1}{n} \leq |x| \leq 1 \\ 0 & \text{if} |x| \geq 1 \end{cases}
$$

Now, consider $(\tilde{\psi}_n, \tilde{\psi}_n) \in \mathcal{H}$. Then $\|(\tilde{\psi}_n, \tilde{\psi}_n)\|_{\mathcal{H}} = 1$. We now choose $\delta > 0$ such that $B_\delta(0) \subset \Omega$ and let $\psi_n(x) = \tilde{\psi}_n(\frac{x}{\delta})$. Then, ψ_n has support in $B_\delta(0)$ and (ψ_n, ψ_n) is such that $\|(\psi_n, \psi_n)\|_{\mathcal{H}} = 1$ for all n. Now, suppose $\rho_0 \geq E(u_\lambda, v_\lambda) + 2\pi$ and we derive a contradiction. This means that for some $t_n,\ s_n > 0$:

$$
E(u_\lambda + t_n\psi_n, v_\lambda + s_n\psi_n) = \sup_{t,s>0} E(u_\lambda + t\psi_n, v_\lambda + s\psi_n) \geq E(u_\lambda, v_\lambda) + 2\pi, \quad \forall n.
$$

Since $E(u_\lambda + tu, v_\lambda + sv) \to -\infty$ as $t, s \to +\infty$, we obtain that (t_n, s_n) is bounded in $\mathbb{R}^2$. Then, using $\|(\psi_n, \psi_n)\|_{\mathcal{H}} = 1$, we obtain

$$
\begin{aligned}
&\frac{t_n^2 + s_n^2}{4} + \int_\Omega \Big(t_n \nabla u_\lambda \nabla\psi_n + s_n \nabla v_\lambda \nabla\psi_n\Big) dx \\
&\geq \int_\Omega \Big(H(u_\lambda + t_n\psi_n, v_\lambda + s_n\psi_n) - H(u_\lambda, v_\lambda)\Big) dx + 2\pi
\end{aligned}
$$

Now, using the fact that (u_λ, v_λ) is a solution, we obtain

$$
\begin{aligned}
\frac{t_n^2 + s_n^2}{4} \geq \int_\Omega \Big[&H(u_\lambda + t_n\psi_n, v_\lambda + s_n\psi_n) \\
&- H(u_\lambda, v_\lambda) - \psi_n\Big(t_n h_1(u_\lambda, v_\lambda) + s_n h_2(u_\lambda, v_\lambda)\Big)\Big] dx + 2\pi.
\end{aligned} \tag{3.2}
$$

Using (H2) we have that $h_1,\ h_2$ are non-decreasing, then there exist $\theta_n \in (0,1)$ such that

$$\begin{aligned}
&\int_\Omega \Big[H(u_\lambda + t_n\psi_n, v_\lambda + s_n\psi_n) - H(u_\lambda, v_\lambda) - \psi_n\Big(t_n h_1(u_\lambda, v_\lambda) + s_n h_2(u_\lambda, v_\lambda)\Big)\Big]dx\\
&= \int_\Omega \psi_n^2\Big[t_n^2 \frac{\partial h_1}{\partial u}(u_\lambda + \theta_n t_n\psi_n, v_\lambda + \theta_n s_n\psi_n) + s_n^2\frac{\partial h_2}{\partial v}(u_\lambda + \theta_n t_n\psi_n, v_\lambda + \theta_n s_n\psi_n)\\
&+ t_n s_n\frac{\partial h_1}{\partial v}(u_\lambda + \theta_n t_n\psi_n, v_\lambda + \theta_n s_n\psi_n) + t_n s_n\frac{\partial h_2}{\partial u}(u_\lambda + \theta_n t_n\psi_n, v_\lambda + \theta_n s_n\psi_n)\Big]dx\\
&\geq 0\,.
\end{aligned}$$

Now, by (3.2), we see that

$$t_n^2 + s_n^2 \geq 8\pi, \quad \text{for all } n \tag{3.3}$$

Since (t_n, s_n) is a critical point of $E(u_\lambda + t\psi, v_\lambda + s\psi)$, we obtain

$$E'(u_\lambda + t\psi_n, v_\lambda + s\psi_n)_{|(t,s)=(t_n,s_n)} = 0.$$

Then

$$\begin{aligned}
t_n^2 + s_n^2 = \int_\Omega \Big[&\Big(h_1(u_\lambda + t_n\psi_n, v_\lambda + s_n\psi_n) - h_1(u_\lambda, v_\lambda)\Big)t_n\\
&+ \Big(h_2(u_\lambda + t_n\psi_n, v_\lambda + s_n\psi_n) - h_2(u_\lambda, v_\lambda)\Big)s_n\Big]\psi_n dx.
\end{aligned}$$

Since $t_n\psi_n \to \infty$, $s_n\psi_n \to \infty$ on $\{|x| \leq \delta/n\}$, we obtain

$$\begin{aligned}
t_n^2 + s_n^2 &\geq \int_{\Omega\cap\{|x|\leq\delta/n\}} e^{(t_n^2+s_n^2)\psi_n^2}(t_n + s_n)\psi_n dx\\
&= \frac{\sqrt{\pi}\delta^2}{2n^2} e^{(t_n^2+s_n^2)\frac{\log n}{4\pi}}(t_n + s_n)(\log n)^{1/2}\\
&= \frac{\sqrt{\pi}\delta^2}{2} e^{(\frac{t_n^2+s_n^2}{4\pi}-2)\log n}(t_n + s_n)(\log n)^{1/2}
\end{aligned} \tag{3.4}$$

This and (3.3) imply that

$$t_n^2 + s_n^2 \to 8\pi, \tag{3.5}$$

and by (3.4) we obtain

$$t_n^2 + s_n^2 \geq (t_n + s_n)(\log n)^{1/2}.$$

This in turn implies that $t_n^2 + s_n^2 \to \infty$ as $n \to \infty$, which contradicts (3.5). $\square$

Definition 3.2. Let $\mathcal{F} \subset \mathcal{H}$ be a closed set. We say that a sequence $(u_n, v_n) \subset \mathcal{H}$ is a Palais-Smale sequence for E at level ρ around $\mathcal{F}$, and we denote $(PS)_{\mathcal{F},\rho}$, if

$$\begin{gathered}
\lim_{n\to+\infty} dist\Big((u_n, v_n), \mathcal{F}\Big) = 0, \quad \lim_{n\to+\infty} E(u_n, v_n) = \rho,\\
\lim_{n\to+\infty} \|E'(u_n, v_n)\|_{\mathcal{H}^{-1}} = 0.
\end{gathered}$$

Lemma 3.3. *Let $\mathcal{F} \subset \mathcal{H}$ be a closed set and $\rho \in \mathbb{R}$. Let $\{(u_n, v_n)\} \subset \mathcal{H}$ be a $(PS)_{\mathcal{F},\rho}$ sequence. Then there exists (u_0, v_0) such that, up to a subsequence, $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$, and*

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega h_1(u_n, v_n)dx &= \int_\Omega h_1(u_0, v_0)dx,\\
\lim_{n\to\infty}\int_\Omega h_2(u_n, v_n)dx &= \int_\Omega h_2(u_0, v_0)dx
\end{aligned}$$

Proof. We have the following relations as $n \to +\infty$

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \int_\Omega H(u_n, v_n) dx = \rho + o_n(1) \tag{3.6}$$

$$\Big|\int_\Omega \nabla u_n.\nabla\varphi\, dx - \int_\Omega h_1(u_n, v_n)\varphi\, dx\Big| \le o_n(1)\|\varphi\|, \quad \forall \varphi \in H_0^1(\Omega) \tag{3.7}$$

$$\Big|\int_\Omega \nabla v_n.\nabla\varphi\, dx - \int_\Omega h_2(u_n, v_n)\varphi\, dx\Big| \le o_n(1)\|\varphi\|, \quad \forall \varphi \in H_0^1(\Omega) \tag{3.8}$$

Step 1: We claim that

$$\begin{gathered}\sup_n \Big(\|u_n\|_{H_0^1(\Omega)} + \|v_n\|_{H_0^1(\Omega)}\Big) < +\infty,\\ \sup_n \int_\Omega h_1(u_n, v_n) u_n dx < +\infty,\\ \sup_n \int_\Omega h_2(u_n, v_n) v_n dx < +\infty.\end{gathered}$$

We note that for all $\varepsilon > 0$, there exists s_ε such that

$$h(t, s) \le \varepsilon\Big(h_1(t, s)t + h_2(t, s)s\Big), \quad \text{for } |s|,\ |t| \ge s_\varepsilon.$$

From (3.6), we obtain

$$\frac{1}{2}\int_\Omega \Big(|\nabla u_n|^2 + |\nabla v_n|^2\Big) dx \le C_\varepsilon + \varepsilon \int_\Omega \Big(h_1(u_n, v_n)u_n + h_2(u_n, v_n)v_n\Big) dx \tag{3.9}$$

From (3.7) with $\varphi = u_n$, and (3.8) with $\varphi = v_n$, we obtain

$$\begin{gathered}\int_\Omega h_1(u_n, v_n)u_n dx \le \int_\Omega |\nabla u_n|^2 dx + o(1)\|u_n\|_{H_0^1(\Omega)},\\ \int_\Omega h_2(u_n, v_n)v_n dx \le \int_\Omega |\nabla v_n|^2 dx + o(1)\|v_n\|_{H_0^1(\Omega)}.\end{gathered}$$

From (3.9) we obtain

$$\begin{aligned}&\int_\Omega \Big(h_1(u_n, v_n)u_n + h_2(u_n, v_n)v_n\Big) dx\\ &\le 2C_\varepsilon + 2\varepsilon \int_\Omega \Big(h_1(u_n, v_n)u_n + h_2(u_n, v_n)v_n\Big) dx + o(1)\\ &\le \frac{2C_\varepsilon}{1 - 2\varepsilon} + o(1)\Big(\|u_n\| + \|v_n\|\Big).\end{aligned}$$

Substituting this in (3.9), we obtain

$$\sup_n \Big(\|u_n\|_{H_0^1(\Omega)} + \|v_n\|_{H_0^1(\Omega)}\Big) < +\infty,$$

which implies

$$\sup_n \int_\Omega h_1(u_n, v_n)u_n dx < \infty, \quad \sup_n \int_\Omega h_2(u_n, v_n)v_n dx < \infty.$$

Step 2: We claim that

$$\lim_{n\to+\infty} \int_\Omega h_1(u_n, v_n) dx = \int_\Omega h_1(u_0, v_0) dx, \tag{3.10}$$

$$\lim_{n\to+\infty} \int_\Omega h_2(u_n, v_n) dx = \int_\Omega h_2(u_0, v_0) dx, \tag{3.11}$$

$$\lim_{n\to+\infty}\int_\Omega H(u_n,v_n)dx=\int_\Omega H(u_0,v_0)dx \tag{3.12}$$

Let $|A|$ denote the Lebesgue measure of $A\subset\mathbb{R}^2$. We show that $\{h_1(u_n,v_n)\}$ and $\{h_2(u_n,v_n)\}$ are equi-integrable in L^1, and then, (3.10) and (3.11) follow from Vitali's convergence Theorem. (3.12) follows from (1.3) and the Lebesgue dominated convergence Theorem. We claim that for all $\varepsilon>0$, there exists $\delta>0$ such that for any $A\subset\Omega$ with $|A|<\delta$, we have

$$\sup_n\int_A|h_2(u_n,v_n)|dx\le\varepsilon.$$

Let $C_1=\sup_n\int_A|h_2(u_n,v_n)v_n|dx$. By step 1, $C_1<+\infty$. Since $\{u_n\}$ and $\{v_n\}$ are bounded in H_0^1, by (3.7) and (3.8) we have

$$\int_\Omega|h_2(u_n,v_n)u_n|dx<\infty,\quad \int_\Omega|h_1(u_n,v_n)v_n|dx<\infty.$$

Let $C_2=\sup_n\int_A|h_2(u_n,v_n)u_n|dx$, and let $C=\max\Big\{C_1,C_2\Big\}$. Define

$$\mu_\varepsilon=\max_{|t|\le\frac{3C}{\varepsilon},\,|s|\le\frac{3C}{\varepsilon}}\Big\{|h_2(t,s)|\Big\}.$$

Then, for any $A\subset\Omega$ with $|A|\le\frac{\varepsilon}{3\mu_\varepsilon}$, we obtain

$$\begin{aligned}
&\int_A|h_2(u_n,v_n)|\,dx\\
&\le\int_{A\cap\{|u_n|,\,|v_n|\le\frac{3C}{\varepsilon}\}}|h_2(u_n,v_n)|\,dx+\int_{A\cap\{|v_n|\ge\frac{3C}{\varepsilon}\}}\frac{|h_2(u_n,v_n)v_n|}{v_n}\,dx\\
&\quad+\int_{A\cap\{|u_n|\ge\frac{3C}{\varepsilon}\}}\frac{|h_2(u_n,v_n)u_n|}{u_n}\,dx\\
&\le|A|\mu_\varepsilon+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\le\varepsilon.
\end{aligned}$$

which shows the equi-integrability of $\{h_2(u_n,v_n)\}$. In a similar way, we can show the equi-integrability of $\{h_1(u_n,v_n)\}$. This completes step 2 and the proof of Lemma 3.3. □

We will also use the following version of Lion's Lemma [14].

Lemma 3.4. *Let $\{(u_n,v_n)\}$ be a sequence in $\mathcal{H}$ such that $\|(u_n,v_n)\|_{\mathcal{H}}=1$, for all n and $u_n\rightharpoonup u$, $v_n\rightharpoonup v$ in H_0^1 for some $(u,v)\ne(0,0)$. Then, for $4\pi<p<4\pi(1-\|u\|_{1,2}^2-\|v\|_{1,2}^2)^{-1}$,*

$$\sup_{n\ge1}\int_\Omega e^{p(u_n^2+v_n^2)}dx<\infty$$

Proof. It is easy to see that

$$\lim_{n\to\infty}\|(u_n-u,v_n-v)\|_{\mathcal{H}}^2=\lim_{n\to\infty}\|u_n-u\|_{1,2}^2+\|v_n-v\|_{1,2}^2=1-\|u\|_{1,2}^2-\|v\|_{1,2}^2,$$
$$u_n^2\le(u_n-u)^2+2\epsilon u_n^2+C_\epsilon u^2\quad\text{for }\epsilon\text{ small.}$$

Then

$$\int_\Omega e^{p(u_n^2+v_n^2)}dx\le\int_\Omega e^{p((u_n-u)^2+(v_n-v)^2)}e^{p\epsilon(u_n^2+v_n^2)}e^{(u^2+v^2)C_\epsilon}dx.$$

Now, using Hölder's inequality with r_1, r_2, r_3 such that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=1$, we obtain

$$\begin{aligned}\int_\Omega e^{p(u_n^2+v_n^2)}dx \le \Big(\int_\Omega e^{pr_1((u_n-u)^2+(v_n-v)^2)}dx\Big)^{1/r_1}\Big(\int e^{\epsilon pr_2(u_n^2+v_n^2)}dx\Big)^{1/r_2}\\ \times\Big(\int e^{pr_3(u^2+v^2)C_\epsilon}dx\Big)^{1/r_3}.\end{aligned}$$

The second and third integrals are finite for ϵ small and using inequality (1.2). For the first integral we have

$$\begin{aligned}&\int_\Omega e^{pr_1((u_n-u)^2+(v_n-v)^2)}dx\\ &=\int_\Omega e^{pr_1(\frac{u_n-u}{\|(u_n-u,v_n-v)\|_{\mathcal{H}}})^2+(\frac{v_n-v}{\|(u_n-u,v_n-v)\|_{\mathcal{H}}})^2\|(u_n-u,v_n-v)\|^2_{\mathcal{H}}}dx.\end{aligned}$$

We can choose $r_1>1$ and close to 1 such that $pr_1(1-\|u\|^2_{1,2}-\|v\|^2_{1,2})<4\pi$ by the hypothesis and the first equation of this proof. Hence, this is bounded again thanks to inequality (1.2). $\square$

Now, we prove our main result.

Theorem 3.5. *For $\lambda\in(0,\Lambda)$, problem (1.1) has a second nontrivial solution $(\overline{u}_\lambda,\overline{v}_\lambda)$ such that $\overline{u}_\lambda\ge u_\lambda>0$ and $\overline{v}_\lambda\ge v_\lambda>0$ in Ω.*

Proof. Let $\{(u_n,v_n)\}$ be a Palais-Smale sequence for E at the level ρ_0 around $\mathcal{F}$. Existence of a such sequence can be obtained using Ekeland Variational principle on $\mathcal{F}$ ([2, 10]). Then, by Lemma 3.3, there exist $(\overline{u}_\lambda,\overline{v}_\lambda)$ and a subsequence denoted again by (u_n,v_n), such that $u_n\rightharpoonup\overline{u}_\lambda$ and $v_n\rightharpoonup\overline{v}_\lambda$ in $H^1_0(\Omega)$. It is easy to verify that $(\overline{u}_\lambda,\overline{v}_\lambda)$ is a solution of (1.1).

It remains to show that $(\overline{u}_\lambda,\overline{v}_\lambda)\not\equiv(u_\lambda,v_\lambda)$. We suppose that $\overline{u}_\lambda\equiv u_\lambda$ and $\overline{v}_\lambda\equiv v_\lambda$ and we derive a contradiction:

Case 1: $\rho_0=E(u_\lambda,v_\lambda)$. In this case, we recall that

$$\mathcal{F}=\{(u,v)\in T:\|(u-u_\lambda,v-v_\lambda)\|_{\mathcal{H}}=\frac{R_0}{2}\},$$

$$E(u_\lambda,v_\lambda)+o(1)=E(u_n,v_n)=\frac12\int_\Omega|\nabla u_n|^2dx+\frac12\int_\Omega|\nabla v_n|^2dx-\int_\Omega H(u_n,v_n)dx.$$

From Lemma 3.3, (equation (3.8)) we have $\int_\Omega H(u_n,v_n)dx\to\int_\Omega H(u_\lambda,v_\lambda)$. Thus, $\|(u_n-u_\lambda,v_n-v_\lambda)\|_{\mathcal{H}}=o(1)$, which contradicts the fact that $(u_n,v_n)\in\mathcal{F}$.

Case 2: $\rho_0\ne E(u_\lambda,v_\lambda)$. In this case $\rho_0-E(u_\lambda,v_\lambda)\in(0,2\pi)$ and $E(u_n,v_n)\to\rho_0$. Let $\beta_0=\int_\Omega H(u_\lambda,v_\lambda)dx$. Then from Lemma 3.3,

$$\frac12\int_\Omega|\nabla u_n|^2dx+\frac12\int_\Omega|\nabla v_n|^2dx\to(\rho_0+\beta_0)\quad\text{as } n\to\infty \tag{3.13}$$

Also, by Fatou's lemma we have that $E(u_\lambda,v_\lambda)\le\liminf_{n\to+\infty}E(u_n,v_n)$. If $\{(u_n,v_n)\}$ does not converge strongly in $\mathcal{H}$, then $E(u_\lambda,v_\lambda)<\rho_0$. By Lemma 3.1, for ϵ small, we have

$$(1+\epsilon)(\rho_0-E(u_\lambda,v_\lambda))<2\pi.$$

Hence, from (3.13) we have

$$(1+\epsilon)\|(u_n,v_n)\|^2_{\mathcal{H}}<4\pi\frac{\rho_0+\beta_0}{\rho_0-E(u_\lambda,v_\lambda)}$$

$$
\begin{aligned}
&< 4\pi \frac{\rho_0+\beta_0}{\rho_0+\beta_0-\frac{1}{2}\|(u_\lambda,v_\lambda)\|_{\mathcal{H}}^2} \\
&< 4\pi\Big(1-\frac{1}{2}\big(\frac{\|(u_\lambda,v_\lambda)\|_{\mathcal{H}}^2}{\rho_0+\beta_0}\big)\Big)^{-1} \\
&< 4\pi\Big(1-\|\frac{u_\lambda}{\sqrt{2(\rho_0+\beta_0)}}\|_{1,2}-\|\frac{v_\lambda}{\sqrt{2(\rho_0+\beta_0)}}\|_{1,2}\Big)^{-1}.
\end{aligned}
$$

Now, choose $p > 4\pi$ such that

$$
(1+\epsilon)\|(u_n,v_n)\|_{\mathcal{H}}^2 \le p < 4\pi(1-\|\frac{u_\lambda}{\sqrt{2(\rho_0+\beta_0)}}\|_{1,2}-\|\frac{v_\lambda}{\sqrt{2(\rho_0+\beta_0)}}\|_{1,2})^{-1}.
$$

Since $\frac{u_n}{\|(u_n,v_n)\|_{\mathcal{H}}} \rightharpoonup \frac{u_\lambda}{\sqrt{2(\rho_0+\beta_0)}}$ and $\frac{v_n}{\|(u_n,v_n)\|_{\mathcal{H}}} \rightharpoonup \frac{v_\lambda}{\sqrt{2(\rho_0+\beta_0)}}$ weakly in $H_0^1(\Omega)$, by Lemma 3.4, we have

$$
\sup_n \int_\Omega \exp\Big(p\big[\big(\frac{u_n}{\|(u_n,v_n)\|_{\mathcal{H}}}\big)^2+\big(\frac{v_n}{\|(u_n,v_n)\|_{\mathcal{H}}}\big)^2\big]\Big)dx < \infty \tag{3.14}
$$

From the definition of h_1, for any $\delta > 0$, there exists a constant $C > 0$ such that

$$
\sup_n h_1(u_n,v_n) \le Ce^{(1+\delta)(u_n^2+v_n^2)}.
$$

Now, it is not difficult to show that $h_1(u_n,v_n) \in L^q(\Omega)$ for some $q > 1$. Indeed, taking δ close to zero and q close to 1 such that $q(1+\delta) < 1+\epsilon$,

$$
\begin{aligned}
\int_\Omega \Big|h_1(u_n,v_n)\Big|^q dx &\le C\int_\Omega e^{q(1+\delta)(u_n^2+v_n^2)}dx \\
&\le C\int_\Omega e^{q(1+\delta)\Big[\Big(\frac{u_n}{\|(u_n,v_n)\|_{\mathcal{H}}}\Big)^2+\Big(\frac{v_n}{\|(u_n,v_n)\|_{\mathcal{H}}}\Big)^2\Big]\|(u_n,v_n)\|_{\mathcal{H}}^2}dx \\
&\le C\int_\Omega e^{p\Big[\Big(\frac{u_n}{\|(u_n,v_n)\|_{\mathcal{H}}}\Big)^2+\Big(\frac{v_n}{\|(u_n,v_n)\|_{\mathcal{H}}}\Big)^2\Big]}dx
\end{aligned}
$$

Now, using (3.14), we obtain that $h_1 \in L^q(\Omega)$. So, by Hölder inequality we have and the assumption that $\overline{u}_\lambda = u_\lambda, \overline{v}_\lambda = v_\lambda$, we have

$$
\int_\Omega h_1(u_n,v_n)u_n dx \longrightarrow \int_\Omega h_1(u_\lambda,v_\lambda) \text{ as } n\to\infty.
$$

Hence,

$$
\begin{aligned}
o(1) = E'(u_n,v_n)(u_n,0) &= \frac{1}{2}\int_\Omega |\nabla u_n|^2dx - \int_\Omega h_1(u_n,v_n)u_n dx \\
&= \frac{1}{2}\int_\Omega |\nabla u_n|^2dx - \int_\Omega |\nabla u_\lambda|^2 + o(1).
\end{aligned}
$$

Similarly, we obtain $\int_\Omega |\nabla v_n|^2dx = \int_\Omega |\nabla v_\lambda|^2 + o(1)$. This is a contradiction to the assumption that $\rho_0 \ne E(u_\lambda,v_\lambda)$. □

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References

[1] Adimurthi; *Existence of Positive Solutions of the Semilinear Dirichlet Problem with Critical Growth for the n-Laplacian,* Ann. Della. Scuola. Norm. Sup. di Pisa, Serie IV, Vol.XVII, Fasc. 3 (1990), pp. 393-413.

[2] Adimurthi, J. Giacomoni; *Multiplicity of positive solutions for a singular and critical elliptic problem in* $\mathbb{R}^2$, Communications in Contemporary Mathematics Vol. 8, No. 5 (2006), pp. 621-656

[3] A. Ambrosetti, H. Brezis, G. Cerami; *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. 122 (1994) pp. 519-543.

[4] F. V. Atkinson and L. A. Peletier; *Ground states and Dirichlet Problems for* $-\Delta u = f(u)$ *in* $\mathbb{R}^2$, Arch. Ration Mech. Anal. 96(2) (1986), pp. 147-166.

[5] M. Badiale and G. Tarantello; *Existence and Multiplicity for elliptic problems with critical growth and discontinuous nonlinearities*, Nonlinear Anal. 29(6) (1997), pp. 639-677.

[6] H. Brezis, L. Nirenberg; H^1 *versus* C^1 *local minimizers*, C. R. Acad. Sci. Paris Ser. I Math. 317 (1993) pp. 465-472.

[7] J. L. Chern, Z. Y. Chen, C. S. Lin; *Uniqueness of Topological Solutions and the Structure of Solutions for the Chern-Simons System with Two Higgs Particles*, Commun. Math. Phys. 296, (2010)323-351.

[8] D. G. de Figueiredo, O. H. Miyagaki, B. Ruf; *Elliptic equations in* R^2 *with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations, Vol. 3, (1995), no.2, pp. 139–153.

[9] D. G. de Figueiredo, J. M. Do Ò , B. Ruf; *Critical and subcritical elliptic systems in dimension two*, Indiana Univ. Math. J. Vol 53, no 4, (2004), pp. 1037–1054.

[10] N. Ghoussoub, D. Preiss; *A general mountain pass principle for locating and classifying critical points,* Anal. Non Lineaire, Vol. 6 no. 5 (1989), pp. 321-330.

[11] Y. Haitao; *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem*, J. Differential Equations, 189, (2003), pp. 487-512.

[12] Joao Marcos B. do Ò; *Semiliear Dirichlet problems for the N-Laplacian in* $\mathbb{R}^N$ *with nonlinearities in critical growth range*, Differential and Integral equations, 9, 5 (1996), pp. 967-979.

[13] B. S. Kaur, K. Sreenadh; *Multiple positive solutions for a singular elliptic equation with Neumann boundary condition in two dimensions*, Electron. J. Diff. Equ., vol. 2009 (2009), No. 43, pp. 1-13.

[14] P. L. Lions; *The concentration compactness principle in the calculus of variations*, part I, Rev. Mat. Iberoamericana 1 (1) (1985), pp. 185-201.

[15] Y. Li, P. Liu, Y. Yang; *Moser-Trudinger inequalities of vector bundle over a compact Riemannian manifold of dimension 2*, Calc. Var. Vol. 28, (2006), pp. 59-83.

[16] C. S. Lin, A. C. Ponce, Y. Yang; *A system of elliptic equations arising in Chern-Simons field theory*. J. Fun. Anal. 247 (2007) 289-350.

[17] C. S. Lin, J. V. Prajapat; *Vortex Condensates for Relativistic Abelian Chern-Simons Model with Two Higgs Scalar Fields and Two Gauge Fields on a Torus*, Commun. Math. Phys. 288, (2009) 311-347.

[18] C. S. Lin, S. Yan; *Bubbling Solutions for Relativistic Abelian Chern-Simons Model on a Torus*, Commun. Math. Phys. 297, (2010) 733-758.

[19] J. Moser; *A sharp form of an inequality by N.Trudinger*, Indiana Univ. Math. Jour., Vol.20, no.11 (1971), pp. 1077-1092.

[20] S. Prashanth and K. Sreenadh; *Multiplicity results in a ball for p-Laplace equation with positive nonlinearity,* Advances in Differential Equations, Vol. 7, no. 7 (2002), pp. 877-896.

[21] S. Prashanth, K. Sreenadh; *Existence of multiple positive solutions for N-Laplacian in a bounded domaine in* $\mathbb{R}^N$, Advanced Nonlinear Studies, Vol. 5, (2005), pp. 13-21.

[22] N. S. Trudinger; *On embedding into Orlicz spaces and some applications*, Journal of Math. Mech., Vol. 17 (1967), pp. 473-484.

Nasreddine Megrez

Chemical and Material Engineering Department, University of Alberta, 9107 - 116 Street, Edmonton (Alberta) T6G 2V4, Canada

E-mail address: nmegrez@gmail.com

K. Sreenadh
Department of Mathematics, Indian Institute of Technology Delhi Hauz Khaz, New Delhi-16, India
E-mail address: sreenadh@gmail.com

Brahim Khaldi
Departement of Sciences, University of Bechar, PB 117, Bechar 08000, Algeria
E-mail address: khaldibra@yahoo.fr