Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 229, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INTEGRAL BOUNDARY-VALUE PROBLEM FOR IMPULSIVE FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this article, we establish a general framework for finding solutions for impulsive fractional integral boundary-value problems. Then, we prove the existence and uniqueness of solutions by applying well known fixed point theorems. The obtained results are illustrated with an example for their feasibility.

1. Introduction

The purpose of this article is to establish the existence and uniqueness of solution to an integral boundary-value problem for impulsive fractional functional integro-differential equation with infinite delay of the form:

$$\begin{gathered}
{}^cD_t^\alpha x(t) = f(t, x_t, Bx(t)), \quad t \in J = [0, T],\ t \neq t_k, \\
\Delta x(t_k) = Q_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\
\Delta x'(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\
x(t) = \phi(t), \quad t \in (-\infty, 0], \\
ax'(0) + bx'(T) = \int_0^T q(x(s))ds,
\end{gathered} \tag{1.1}$$

where $T > 0$, $\alpha \in (1, 2)$, $a, b \in \mathbb{R}$ such that $a + b \neq 0$. ${}^cD_t^\alpha$ is the Caputo fractional derivative. The functions $f : J \times \mathfrak{B}_h \times X \to X$ and $q : X \to X$ are given functions that satisfy certain assumptions, where $\mathfrak{B}_h$ is a phase space defined in details in Section 2. Here $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $Q_k, I_k \in C(X, X)$, $(k = 1, 2, \dots, m)$, are bounded functions, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$. We assume that $x_t : (-\infty, 0] \to X$, $x_t(s) = x(t+s)$, $s \leq 0$, belong to an abstract phase space $\mathfrak{B}_h$. The term $Bx(t)$ is given by

$$Bx(t) = \int_0^t K(t, s)x(s)ds,$$

2000 *Mathematics Subject Classification.* 26A33, 34K05, 34A12, 34A37.
Key words and phrases. Fractional differential equation; integral boundary condition; impulsive conditions; infinite delay.

Submitted May 29, 2012. Published December 18, 2012.

where $K \in C(D, \mathbb{R}^+)$, the set of all positive functions which are continuous on $D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t < T\}$ and $B^* = \sup_{t\in[0,t]} \int_0^t K(t,s)ds < \infty$.

Fractional differential equations have attracted considerable interest because of their ability to model complex phenomena. Due to the extensive applications of fractional differential equations in engineering and science, research in this area has grown significantly all around the world. For more details about fractional calculus and fractional differential equations we refer the interested readers to the books by Podlubny [14], Hilfer [9] and the papers [1, 3, 4, 10, 13, 15, 16, 17, 20] and references there in.

The impulsive differential equations arising from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occurs. Such processes are naturally seen in biology, physics, engineering, etc. Due to their significance, many authors have been established the solvability of impulsive differential equations. For the general theory and applications of such equations we refer the interested reader to see the papers [4, 16, 20] and references therein.

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics etc. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers [2, 3, 4, 6] and the references therein. On the other hand, we know that delay arises naturally in practical systems due to the transmission of signal or the mechanical transmission. Moreover, the Cauchy problem for various delay equations in Banach spaces has been receiving more and more attention during the past decades, see [10, 11, 13, 15, 19] and for boundary value problem with infinite delay one can see these papers[5, 12] and references therein.

Recently, Michal Feckan et al [7] gave a counter example to show that the formula of solutions for impulsive fractional differential equations used in previous papers are incorrect. Since the authors used that the Caputo derivative ${}^cD_t^\alpha$ restricted on $(a,b]$, $0 < a < b$, is ${}^cD_{a,t}^\alpha$, but unfortunately it does not hold. In [7], the authors introduced a correct formula of solutions for a impulsive Cauchy problem with Caputo fractional derivative. In [18], the author discussed some existence results for boundary value problems for impulsive fractional differential equations. However, the theory of boundary value problem for impulsive fractional differential equations is still in the initial stages. Our work is motivated by these papers [7, 8, 18].

To the best of our knowledge, this is the first paper dealing with integral boundary value problem involving impulsive nonlinear integro-differential equations of fractional order $\alpha \in (1,2)$ with infinite delay. We organize the rest of this paper as follows: in Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. The proofs of our main results are given in Section 3. Section 4 contains an illustrative example.

2. Preliminaries and assumptions

In this section, we shall introduce some basic definitions, properties and lemmas which are required for establishing our results. Let $(X, \|\cdot\|_X)$ be a real Banach space.

To describe fractional order functional differential equations with infinite delay, we need to discuss the abstract phase space $\mathfrak{B}_h$ in a convenient way (see for instance in [20]). Assume that $h : (-\infty, 0] \to (0, \infty)$ is a continuous functions with $l = \int_{-\infty}^{0} h(t)dt < \infty$. For any $a > 0$, we define $\mathfrak{B} = \{\psi : [-a, 0] \to X$ such that $\psi(t)$ is bounded and measurable$\}$ and equip the space $\mathfrak{B}$ with the norm

$$\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|_X, \quad \forall\, \psi \in \mathfrak{B}.$$

Let us define $\mathfrak{B}_h = \{\psi : (-\infty, 0] \to X$, such that for any $c > 0$, $\psi|_{[-c,0]} \in \mathfrak{B}$ and $\int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds < \infty\}$. If $\mathfrak{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds, \quad \forall\, \psi \in \mathfrak{B}_h,$$

then it is clear that $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a Banach space. Now we consider the space

$$\begin{gathered}\mathfrak{B}_h' = \{x : (-\infty, T] \to X \text{ such that } x|_{J_k} \in C(J_k, X) \text{ and there exist}\\ x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-),\ x_0 = \phi \in \mathfrak{B}_h, k = 1, \dots, m\},\end{gathered}$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}], k = 0, 1, 2, \dots, m$. Set $\|\cdot\|_{B_h'}$ to be a seminorm in $\mathfrak{B}_h'$ defined by

$$\|x\|_{\mathfrak{B}_h'} = \sup\{\|x(s)\|_X : s \in [0, T]\} + \|\phi\|_{\mathfrak{B}_h}, x \in \mathfrak{B}_h'.$$

Assume that $x \in \mathfrak{B}_h'$, then for $t \in J$, $x_t \in \mathfrak{B}_h$. Moreover,

$$l\|x(t)\|_X \le \|x_t\|_{\mathfrak{B}_h} \le l \sup_{0<s<t} \|x(s)\|_X + \|x_0\|_{\mathfrak{B}_h},$$

where $l = \int_{-\infty}^{0} h(t)dt$.

Definition 2.1. The Riemann-Liouville fractional integral operator for order $\alpha > 0$, of a function $f : \mathbb{R}_+ \to \mathbb{R}$ and $f \in L^1(\mathbb{R}_+, X)$ is defined by

$$J_t^0 f(t) = f(t),\ J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,\ t > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. Caputo's derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = J^{n-\alpha} f^{(n)}(t), \tag{2.2}$$

for $n - 1 \le \alpha < n,\ n \in N$. If $0 < \alpha \le 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s)ds. \tag{2.3}$$

Obviously, Caputo's derivative of a constant is equal to zero.

Definition 2.3. A function $x \in \mathfrak{B}_h'$ is said to be a solution of the problem-(1.1) if x satisfies the differential equation ${}^cD_t^\alpha x(t) = f(t, x_t, Bx(t))$ a.e. on $J \setminus \{t_1, \dots, t_m\}$ and the following conditions:

$$\begin{gathered}\Delta x(t_k) = Q_k(x(t_k^-)), \quad k = 1, 2, \dots, m,\\ \Delta x'(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m,\\ x(t) = \phi(t), \quad t \in (-\infty, 0],\\ ax'(0) + bx'(T) = \int_0^T q(x(s))ds.\end{gathered} \tag{2.4}$$

Lemma 2.4 ([18, lemma 2.5]). *For $\alpha > 0$, the general solution of fractional differential equation ${}^cD_t^\alpha x(t) = 0$ is given by $x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ ($n = [\alpha] + 1$) and $[\alpha]$ denotes the integer part of the real number α.*

Note that

$$x(t) = x_0 - ct - \int_0^\omega \frac{(\omega - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \tag{2.5}$$

is the solution of the Cauchy problem

$$\begin{gathered} {}^cD_t^\alpha x(t) = h(t), \quad t \in J, \ \alpha \in (1,2), \\ x(0) = x_0 - \int_0^\omega \frac{(\omega - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds. \end{gathered} \tag{2.6}$$

Now, we can obtain the following result.

Lemma 2.5 ([7, lemma 2.6]). *Let $\alpha \in (1,2)$, $c \in \mathbb{R}$ and $h : J \to \mathbb{R}$ be continuous function. A function $x \in C(J, \mathbb{R})$ is a solution of the fractional integral equation* (2.7)

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - \int_0^\omega \frac{(\omega - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + x_0 - c(t - \omega) \tag{2.7}$$

if and only if x is a solution of the fractional Cauchy problem

$$\begin{gathered} {}^cD_t^\alpha x(t) = h(t), \quad t \in J, \\ x(\omega) = x_0, \quad \omega \geq 0. \end{gathered} \tag{2.8}$$

Lemma 2.6. *Let $\alpha \in (1,2)$ and $f : J \times \mathfrak{B}_h \times X \to \mathbb{R}$ be continuously differentiable function. A piecewise continuously differentiable function $x \in \mathfrak{B}'_h$ is a solution of system* (1.1) *if and only if $x \in \mathfrak{B}'_h$ is a solution of the fractional integral equation*

$$x(t) = \begin{cases} \phi(t), \quad \text{if } t \in (-\infty, 0], \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) - \frac{bt}{a+b} \sum_{i=1}^m I_i(x(t_i^-)) \\ -\frac{bt}{a+b} \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x_s, Bx(s))ds + \frac{t}{a+b} \int_0^T q(x(s))ds, \\ \quad \text{if } t \in [0, t_1], \\ \dots, \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) + \sum_{i=1}^k (t - t_i) I_i(x(t_i^-)) \\ + \sum_{i=1}^k Q_i(x(t_i^-)) - \frac{bt}{a+b} \sum_{i=1}^m I_i(x(t_i^-)) \\ -\frac{bt}{a+b} \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x_s, Bx(s))ds + \frac{t}{a+b} \int_0^T q(x(s))ds, \\ \quad \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{2.9}$$

where $k = 1, \ldots, m$.

Proof. Assume x satisfies (1.1). If $t \in [0, t_1]$, then

$${}^cD_t^\alpha x(t) = f(t, x_t, Bx(t)), \ t \in (0, t_1], \ x(0) = \phi(0). \tag{2.10}$$

By using Lemma 2.5, we can write the solution of (2.10) as

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) - ct. \tag{2.11}$$

If $t \in (t_1, t_2]$, then

$$\begin{gathered} {}^cD_t^\alpha x(t) = f(t, x_t, Bx(t)),\ t \in (t_1, t_2], \\ x(t_1^+) = x(t_1^-) + Q_1(x(t_1^-)),\ x'(t_1^+) = x'(t_1^-) + I_1(x(t_1^-)). \end{gathered} \tag{2.12}$$

Again by lemma 2.5, we have the following form of the solution

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds \\ &\quad + x(t_1^-) + Q_1(x(t_1^-)) - d(t-t_1) \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) - ct_1 + Q_1(x(t_1^-)) - d(t-t_1). \end{aligned}$$

Since $x'(t_1^+) = x'(t_1^-) + I_1(x(t_1^-))$, we obtain $d = c - I_1(x(t_1^-))$. Thus

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) + (t-t_1)I_1(x(t_1^-)) + Q_1(x(t_1^-)) - ct.$$

If $t \in (t_2, t_3]$, then by similar way using the lemma 2.5, we have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds \\ &\quad + x(t_2^-) + Q_2(x(t_2^-)) - e(t-t_2). \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) - ct_2 + (t_2-t_1)I_1(x(t_1^-)) \\ &\quad + Q_1(x(t_1^-)) + Q_2(x(t_2^-)) - e(t-t_2). \end{aligned}$$

Since $x'(t_2^+) = x'(t_2^-) + I_2(x(t_2^-))$, we obtain $e = c - I_1(x(t_1^-)) - I_2(x(t_2^-))$. Thus

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) + (t-t_1)I_1(x(t_1^-)) \\ &\quad + (t-t_2)I_2(x(t_2^-)) + Q_1(x(t_1^-)) + Q_2(x(t_2^-)) - ct. \end{aligned}$$

Similarly, if $t \in (t_k, t_{k+1}]$, then again from lemma 2.5, we have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) + \sum_{i=1}^k (t-t_i)I_i(x(t_i^-)) \\ &\quad + \sum_{i=1}^k Q_i(x(t_i^-)) - ct. \end{aligned}$$

By using the integral boundary condition $ax'(0)+bx'(T) = \int_0^T q(x(s))ds$, we obtain

$$\begin{aligned} c &= \frac{b}{a+b} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha-1)} f(s, x_s, Bx(s))ds \\ &\quad + \frac{b}{a+b} \sum_{i=1}^m I_i(x(t_i^-)) - \frac{1}{a+b} \int_0^T q(x(s))ds. \end{aligned}$$

Thus for $t \in [0, t_1]$,

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) - \frac{bt}{a+b} \sum_{i=1}^m I_i(x(t_i^-))$$

$$
- \frac{bt}{a+b}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x_s,Bx(s))ds + \frac{t}{a+b}\int_0^T q(x(s))ds,
$$

and for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, we have

$$
\begin{aligned}
x(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_s,Bx(s))ds + \phi(0) + \sum_{i=1}^{k}(t-t_i)I_i(x(t_i^-)) \\
&+ \sum_{i=1}^{k} Q_i(x(t_i^-)) - \frac{bt}{a+b}\sum_{i=1}^{m} I_i(x(t_i^-)) \\
&- \frac{bt}{a+b}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x_s,Bx(s))ds + \frac{t}{a+b}\int_0^T q(x(s))ds.
\end{aligned}
$$

Conversely, assume that x satisfies (2.9). By a direct computation, it follows that the solution given in (2.9) satisfies the system (1.1). This completes the proof of the lemma. $\square$

Further we introduce the following assumptions to establish our results.

(H1) There exists constants $\mu_1, \mu_2 > 0$, such that

$$
\|f(t,\varphi,x) - f(t,\psi,y)\|_X \le \mu_1\|\varphi-\psi\|_{\mathfrak{B}_h} + \mu_2\|x-y\|_X,
$$

$t \in J$, $\varphi,\psi \in \mathfrak{B}_h$, $x, y \in X$.

(H2) The function $q : X \to X$ is continuous and there exists constant $L_q > 0$, such that

$$
\|q(x(s)) - q(y(s))\|_X \le L_q\|x-y\|_X.
$$

(H3) For each $k = 1, \dots, m$, there exists $\overline{L}, L > 0$, such that

$$
\begin{aligned}
\|Q_k(x) - Q_k(y)\|_X &\le \overline{L}\|x-y\|_X, \quad \forall x, y \in X. \\
\|I_k(x) - I_k(y)\|_X &\le L\|x-y\|_X, \quad \forall x, y \in X.
\end{aligned}
$$

(H4) The function $f : J \times \mathfrak{B}_h \times X \to X$ is continuous and there exist two continuous functions $\mu_1, \mu_2 : J \to (0,\infty)$ such that $\|f(t,\psi,x)\|_X \le \mu_1(t)\|\psi\|_{\mathfrak{B}_h} + \mu_2(t)\|x\|_X$ and $\mu_1^* = \sup_{t\in[0,T]}\mu_1(t)$, $\mu_2^* = \sup_{t\in[0,T]}\mu_2(t)$.

(H5) The functions $q : X \to X$, $I_k : X \to X$ and $Q_k : X \to X$, $k = 1, \dots, m$ are continuous and there exist constants C, ρ, Ω such that $\|q(x)\|_X \le C$, $x \in X$, $\rho = \max_{1\le k\le m, x\in B_r}\{\|I_k(x)\|_X\}$ and $\Omega = \max_{1\le k\le m, x\in B_r}\{\|Q_k(x)\|_X\}$.

3. Existence and uniqueness results

Theorem 3.1. *Suppose that the assumptions* (H1)–(H3) *hold and*

$$
\Lambda = \Big[\frac{(a+(1+\alpha)b)(\mu_1 l + \mu_2 B^*)T^\alpha}{(a+b)\Gamma(\alpha+1)} + \frac{(a+2b)LTm + L_qT^2}{a+b} + \overline{L}m\Big] < 1.
$$

Then (1.1) *has an unique solution.*

Proof. Consider the operator $N : B'_h \to B'_h$ defined by

$$Nx(t) = \begin{cases} \phi(t), \quad \text{if } t \in (-\infty, 0], \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) - \frac{bt}{a+b} \sum_{i=1}^m I_i(x(t_i^-)) \\ -\frac{bt}{a+b} \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x_s, Bx(s))ds + \frac{t}{a+b} \int_0^T q(x(s))ds, \\ \quad \text{if } t \in [0, t_1], \\ \dots, \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s, Bx(s))ds + \phi(0) + \sum_{i=1}^k (t - t_i) I_i(x(t_i^-)) \\ + \sum_{i=1}^k Q_i(x(t_i^-)) - \frac{bt}{a+b} \sum_{i=1}^m I_i(x(t_i^-)) \\ -\frac{bt}{a+b} \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x_s, Bx(s))ds + \frac{t}{a+b} \int_0^T q(x(s))ds, \\ \quad \text{if } t \in (t_k, t_{k+1}], \end{cases}$$

where $k = 1, 2, \dots, m$. Let $y(.) : (-\infty, T] \to X$ be the function defined by

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ 0, & t \in J, \end{cases}$$

then $y_0 = \phi$. For each $z \in C([0, T], \mathbb{R})$ with $z(0) = 0$, we denote

$$\overline{z}(t) = \begin{cases} 0, & t \in (-\infty, 0]; \\ z(t), & t \in J. \end{cases}$$

If $x(.)$ satisfies (2.9) then we can decompose $x(\cdot)$ as $x(t) = y(t) + \overline{z}(t)$, which implies $x_t = y_t + \overline{z}_t$ for $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds + \phi(0) \\ -\frac{bt}{a+b} \sum_{i=1}^m I_i(z(t_i^-)) + \frac{t}{a+b} \int_0^T q(y(s) + \overline{z}(s))ds \\ -\frac{bt}{a+b} \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds, \quad t \in [0, t_1], \\ \dots, \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds + \phi(0) \\ + \sum_{i=1}^k (t - t_i) I_i(z(t_i^-)) + \sum_{i=1}^k Q_i(z(t_i^-)) \\ -\frac{bt}{a+b} \sum_{i=1}^m I_i(z(t_i^-)) + \frac{t}{a+b} \int_0^T q(y(s) + \overline{z}(s))ds \\ -\frac{bt}{a+b} \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds, \quad t \in (t_k, t_{k+1}], \end{cases}$$

where $k = 1, 2, \dots, m$. Set $\mathfrak{B}''_h = \{z \in \mathfrak{B}'_h \text{ such that } z_0 = 0\}$ and let $\|\cdot\|_{\mathfrak{B}''_h}$ be the seminorm in $\mathfrak{B}''_h$ defined by

$$\|z\|_{\mathfrak{B}''_h} = \sup_{t \in J} \|z(t)\|_X + \|z_0\|_{\mathfrak{B}_h} = \sup_{t \in J} \|z(t)\|_X, \; z \in \mathfrak{B}''_h.$$

Thus $(\mathfrak{B}_h'', \|\cdot\|_{\mathfrak{B}_h''})$ is a Banach space. We define the operator $P : \mathfrak{B}_h'' \to \mathfrak{B}_h''$ by

$$Pz(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds + \phi(0) \\ -\frac{bt}{a+b}\sum_{i=1}^m I_i(z(t_i^-)) + \frac{t}{a+b}\int_0^T q(y(s) + \overline{z}(s))ds \\ -\frac{bt}{a+b}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds, \quad t \in [0, t_1], \\ \dots, \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds + \phi(0) \\ +\sum_{i=1}^k (t - t_i) I_i(z(t_i^-)) + \sum_{i=1}^k Q_i(z(t_i^-)) \\ -\frac{bt}{a+b}\sum_{i=1}^m I_i(z(t_i^-)) + \frac{t}{a+b}\int_0^T q(y(s) + \overline{z}(s))ds \\ -\frac{bt}{a+b}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds, \quad t \in (t_k, t_{k+1}], \end{cases}$$

where $k = 1, 2, \dots, m$. It is clear that the operator N has a unique fixed point if and only if P has a unique fixed point. So let us prove that P has a unique fixed point. Let $z, z^* \in \mathfrak{B}_h''$ and $t \in [0, t_1]$ we have

$$\begin{aligned} &\|(Pz)(t) - (Pz^*)(t)\|_X \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) - f(s, y_s + \overline{z}_s^*, B(y(s) \\ &\quad + \overline{z}^*(s)))\|_X \, ds + \frac{bt}{a+b}\sum_{i=1}^m \|I_i(z(t_i^-)) - I_i(z^*(t_i^-))\|_X \\ &\quad + \frac{t}{a+b}\int_0^T \|q(y(s) + \overline{z}(s)) - q(y(s) + \overline{z}^*(s))\|_X \, ds \\ &\quad + \frac{bt}{a+b}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) - f(s, y_s + \overline{z}_s^*, B(y(s) \\ &\quad + \overline{z}^*(s))\|_X \, ds \\ &\leq \Big[\frac{(\mu_1 l + \mu_B^*)(a + (1+\alpha)b)T^\alpha}{(a+b)\Gamma(\alpha+1)} + \frac{T(bLm + L_q T)}{a+b}\Big]\|z - z^*\|_{\mathfrak{B}_h''}. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, then

$$\begin{aligned} &\|(Pz)(t) - (Pz^*)(t)\|_X \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) - f(s, y_s + \overline{z}_s^*, B(y(s) \\ &\quad + \overline{z}^*(s)))\|_X \, ds + \sum_{i=1}^k (t - t_i)\|I_i(z(t_i^-)) - I_i(z^*(t_i^-))\|_X \\ &\quad + \sum_{i=1}^k \|Q_i(z(t_i^-)) - Q_i(z^*(t_i^-))\|_X \\ &\quad + \frac{bt}{a+b}\sum_{i=1}^m \|I_i(z(t_i^-)) - I_i(z^*(t_i^-))\|_X \\ &\quad + \frac{t}{a+b}\int_0^T \|q(y(s) + \overline{z}(s)) - q(y(s) + \overline{z}^*(s))\|_X \, ds \\ &\quad + \frac{bt}{a+b}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) - f(s, y_s + \overline{z}_s^*, B(y(s) \end{aligned}$$

$$
\begin{aligned}
&+ \overline{z}^*(s)))\|_X\, ds \\
&\le \Big[\frac{(a+(1+\alpha)b)(\mu_1 l + \mu_2 B^*)T^\alpha}{(a+b)\Gamma(\alpha+1)} + \frac{(a+2b)LTm + L_q T^2}{a+b} + \overline{L}m\Big]\|z - z^*\|_{\mathfrak{B}''_h}.
\end{aligned}
$$

Thus for all $t \in [0,T]$, we have the estimate

$$
\begin{aligned}
&\|P(z) - P(z^*)\|_{B''_h} \\
&\le \Big[\frac{(a+(1+\alpha)b)(\mu_1 l + \mu_2 B^*)T^\alpha}{(a+b)\Gamma(\alpha+1)} + \frac{(a+2b)LTm + L_q T^2}{a+b} + \overline{L}m\Big]\|z - z^*\|_{B''_h} \\
&\le \Lambda \|z - z^*\|_{\mathfrak{B}''_h}.
\end{aligned}
$$

Since $\Lambda < 1$, the map P is a contraction map and has a unique fixed point $z \in \mathfrak{B}''_h$, which is obviously a solution of the system (1.1) on $(-\infty, T]$. This completes the proof of the theorem. □

Our second existence result is based on the following Krasnoselkii's fixed point theorem.

Theorem 3.2. *Let B be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operator such that (i) $Px + Qy \in B$, whenever $x, y \in B$. (ii) P is compact and continuous. (iii) Q is a contraction mapping. Then there exists $z \in B$ such that $z = Pz + Qz$.*

Theorem 3.3. *Suppose that assumptions* (H1), (H4), (H5) *are satisfied with*

$$
\Delta = \frac{(\mu_1 l + \mu_2 B^*)(a+(1+\alpha)b)T^\alpha}{(a+b)\Gamma(\alpha+1)} < 1.
$$

Then (1.1) *has at least one solution on* $(-\infty, T]$.

Proof. Choose

$$
\begin{aligned}
r \ge \Big[&\|\phi(0)\| + (\rho T + \Omega)m + \frac{(b\rho m + TC)T}{a+b} \\
&+ \frac{(\mu_1^*(\|\phi\| + lr) + \mu_2^* B^* r)(a+(1+\alpha)b)T^\alpha}{(a+b)\Gamma(\alpha+1)}\Big].
\end{aligned}
$$

Define $B_r = \{z \in \mathfrak{B}''_h : \|z\|_{\mathfrak{B}''_h} \le r\}$, then B_r is a bounded, closed convex subset in $\mathfrak{B}''_h$. Let $P_1 : B_r \to B_r$ and $P_2 : B_r \to B_r$ be defined as

$$
\begin{aligned}
(P_1 z)(t) &= \phi(0) + \frac{t}{a+b}\int_0^T q(y(s) + \overline{z}(s))ds + \sum_{i=1}^{k} Q_i(z(t_i^-)) \\
&\quad - \frac{bt}{a+b}\sum_{i=1}^{m} I_i(z(t_i^-)) + \sum_{i=1}^{k}(t - t_i) I_i(z(t_i^-)), \quad t \in J_k. \\
(P_2 z)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds \\
&\quad - \frac{bt}{a+b}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s)))ds, \quad t \in J_k,
\end{aligned}
$$

where $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m$. Now, we proceed the proof in following steps:

Step 1. Let $z, z^* \in B_r$, then we show that $P_1 z + P_2 z^* \in B_r$ for $t \in J_k, k = 0, 1, \dots, m$. We have

$$\begin{aligned}
&\|(P_1 z)(t) + (P_2 z^*)(t)\|_X \\
&\le \|\phi(0)\|_X + \frac{t}{a+b}\int_0^T \|q(y(s)+\overline{z}(s))\|_X\,ds \\
&\quad + \sum_{i=1}^{k}\|Q_i(z(t_i^-))\|_X + \frac{bt}{a+b}\sum_{i=1}^{m}\|I_i(z(t_i^-))\|_X + \sum_{i=1}^{k}(t-t_i)\|I_i(z(t_i^-))\|_X \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s, y_s+\overline{z}_s^*, B(y(s)+\overline{z}^*(s)))\|_X\,ds \\
&\quad + \frac{bt}{a+b}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s, y_s+\overline{z}_s^*, B(y(s)+\overline{z}^*(s)))\|_X\,ds,
\end{aligned} \tag{3.1}$$

we estimate the inequality (3.1), by using (H4) and (H5), as

$$\begin{aligned}
\|(P_1 z)(t) + (P_2 z^*)(t)\|_X \le \Big[&\|\phi(0)\| + (\rho T + \Omega)m + \frac{(b\rho m + TC)T}{a+b} \\
&+ \frac{(\mu_1^*(\|\phi\| + lr) + \mu_2^* B^* r)(a+(1+\alpha)b)T^\alpha}{(a+b)\Gamma(\alpha+1)}\Big],
\end{aligned}$$

which implies that $\|P_1 z + P_2 z^*\|_{B_h''} \le r$.

Step 2. Now, we shall show that the mapping $(P_1 z)(t)$ is continuous on B_r. For this purpose, let $\{z^n\}_{n=1}^\infty$ be a sequence in B_r with $\lim z^n \to z$ in B_r. Then for $t \in J_k, k = 0, 1, \dots, m$, we have

$$\begin{aligned}
&\|(P_1 z^n)(t) - (P_1 z)(t)\|_X \\
&\le \frac{t}{a+b}\int_0^T \|q(y(s)+\overline{z}^n(s)) - q(y(s)+\overline{z}(s))\|_X\,ds \\
&\quad + \sum_{i=1}^{k}\|Q_i(z^n(t_i^-)) - Q_i(z(t_i^-))\|_X + \frac{bt}{a+b}\sum_{i=1}^{m}\|I_i(z^n(t_i^-)) - I_i(z(t_i^-))\|_X \\
&\quad + \sum_{i=1}^{k}(t-t_i)\|I_i(z^n(t_i^-)) - I_i(z(t_i^-))\|_X.
\end{aligned}$$

Since the functions q, Q_k, I_k, $k = 0, 1, \dots, m$, are continuous, hence $\lim_{n\to\infty} P_1 z^n = P_1 z$ in B_r. Which implies that the mapping P_1 is continuous on B_r.

Step 3 $(P_1 z)(t)$ is uniformly bounded follows by the following inequality. For $t \in J_k, k = 0, 1, \dots, m$, we have

$$\begin{aligned}
\|(P_1 z)(t)\|_X &\le \|\phi(0)\|_X + \frac{t}{a+b}\int_0^T \|q(y(s)+\overline{z}(s))\|_X\,ds + \sum_{i=1}^{k}\|Q_i(z(t_i^-))\|_X \\
&\quad + \frac{bt}{a+b}\sum_{i=1}^{m}\|I_i(z(t_i^-))\|_X + \sum_{i=1}^{k}(t-t_i)\|I_i(z(t_i^-))\|_X \\
&\le \|\phi(0)\|_X + \frac{T(TC + b\rho m)}{a+b} + \Omega m + \rho m T.
\end{aligned}$$

Step 4 To show that $P_1(B_r)$ is equicontinuous. Let $\tau_1, \tau_2 \in J_k$, $t_k \le \tau_1 < \tau_2 \le t_{k+1}$, $k = 0, 1, \dots, m$, $z \in B_r$, we have

$$\begin{aligned} \|(P_1 z)(\tau_2) - (P_2 z)(\tau_1)\|_X \le (\tau_2 - \tau_1)[\frac{1}{a+b}\int_0^T \|q(y(s) + \overline{z}(s))\|_X\, ds \\ + \sum_{i=1}^{k} \|I_i(z(t_i^-))\|_X + \frac{b}{a+b}\sum_{i=1}^{m} \|I_i(z(t_i^-))\|_X], \end{aligned}$$

which implies that $P_1(B_r)$ is equicontinuous. Finally, combing Step 2 to Step 4 together with the Ascol's theorem, we conclude that the operator P_1 is a compact.

Step 5. Now, we show that P_2 is a contraction mapping. Let $z, z^* \in B_r$ and $t \in J_k,\ k = 0, 1, \dots, m$, we have

$$\begin{aligned} &\|(P_2 z)(t) - (P_2 z^*)(t)\|_X \\ &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, y_s + \overline{z}_s, B(y(s) + \overline{z}(s))) - f(s, y_s + \overline{z}^*_s, B(y(s) \\ &\quad + \overline{z}^*(s)))\|_X\, ds + \frac{bt}{a+b}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(s, y_s + \overline{z}_s, B(y(s) \\ &\quad + \overline{z}(s))) - f(s, y_s + \overline{z}^*_s, B(y(s) + \overline{z}^*(s)))\|_X\, ds \\ &\le \frac{(\mu_1 l + \mu_2 B^*)(a + (1+\alpha)b)T^\alpha}{(a+b)\Gamma(\alpha+1)} \|z - z^*\|_{B''_h} \\ &\le \Delta \|z - z^*\|_{\mathfrak{B}''_h}, \end{aligned}$$

where

$$\Delta = \frac{(\mu_1 l + \mu_2 B^*)(a + (1+\alpha)b)T^\alpha}{(a+b)\Gamma(\alpha+1)}.$$

As $\Delta < 1$, then P_2 is a contraction map. Thus all the assumptions of the Theorem 3.2 are satisfied and the conclusion of the Theorem 3.2 implies that the system (1.1) has at least one solution on $(-\infty, 0]$. This completes the proof of the theorem. □

4. Application

We consider the model

$$\begin{gathered} {}^C D_t^{3/2} u(t) = \frac{1}{(t+9)^2}\int_{-\infty}^0 e^{2\theta} \sin(\|u(t+\theta)\|_X) d\theta \\ + \frac{1}{(t+7)^2} \sin\big(\|\int_0^t (t-s)u(s)ds\|_X\big), \quad t \in [0,1],\ t \ne t_i, i = 1, 2, 3, \\ u(t) = \phi(t), \quad t \in (-\infty, 0], \\ \Delta u(t_i) = \int_{-\infty}^0 e^{2\theta} \frac{\|u(t_i + \theta)\|_X}{25 + \|u(t_i + \theta)\|_X} d\theta, \\ \Delta u'(t_i) = \int_{-\infty}^0 e^{2\theta} \frac{\|u(t_i + \theta)\|_X}{27 + \|u(t_i + \theta)\|_X} d\theta, \\ u'(0) + u'(1) = \int_0^1 \sin(\frac{1}{2}\|u(s)\|_X) ds, \end{gathered} \tag{4.1}$$

where X is a real Banach space, $0 < t_1 < t_2 < t_3 < 1$ are prefixed numbers and $\phi \in \mathfrak{B}_h$. Let $h(s) = e^{2s}, s < 0$ then $l = \int_{-\infty}^{0} h(s)ds = 1/2$ and define $\|\phi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s<\theta<0} \|\phi(\theta)\|_X \, ds$. Hence for $t \in [0,1]$ and $\phi \in \mathfrak{B}_h$, we set

$$f(t,\phi,Bu(t)) = \frac{1}{(t+9)^2}\int_{-\infty}^{0} h(\theta)\sin(\|\phi(\theta)\|_X)d\theta + \frac{1}{(t+7)^2}\sin(\|Bu(t)\|_X),$$

$$Q_i(\phi) = \int_{-\infty}^{0} h(\theta)\frac{\|\phi(\theta)\|_X}{25+\|\phi(\theta)\|_X}d\theta,$$

$$I_i(\phi) = \int_{-\infty}^{0} h(\theta)\frac{\|\phi(\theta)\|_X}{27+\|\phi(\theta)\|_X}d\theta,$$

where $Bu(t) = \int_0^t (t-s)u(s)ds$, now $B^* = \sup_{t\in[0,1]} \int_0^t (t-s)ds = \frac{1}{2} < \infty$. Then the above equations (4.1) can be written in the abstract form as (1.1). Moreover,

$$\|f(t,\phi,Bu(t)) - f(t,\psi,Bv(t))\|_X \le \frac{1}{81}\|\phi-\psi\|_{\mathfrak{B}_h} + \frac{1}{49}\|Bu(t)-Bv(t)\|_X,$$

$$\|Q_i(\phi) - Q_i(\psi)\|_X \le \frac{1}{25}\|\phi-\psi\|_{\mathfrak{B}_h},$$

$$\|I_i(\phi) - I_i(\psi)\|_X \le \frac{1}{27}\|\phi-\psi\|_{\mathfrak{B}_h},$$

$$\|q(u) - q(v)\|_X \le \frac{1}{2}\|u-v\|_X,$$

therefore, (H1),(H2) and (H3) are satisfied with $\mu_1 = 1/81$, $\mu_2 = 1/49$, $L_q = 1/2$, $L = 1/27$, $\overline{L} = 1/25$. Further,

$$\frac{(a+(1+\alpha)b)(\mu_1 l + \mu_2 B^*)T^\alpha}{(a+b)\Gamma(\alpha+1)} + \frac{(a+2b)LTm + L_q T^2}{a+b} + \overline{L}m \approx 0.558 < 1.$$

Thus, all the assumptions of Theorem 3.1 are satisfied. Hence, the impulsive fractional boundary-value problem (4.1) has a unique solution.

Acknowledgements. The authors are grateful to the anonymous referees for their valuable suggestions that led to the improvement of the original manuscript. J. Dabas has been partially supported by Sponsored Research & Industrial Consultancy, Indian Institute of Technology Roorkee, project IITR/SRIC/247/F.I.G (Scheme-A).

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