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GLOBAL STABILITY FOR DELAY SIR AND SEIR EPIDEMIC MODELS WITH SATURATED INCIDENCE RATES

ABDELHADI ABTA, ABDELILAH KADDAR, HAMAD TALIBI ALAOUI

Abstract. In this article we propose a comparison of a delayed SIR model and its corresponding SEIR model in terms of global stability. We consider a saturated incidence rate and we determine, using Lyapunov functionals, conditions by which the disease-free equilibrium and the endemic equilibrium are globally asymptotically stable. Also some numerical simulations are given to compare a global behaviour of a delayed SIR model and its corresponding SEIR model.

1. Introduction

In this article, we propose the following delay SIR epidemic model with a saturated incidence rate (see, [15]):

$$\begin{aligned}\frac{dS}{dt} &= A - \mu S(t) - \frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)},\\ \frac{dI}{dt} &= \frac{\beta e^{-\mu\tau} S(t-\tau)I(t-\tau)}{1+\alpha_1 S(t-\tau)+\alpha_2 I(t-\tau)} - (\mu+\alpha+\gamma)I(t).\end{aligned} \tag{1.1}$$

The initial condition for the above system is

$$S(\theta)=\varphi_1(\theta), \quad I(\theta)=\varphi_2(\theta), \quad \theta\in[-\tau,0] \tag{1.2}$$

with $\varphi=(\varphi_1,\varphi_2)\in C^+\times C^+$, such that $\varphi_i(\theta)\geq 0$ $(-\tau\leq\theta\leq 0,\ i=1,2)$. Here C denotes the Banach space $C([-\tau,0],\mathbb{R})$ of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}$, equipped with the supremum norm. The nonnegative cone of C is defined as $C^+=C([-\tau,0],\mathbb{R}^+)$. where S is the number of susceptible individuals, I is the number of infectious individuals, A is the recruitment rate of the population, μ is the natural death of the population, α is the death rate due to disease, β is the transmission rate, α_1 and α_2 are the parameters that measure the inhibitory effect, γ is the recovery rate of the infectious individuals, and τ is the incubation period.

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The corresponding SEIR model of system (1.1) is described in [15] as

$$\begin{aligned} \frac{dS}{dt} &= A - \mu S(t) - \frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}, \\ \frac{dE}{dt} &= \frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)} - (\sigma+\mu)E(t), \\ \frac{dI}{dt} &= \sigma E(t) - (\mu+\alpha+\gamma)I(t). \end{aligned} \tag{1.3}$$

where E is the number of exposed individuals, and σ is the rate at which exposed individuals become infectious. Thus $\frac{1}{\sigma}$ is the mean latent period.

In models (1.1) and (1.3) the formulation of the incidence rate $\frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}$; i.e., the infection rate of susceptible individuals through their contacts with infectious (see, for example, [10, 24]), includes the three forms: The first one is the bilinear incidence rate βSI, [9, 27]. The second one is the saturated incidence rate of the form $\frac{\beta S(t)I(t)}{1+\alpha_1 S(t)}$ [1, 26]. The third one is the saturated incidence rate of the form $\frac{\beta S(t)I(t)}{1+\alpha_2 I(t)}$ [14, 2, 23].

In [15], we considered a local properties of a delayed SIR model (system (1.1)) and its corresponding SEIR model (system (1.3)), and we observed that if $\mu\tau$ is close enough to 0, then the two above models generate identical local asymptotic behavior.

For, the model (1.1) with $\tau = 0$ and $\mu = A$, Korobeinikov [16] proved that the endemic equilibrium is globally asymptotically stable. Huang and al [13] studied the global asymptotic stability of the delay SIR model

$$\begin{aligned} \frac{dS}{dt} &= \mu - \mu S(t) - f(S(t), I(t-\tau)), \\ \frac{dI}{dt} &= f(S(t), I(t-\tau)) - (\sigma+\mu)I(t). \end{aligned} \tag{1.4}$$

The fundamental difference of this model with our model (1.1) is the presence of the fraction $e^{-\mu\tau}$ in the incidence rate in the second equation of (1.1). The Lyapunov functional proposed in [13] in not valid for (1.1).

For, the SEIR model, Sun and al [20, 17] proposed nonlinear incidence of the form $\beta I^p S^q$ and constructed an explicit Lyapunov function and established a global stability of this model.

In this paper, by constricting the suitable Lyapunov functionals, we determine the global asymptotic stability of a delayed SIR model (1.1) and its corresponding SEIR model (1.3). The rest of the paper is organized as follows. In Section 2, global stability of the delayed SIR epidemiological model (1.1) is established. In Section 3, global stability of the SEIR epidemiological model (1.3) is determined. In Section 4, numerical simulations and concluding remarks are provided. In the appendix, some results on the global stability are stated.

2. Global stability analysis of delayed SIR model

In this section, we discuss the global stability of a disease-free equilibrium and an endemic equilibrium of system (1.1). With the change of variables $i(t) = I(t+\tau)$

and $s(t) = S(t)$, the system (1.1) becomes

$$
\begin{aligned}
\frac{ds(t)}{dt} &= A - \mu s - \frac{\beta s(t) i(t-\tau)}{1+\alpha_1 s(t) + \alpha_2 i(t-\tau)} \\
\frac{di(t)}{dt} &= \frac{e^{-\mu\tau}\beta s(t) i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)} - (\mu+\alpha+\gamma) i(t).
\end{aligned}
\tag{2.1}
$$

In the rest of this paper, we set $\mu_1 = \mu+\alpha$. Since $\frac{d}{dt}(s(t)+i(t)) \le A - \mu(s(t)+i(t))$, we have $\limsup(s(t)+i(t)) \le \frac{A}{\mu}$. Hence we discuss system (2.1) in the closed set

$$
\Omega =: \big\{(\varphi_1,\varphi_2) \in C^+ \times C^+ : \|\varphi_1+\varphi_2\| \le A/\mu\big\}.
$$

It is easy to show that Ω is positively invariant with respect to system (2.1). System (2.1) always has a disease-free equilibrium $P_1 = (A/\mu, 0)$. Further, if

$$
R_{01} := \frac{A\beta e^{-\mu\tau}}{(\alpha_1 A+\mu)(\mu_1+\gamma)} > 1,
$$

system (2.1) admits a unique endemic equilibrium $P_1^* = (S^*, I^*)$, with

$$
\begin{aligned}
S^* &= \frac{A[(\mu_1+\gamma)+\alpha_2 A e^{-\mu\tau}]}{(\mu_1+\gamma)[\alpha_1 A(R_{01}-1)+\mu R_{01}] + \mu\alpha_2 A e^{-\mu\tau}}, \\
I^* &= \frac{A(R_{01}-1)e^{-\mu\tau}(\alpha_1 A+\mu)}{(\mu_1+\gamma)[\alpha_1 A(R_{01}-1)+\mu R_{01}] + \mu\alpha_2 A e^{-\mu\tau}}.
\end{aligned}
$$

Next we consider the global asymptotic stability of the disease-free equilibrium P_1 and the endemic equilibrium P_1^* of (2.1) by Lyapunov functionals, respectively.

Proposition 2.1. *If $R_{01} \le 1$, then the disease-free equilibrium P_1 is globally asymptotically stable.*

Proof. Define a Lyapunov functional $V(t) = V_1(t) + i(t) + V_2(t)$, with

$$
\begin{gathered}
V_1(t) = e^{-\mu\tau} \int_{\frac{A}{\mu}}^{s(t)} \Big(1 - \frac{A(1+\alpha_1 u)}{(\mu+\alpha_1 A)u}\Big) du, \\
V_2(t) = (\mu_1+\gamma) \int_0^{\tau} i(t-u) du.
\end{gathered}
$$

We will show that $\frac{dV(t)}{dt} \le 0$ for all $t \ge 0$. We have

$$
\begin{aligned}
\frac{dV_1(t)}{dt} &= e^{-\mu\tau}\Big(1 - \frac{A(1+\alpha_1 s(t))}{(\mu+\alpha_1 A)s(t)}\Big)\Big(A - \mu s(t) - \frac{\beta s(t) i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}\Big) \\
&= e^{-\mu\tau}\Big(1 - \frac{A(1+\alpha_1 s(t))}{(\mu+\alpha_1 A)s(t)}\Big)\big(A - \mu s(t)\big) \\
&\quad - e^{-\mu\tau}\Big(1 - \frac{A(1+\alpha_1 s(t))}{(\mu+\alpha_1 A)s(t)}\Big)\Big(\frac{\beta s(t) i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}\Big),
\end{aligned}
$$

and

$$
\frac{dV_2(t)}{dt} = (\mu_1+\gamma)[i(t) - i(t-\tau)]
$$

Therefore,

$$
\begin{aligned}
\frac{dV(t)}{dt} &= e^{-\mu\tau}\Big(1 - \frac{A(1+\alpha_1 s(t))}{(\mu+\alpha_1 A)s(t)}\Big)\big(A - \mu s(t)\big) \\
&\quad - e^{-\mu\tau}\Big(1 - \frac{A(1+\alpha_1 s(t))}{(\mu+\alpha_1 A)s(t)}\Big)\Big(\frac{\beta s(t) i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{e^{-\mu\tau}\beta s(t)i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}-(\mu_1+\gamma)i(t)+(\mu_1+\gamma)[i(t)-i(t-\tau)]\\
&=-\frac{e^{-\mu\tau}(A-\mu s(t))^2}{(\mu+\alpha_1 A)s(t)}+(\mu_1+\gamma)\Big(\frac{R_{01}(1+\alpha_1 s(t))}{(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}-1\Big)i(t-\tau)
\end{aligned}
$$

Since $\frac{1+\alpha_1 s(t)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}\le 1$ for all $t\ge 0$, it follows that

$$
\frac{dV(t)}{dt}\le -\frac{e^{-\mu\tau}(A-\mu s(t))^2}{(\mu+\alpha_1 A)s(t)}+(\mu_1+\gamma)\left(R_{01}-1\right)i(t-\tau).
$$

Therefore, $R_{01}\le 1$ ensures that $\frac{dV(t)}{dt}\le 0$ for all $t\ge 0$, where $\frac{dV(t)}{dt}=0$ holds if $s(t)=\frac{A}{\mu}$ and $i(t)=0$. Hence, it follows from system (2.1) that $\{P_1\}$ is the largest invariant set in $\Big\{(s(t),i(t))|\frac{dV(t)}{dt}=0\Big\}$. From the Lyapunov-LaSalle asymptotic stability, we obtain that P_1 is globally asymptotically stable. This completes the proof. $\square$

Proposition 2.2. *If $R_{01}>1$, then the endemic equilibrium P_1^* is globally asymptotically stable.*

Proof. To prove global stability of the endemic equilibrium, we define a Lyapunov functional $V(t)=V_1(t)+V_2(t)$, with

$$
V_1(t)=e^{-\mu\tau}\int_{s^*}^{s(t)}\Big(1-\frac{s^*(1+\alpha_1 u+\alpha_2 i^*)}{u(1+\alpha_1 s^*+\alpha_2 i^*)}\Big)du+i(t)-i^*-i^*\ln\big(\frac{i(t)}{i^*}\big),
$$

and

$$
V_2(t)=\frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\int_0^{\tau}\Big[\frac{i(t-u)}{i^*}-1-\ln\big(\frac{i(t-u)}{i^*}\big)\Big]du.
$$

We here note that

$$
\frac{\partial V_1}{\partial s}=1-\frac{s^*(1+\alpha_1 s+\alpha_2 i^*)}{s(1+\alpha_1 s^*+\alpha_2 i^*)},\quad \frac{\partial V_1}{\partial i}=1-\frac{i^*}{i},
$$

which implies that the point (s^*,i^*) is a stationary point of the function $V_1(t)$ and it is the unique stationary point and the global minimum of this function. Using the relations

$$
A=\mu s^*+\frac{\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*},\quad \mu_1+\gamma=\frac{e^{-\mu\tau}\beta s^*}{1+\alpha_1 s^*+\alpha_2 i^*},
$$

the time derivative of the function $V_1(t)$ along the positive solution of system (2.1) becomes

$$
\begin{aligned}
\frac{dV_1(t)}{dt}&=e^{-\mu\tau}\Big(1-\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big)\Big(A-\mu s(t)-\frac{\beta s(t)i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}\Big)\\
&\quad+\big(1-\frac{i^*}{i(t)}\big)\Big(\frac{e^{-\mu\tau}\beta s(t)i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}-(\mu_1+\gamma)i(t)\Big)\\
&=e^{-\mu\tau}\Big(1-\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big)\\
&\quad\times\Big(\mu(s^*-s(t))+\frac{\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}-\frac{\beta s(t)i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}\Big)\\
&\quad+e^{-\mu\tau}\big(1-\frac{i^*}{i(t)}\big)\Big(\frac{\beta s(t)i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}-\frac{\beta s^*}{1+\alpha_1 s^*+\alpha_2 i^*}i(t)\Big),
\end{aligned}
\tag{2.2}
$$

and the time derivative of the function $V_2(t)$ becomes

$$\frac{dV_2(t)}{dt} = \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^* + \alpha_2 i^*}\Big[-\frac{i(t-\tau)}{i^*} + \frac{i(t)}{i^*} + \ln\big(\frac{i(t-\tau)}{i(t)}\big)\Big]. \tag{2.3}$$

From (2.2) and (2.3), we obtain

$$\begin{aligned}
&\frac{dV(t)}{dt}\\
&= -\frac{e^{-\mu\tau}\mu(s(t)-s^*)^2(1+\alpha_2 i^*)}{(1+\alpha_1 s^*+\alpha_2 i^*)} + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\\
&\quad \times \Big(1-\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big)\Big(1-\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i^*(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\Big)\\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\big(1-\frac{i^*}{i(t)}\big)\Big(\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i^*(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))} - \frac{i(t)}{i^*}\Big)\\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[-\frac{i(t-\tau)}{i^*} + \frac{i(t)}{i^*} + \ln\big(\frac{i(t-\tau)}{i(t)}\big)\Big]\\
&= -\frac{e^{-\mu\tau}\mu(s(t)-s^*)^2(1+\alpha_2 i^*)}{(1+\alpha_1 s^*+\alpha_2 i^*)} + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[2-\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\\
&\quad + \frac{i(t-\tau)(1+\alpha_1 s(t)+\alpha_2 i^*)}{i^*(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))} - \frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\\
&\quad - \frac{i(t-\tau)}{i^*} + \ln\big(\frac{i(t-\tau)}{i(t)}\big)\Big]\\
&= -\frac{e^{-\mu\tau}\mu(s(t)-s^*)^2(1+\alpha_2 i^*)}{(1+\alpha_1 s^*+\alpha_2 i^*)}\\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[1-\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)} + \ln\Big(\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big)\Big]\\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[1-\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\\
&\quad + \ln\Big(\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\Big)\Big]\\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[1-\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*} + \ln\Big(\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}\Big)\Big]\\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[-1+\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}\\
&\quad + \frac{i(t-\tau)(1+\alpha_1 s(t)+\alpha_2 i^*)}{i^*(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))} - \frac{i(t-\tau)}{i^*}\Big] + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[\ln\Big(\frac{i(t-\tau)}{i(t)}\Big)\\
&\quad - \ln\Big(\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big) - \ln\Big(\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\Big)\\
&\quad - \ln\Big(\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}\Big)\Big].
\end{aligned}$$

Therefore,

$$\frac{dV(t)}{dt}$$

$$
\begin{aligned}
&= -\frac{e^{-\mu\tau}\mu(s(t)-s^*)^2(1+\alpha_2 i^*)}{(1+\alpha_1 s^*+\alpha_2 i^*)} + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*} \\
&\quad \times \Big(1-\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big)\Big(1-\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i^*(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\Big) \\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\big(1-\frac{i^*}{i(t)}\big)\Big(\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i^*(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))} - \frac{i(t)}{i^*}\Big) \\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[-\frac{i(t-\tau)}{i^*}+\frac{i(t)}{i^*}+\ln\big(\frac{i(t-\tau)}{i(t)}\big)\Big] \\
&= -\frac{e^{-\mu\tau}\mu(s(t)-s^*)^2(1+\alpha_2 i^*)}{(1+\alpha_1 s^*+\alpha_2 i^*)} + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[2-\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)} \\
&\quad + \frac{i(t-\tau)(1+\alpha_1 s(t)+\alpha_2 i^*)}{i^*(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))} - \frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))} \\
&\quad - \frac{i(t-\tau)}{i^*} + \ln\big(\frac{i(t-\tau)}{i(t)}\big)\Big] \\
&= -\frac{e^{-\mu\tau}\mu(s(t)-s^*)^2(1+\alpha_2 i^*)}{(1+\alpha_1 s^*+\alpha_2 i^*)} \\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[1-\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}+\ln\Big(\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big)\Big] \\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[1-\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))} \\
&\quad + \ln\Big(\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\Big)\Big] \\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[1-\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}+\ln\Big(\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}\Big)\Big] \\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[-1+\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}+\frac{i(t-\tau)(1+\alpha_1 s(t)+\alpha_2 i^*)}{i^*(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))} \\
&\quad - \frac{i(t-\tau)}{i^*}\Big] + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[\ln\big(\frac{i(t-\tau)}{i(t)}\big) - \ln\Big(\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big) \\
&\quad - \ln\Big(\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\Big) - \ln\Big(\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}\Big)\Big].
\end{aligned}
$$

Since

$$
\begin{aligned}
\ln\big(\frac{i(t-\tau)}{i(t)}\big) &= \ln\Big(\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big) + \ln\Big(\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\Big) \\
&\quad + \ln\Big(\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&-1+\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}+\frac{i(t-\tau)(1+\alpha_1 s(t)+\alpha_2 i^*)}{i^*(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}-\frac{i(t-\tau)}{i^*} \\
&= -\frac{\alpha_2(1+\alpha_1 s(t))(i(t-\tau)-i^*)^2}{i^*(1+\alpha_1 s(t)+\alpha_2 i^*)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{dV(t)}{dt} \\
&= -\frac{e^{-\mu\tau}\mu(s(t)-s^*)^2(1+\alpha_2 i^*)}{(1+\alpha_1 s^*+\alpha_2 i^*)} + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*} \\
&\quad \times \Big[1 - \frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)} + \ln\Big(\frac{s^*(1+\alpha_1 s(t)+\alpha_2 i^*)}{s(t)(1+\alpha_1 s^*+\alpha_2 i^*)}\Big)\Big] \\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[1 - \frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))} \\
&\quad + \ln\Big(\frac{s(t)i(t-\tau)(1+\alpha_1 s^*+\alpha_2 i^*)}{s^* i(t)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}\Big)\Big] \\
&\quad + \frac{e^{-\mu\tau}\beta s^* i^*}{1+\alpha_1 s^*+\alpha_2 i^*}\Big[1 - \frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*} \\
&\quad + \ln\Big(\frac{1+\alpha_1 s(t)+\alpha_2 i(t-\tau)}{1+\alpha_1 s(t)+\alpha_2 i^*}\Big)\Big] \\
&\quad - \frac{e^{-\mu\tau}\beta s^* i^* \alpha_2(1+\alpha_1 s(t))(i(t-\tau)-i^*)^2}{i^*(1+\alpha_1 s^*+\alpha_2 i^*)(1+\alpha_1 s(t)+\alpha_2 i^*)(1+\alpha_1 s(t)+\alpha_2 i(t-\tau))}.
\end{aligned} \tag{2.4}
$$

It is easy to see that the first and the last terms in (2.4) are non-positive and since the function $g(x) = 1-x+\ln(x)$ is always non-positive for any $x > 0$, and $g(x) = 0$ if and only if $x = 1$, then the second term, the third term and the fourth term in (2.4) are non-positive. Therefore, $\frac{dV(t)}{dt} \le 0$ for all $t \ge 0$, where the equality holds only at the equilibrium point (s^*, i^*). Hence, the functional V satisfies all the conditions of Theorem 5.2. This proves that P_1^* is globally asymptotically stable. $\square$

3. Global Stability analysis of SEIR model

In this section, we discuss the global stability of a disease-free equilibrium and an endemic equilibrium of system (1.3). Since $\frac{d}{dt}(S+E+I) \le A - \mu(S+E+I)$, we have that $\limsup(S+E+I) \le \frac{A}{\mu}$. Hence we discuss system (1.3) in the closed set:

$$
\Omega =: \big\{(S,E,I) \in (\mathbb{R}^+)^3 | S+E+I \le \frac{A}{\mu}\big\}.
$$

It is easy to show that Ω is positively invariant with respect to system (1.3), which always has a disease-free equilibrium $P_2 = (\frac{A}{\mu}, 0, 0)$. Further, if

$$
R_{02} := \frac{A\beta\sigma}{(\sigma+\mu)(\mu_1+\gamma)(\alpha_1 A+\mu)} > 1,
$$

then (1.3) admits a unique endemic equilibrium $P_2^* = (S^*, I^*, E^*)$, with

$$
\begin{gathered}
S^* = \frac{A[(\sigma+\mu)(\mu_1+\gamma)+\alpha_2\sigma A]}{\alpha_2\sigma\mu A + (\sigma+\mu)(\mu_1+\gamma)[(\alpha_1 A+\mu)(R_{02}-1)+\mu]}, \quad E^* = \frac{\mu_1+\gamma}{\sigma} I^*, \\
I^* = \frac{\sigma A(R_{02}-1)(\alpha_1 A+\mu)}{\alpha_2\sigma\mu A + (\sigma+\mu)(\mu_1+\gamma)[(\alpha_1 A+\mu)(R_{02}-1)+\mu]}.
\end{gathered}
$$

Now we consider the global asymptotic stability of the disease-free equilibrium P_2 and the endemic equilibrium P_2^* by Lyapunov functionals, respectively.

Proposition 3.1. *If $R_{02} \leq 1$, then the disease-free equilibrium P_2 is globally asymptotically stable.*

Proof. Define a Lyapunov functional $V(S,E,I) = V_1(S,E,I) + V_2(S,E,I)$ with

$$V_1(t) = \int_{\frac{A}{\mu}}^{S(t)} \Big(1 - \frac{A(1+\alpha_1 u))}{(\mu+\alpha_1 A)u}\Big) du$$

and

$$V_2(t) = E + \frac{\sigma+\mu}{\sigma} I.$$

We will show that $\frac{dV(t)}{dt} \leq 0$ for all $t \geq 0$. We have

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= \Big(1 - \frac{A(1+\alpha_1 S(t))}{(\mu+\alpha_1 A)S(t)}\Big)\Big(A - \mu S(t) - \frac{\beta S(t) I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}\Big) \\
&= \Big(1 - \frac{A(1+\alpha_1 S(t))}{(\mu+\alpha_1 A)S(t)}\Big)(A - \mu S(t)) \\
&\quad - \Big(1 - \frac{A(1+\alpha_1 S(t))}{(\mu+\alpha_1 A)S(t)}\Big)\Big(\frac{\beta S(t) I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}\Big),
\end{aligned}$$

and

$$\frac{dV_2(t)}{dt} = \dot{E} + \frac{\sigma+\mu}{\sigma}\dot{I} = \frac{\beta S(t) I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)} - \frac{(\sigma+\mu)(\mu_1+\gamma)}{\sigma} I(t).$$

Therefore,

$$\begin{aligned}
\frac{dV(t)}{dt} &= \Big(1 - \frac{A(1+\alpha_1 S(t))}{(\mu+\alpha_1 A)S(t)}\Big)(A - \mu S(t)) \\
&\quad + \Big[\frac{\beta A}{(\mu+\alpha_1 A)} \frac{1+\alpha_1 S(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)} - \frac{(\sigma+\mu)(\mu_1+\gamma)}{\sigma}\Big] I(t) \\
&= \Big(1 - \frac{A(1+\alpha_1 S(t))}{(\mu+\alpha_1 A)S(t)}\Big)(A - \mu S(t)) \\
&\quad + \frac{(\sigma+\mu)(\mu_1+\gamma)}{\sigma}\Big[R_{02}\frac{1+\alpha_1 S(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)} - 1\Big] I(t).
\end{aligned}$$

Hence

$$\frac{dV(t)}{dt} \leq -\frac{(A-\mu S(t))^2}{(\mu+\alpha_1 A)S(t)} + \frac{(\sigma+\mu)(\mu_1+\gamma)}{\sigma}\left[R_{02} - 1\right] I(t).$$

Therefore, $R_{02} \leq 1$ ensures that $\frac{dV(t)}{dt} \leq 0$ for all $t \geq 0$, where $\frac{dV(t)}{dt} = 0$ holds if $S(t) = \frac{A}{\mu}$, $E(t) = 0$ and $I(t) = 0$. Hence, it follows from system (1.3) that $\{P_1\}$ is the largest invariant set in $\Big\{(S,E,I) | \frac{dV(t)}{dt} = 0\Big\}$. From the Lyapunov-LaSalle asymptotic stability, we obtain that P_1 is globally asymptotically stable. This completes the proof. □

Proposition 3.2. *If $R_{02} > 1$, then the disease free equilibrium P_2^* is globally asymptotically stable.*

Proof. Define a Lyapunov functional $V(t) = V_1(t) + V_2(t)$ with

$$V_1(t) = \int_{S^*}^{S(t)} \Big(1 - \frac{S^*(1+\alpha_1 u + \alpha_2 I^*)}{u(1+\alpha_1 S^* + \alpha_2 I^*)}\Big) du$$

and

$$V_2(t) = E(t) - E^* + E^* \ln\big(\frac{E(t)}{E^*}\big) + \frac{\sigma+\mu}{\sigma}\big[I(t) - I^* + I^* \ln\big(\frac{I(t)}{I^*}\big)\big].$$

Using the relations $A = \mu S^* + \frac{\beta S^* I^*}{1+\alpha_1 S^* + \alpha_2 I^*}$, $\frac{\beta S^* I^*}{1+\alpha_1 S^*+\alpha_2 I^*} = (\sigma+\mu)E^*$ and $\sigma E^* = (\mu_1+\gamma)I^*$, the time derivative of the function $V_1(t)$ along the positive solution of system (1.3) becomes

$$\begin{aligned}
&\frac{dV_1(t)}{dt}\\
&= \Big(1 - \frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\Big)\Big(A - \mu S(t) - \frac{\beta S(t) I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}\Big)\\
&= \Big(1 - \frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\Big)\Big(\mu(S^* - S(t)) + \frac{\beta S^* I^*}{1+\alpha_1 S^*+\alpha_2 I^*}\\
&\quad - \frac{\beta S(t) I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}\Big)\\
&= \Big(1 - \frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\Big)(\mu(S^* - S(t)))\\
&\quad + \Big(1 - \frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\Big)\Big(\frac{\beta S^* I^*}{1+\alpha_1 S^*+\alpha_2 I^*} - \frac{\beta S(t) I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}\Big)\\
&= \Big(1 - \frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\Big)(\mu(S^* - S(t)))\\
&\quad + (\sigma+\mu)E^*\Big(1 - \frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\Big)\Big(1 - \frac{S(t)I(t)(1+\alpha_1 S^*+\alpha_2 I^*)}{S^* I^*(1+\alpha_1 S(t)+\alpha_2 I(t))}\Big).
\end{aligned}$$

The time derivative of the function $V_2(t)$ becomes

$$\begin{aligned}
\frac{dV_2(t)}{dt} &= \big(1 - \frac{E^*}{E(t)}\big)\dot{E}(t) + \frac{\sigma+\mu}{\sigma}\big(1 - \frac{I^*}{I(t)}\big)\dot{I}(t)\\
&= \big(1 - \frac{E^*}{E(t)}\big)\Big(\frac{\beta S(t) I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)} - (\sigma+\mu)E(t)\Big)\\
&\quad + \frac{\sigma+\mu}{\sigma}\big(1 - \frac{I^*}{I(t)}\big)(\sigma E(t) - (\mu_1+\gamma)I(t))\\
&= (\sigma+\mu)E^*\Big[\big(1 - \frac{E^*}{E(t)}\big)\Big(\frac{S(t)I(t)(1+\alpha_1 S^*+\alpha_2 I^*)}{S^* I^*(1+\alpha_1 S(t)+\alpha_2 I(t))} - \frac{E(t)}{E^*}\Big)\\
&\quad + \big(1 - \frac{I^*}{I(t)}\big)\big(\frac{E(t)}{E^*} - \frac{I(t)}{I^*}\big)\Big]
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{dV(t)}{dt}\\
&= \Big(1 - \frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\Big)(\mu(S^* - S(t))) + (\sigma+\mu)E^*\\
&\quad \times \Big[3 + \frac{I(t)(1+\alpha_1 S(t)+\alpha_2 I^*)}{I^*(1+\alpha_1 S(t)+\alpha_2 I(t))} - \frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*}\\
&\quad - \frac{E^* S(t) I(t)(1+\alpha_1 S^*+\alpha_2 I^*)}{E(t) S^* I^*(1+\alpha_1 S(t)+\alpha_2 I(t))} - \frac{I(t)}{I^*} - \frac{I^* E(t)}{I(t) E^*}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \Big(1-\frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\Big)(\mu(S^*-S(t)))\\
&\quad + (\sigma+\mu)E^*\Big[\Big(\frac{1+\alpha_1 S(t)+\alpha_2 I(t)}{1+\alpha_1 S(t)+\alpha_2 I^*}-\frac{I(t)}{I^*}-1+\frac{I(t)(1+\alpha_1 S(t)+\alpha_2 I^*)}{I^*(1+\alpha_1 S(t)+\alpha_2 I(t))}\Big)\\
&\quad + \Big(4-\frac{1+\alpha_1 S(t)+\alpha_2 I(t)}{1+\alpha_1 S(t)+\alpha_2 I^*}-\frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\\
&\quad - \frac{E^*S(t)I(t)(1+\alpha_1 S^*+\alpha_2 I^*)}{E(t)S^*I^*(1+\alpha_1 S(t)+\alpha_2 I(t))}-\frac{I^*E(t)}{I(t)E^*}\Big)\Big]\\
&= \Big(1-\frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\Big)(\mu(S^*-S(t)))\\
&\quad + (\sigma+\mu)E^*\Big[\Big(\frac{1+\alpha_1 S(t)+\alpha_2 I(t)}{1+\alpha_1 S(t)+\alpha_2 I^*}-1\Big)\Big(1-\frac{I(t)(1+\alpha_1 S(t)+\alpha_2 I^*)}{I^*(1+\alpha_1 S(t)+\alpha_2 I(t))}\Big)\\
&\quad + \Big(4-\frac{1+\alpha_1 S(t)+\alpha_2 I(t)}{1+\alpha_1 S(t)+\alpha_2 I^*}-\frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\\
&\quad - \frac{E^*S(t)I(t)(1+\alpha_1 S^*+\alpha_2 I^*)}{E(t)S^*I^*(1+\alpha_1 S(t)+\alpha_2 I(t))}-\frac{I^*E(t)}{I(t)E^*}\Big)\Big]\\
&= -\frac{(1+\alpha_2 I^*)(S(t)-S^*)^2}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}-\frac{\alpha_2(1+\alpha_1 S(t))(I(t)-I^*)^2(\sigma+\mu)E^*}{I^*(1+\alpha_1 S(t)+\alpha_2 I(t))(1+\alpha_1 S(t)+\alpha_2 I^*)}\\
&\quad + (\sigma+\mu)E^*\Big(4-\frac{1+\alpha_1 S(t)+\alpha_2 I(t)}{1+\alpha_1 S(t)+\alpha_2 I^*}-\frac{S^*(1+\alpha_1 S(t)+\alpha_2 I^*)}{S(t)(1+\alpha_1 S^*+\alpha_2 I^*)}\\
&\quad - \frac{E^*S(t)I(t)(1+\alpha_1 S^*+\alpha_2 I^*)}{E(t)S^*I^*(1+\alpha_1 S(t)+\alpha_2 I(t))}-\frac{I^*E(t)}{I(t)E^*}\Big).
\end{aligned}
$$

Here $\frac{-(1+\alpha_2 I^*)(S-S^*)^2}{S(1+\alpha_1 S^*+\alpha_2 I^*)} \le 0$ and $\frac{-\alpha_2(1+\alpha_1 S)(I-I^*)^2(\sigma+\mu)E^*}{I^*(1+\alpha_1 S+\alpha_2 I)(1+\alpha_1 S+\alpha_2 I^*)} \le 0$ for all $t \ge 0$. Since the arithmetic mean is greater than or equal to the geometric mean,

$$
4-\frac{1+\alpha_1 S+\alpha_2 I}{1+\alpha_1 S+\alpha_2 I^*}-\frac{S^*(1+\alpha_1 S+\alpha_2 I^*)}{S(1+\alpha_1 S^*+\alpha_2 I^*)}-\frac{E^*SI(1+\alpha_1 S^*+\alpha_2 I^*)}{ES^*I^*(1+\alpha_1 S+\alpha_2 I)}-\frac{I^*E}{IE^*} \le 0
$$

for all $t \ge 0$. Therefore, $\frac{dV(t)}{dt} \le 0$ for all $t \ge 0$, where the equality holds only at the equilibrium point $(S,E,I)=(S^*,E^*,I^*)$. Thus $\{P_2^*\}$ is the largest invariant set in $\Big\{(S,E,I)|\frac{dV(t)}{dt}=0\Big\}$. Consequently, we obtain, by the Lyapunov-LaSalle asymptotic stability theorem, that P_2^* is globally asymptotically stable. This completes the proof. □

4. Numerical Simulations and Concluding Remarks

In this section, we give a numerical simulation supporting the theoretical analysis given in section 2 and 3. We take the parameters of the system (1.1) as follows:

$$
\begin{gathered}
A=0.04,\quad \alpha_1=0.01,\quad \alpha_2=0.01,\quad \mu=0.05,\\
\gamma=0.05,\quad \alpha=0.09,\quad \beta=2.5,\quad \tau=100.
\end{gathered}
$$

Then $R_{01}=0.07$. Therefore, by Proposition 2.1, the free-disease equilibrium P_1 is globally asymptotically stable; see Figure 1.

Now we take the parameters of the system (1.3) as follows:

$$
\begin{gathered}
A=0.04,\quad \alpha_1=0.01,\quad \alpha_2=0.01,\quad \mu=0.05,\\
\gamma=0.05,\quad \alpha=0.09,\quad \beta=2.5,\quad \sigma=0.01.
\end{gathered}
$$

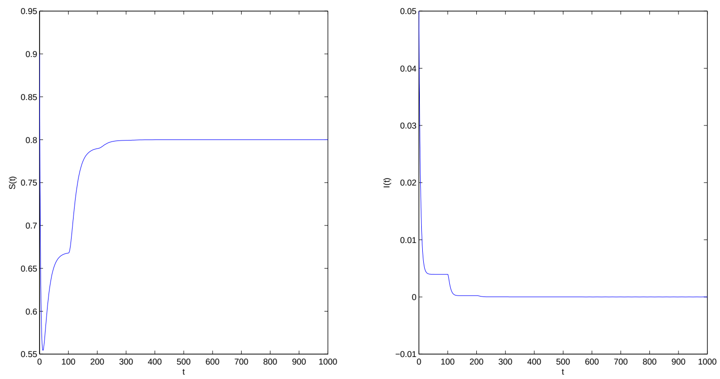

FIGURE 1. Solutions (S, I) of the delay SIR model (1.1) are globally asymptotically stable and converge to the free-disease equilibrium P_1

Then $R_{02} = 1.74$. Therefore, by Proposition 3.2, the endemic equilibrium P_2^* is globally asymptotically stable; see Figure 2.

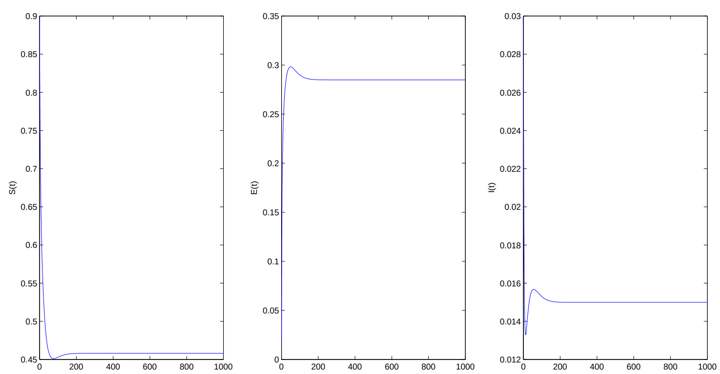

FIGURE 2. Solutions (S, E, I) of the SEIR model (1.3) are globally asymptotically stable and converge to the endemic equilibrium P_2^*

In epidemiological research literatures, a latent or incubation period can be medelled by incorporating it as a delay effect (delayed SIR models) [4], or by introducing an exposed class (SEIR models) [12]. In this paper we consider the global stability for a delayed SIR model with saturated incidence rate (system (1.1)) and for its corresponding SEIR model (system (1.3)).

In the section 2, by modifying Lyapunov functional techniques in Huang and al [13], we proved that if $R_{01} \leq 1$, the disease-free equilibrium is globally asymptotically stable, and this is the only equilibrium. On the contrary, if $R_{01} > 1$, then an endemic equilibrium appears which is globally asymptotically stable. In the section 3, by proposing Lyapunov functional, we showed that if $R_{02} \leq 1$, the disease-free equilibrium is globally asymptotically stable, and this is the only equilibrium. On the contrary, if $R_{02} > 1$, then an endemic equilibrium appears which is globally asymptotically stable.

Finally, numerical simulations are given to support the theoretical analysis and to show that the delayed SIR model (1.1) and its corresponding SEIR model (1.3) can generate different global asymptotic behavior, for example if the incubation period $\tau = 100$ (thus $\sigma = \frac{1}{\tau} = 0.01$), the system (1.1) has only a disease free equilibrium P_1 which is globally asymptotically stable but the system (1.3) has a disease free equilibrium P_2 which is unstable and an endemic equilibrium P_2^* which is globally asymptotically stable (see Figure 1 and Figure 2). In this case we ask the following question: Which model can be adopted for modeling the incubation period in the case of human immunodeficiency virus?

5. Appendix: The Lyapunov-LaSalle theorem

In the following, we present the method of Lyapunov functionals in the context of a delay differential equations,

$$\frac{dx}{dt} = f(x_t), \tag{5.1}$$

where $f : C \to \mathbb{R}^n$ is completely continuous and solutions of (5.1) are unique and continuously dependent on the initial data. We denote by $x(\phi)$ the solution of (5.1) through $(0, \phi)$. For a continuous functional $V : C \to \mathbb{R}$, we define

$$\dot{V} = \limsup_{h\to 0^+} \frac{1}{h}[V(x_h(\phi) - V(\phi)],$$

the derivative of V along a solution of (5.1). To state the Lyapunov-LaSalle type theorem for (5.1), we need the following definition.

Definition 5.1 ([8, p. 30])**.** We say $V : C \to \mathbb{R}$ is a Lyapunov functional on a set G in C for (5.1) if it is continuous on $\overline{G}$ (the closure of G) and $\dot{V} \le 0$ on G. We also define $E = \{\phi \in \overline{G} : \dot{V}(\phi) = 0\}$, and M is the largest set in E which is invariant with respect to (5.1).

The following result is the Lyapunov-LaSalle type theorem for (5.1).

Theorem 5.2 ([8, p. 30])**.** *If V is a Lyapunov functional on G and $x_t(\phi)$ is a bounded solution of (5.1) that stays in G, then ω-limit set $\omega(\phi) \subset M$; that is, $x_t(\phi) \to M$ as $t \to +\infty$.*

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