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OBLIQUE DERIVATIVE PROBLEMS FOR DEGENERATE LINEAR SECOND-ORDER ELLIPTIC EQUATIONS IN A 3-DIMENSIONAL BOUNDED DOMAIN WITH A BOUNDARY CONICAL POINT

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Abstract. We investigate the behavior of strong solutions to oblique derivative problems for degenerate linear second-order elliptic equations in a 3-dimensional bounded domain with a boundary conical point. We obtain estimates for the local and global solutions and find the best exponents of the continuity at the conical boundary point.

1. Introduction

We investigate the behavior of strong solutions to the oblique derivative problem for degenerate linear second-order elliptic equations in a 3-dimensional bounded domain with the boundary conical point. Such problem was studied for (1.1) in a 2-dimensional bounded domain with a boundary conical point by Borsuk [4], and for the Laplace operator in a 2-dimensional domain by Solonnikov et al [8]-[10], [15]-[17]. They established a-priori estimates for weak solutions in the Sobolev - Kondratiev weighted spaces. Some regularity results were obtained by Lieberman in [12]-[14] for such problems in *smooth* domains.

Let $G \subset \mathbb{R}^3$ be a bounded domain with boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$. We consider the elliptic boundary value problem

$$\begin{gathered}\mathcal{L}[u] \equiv a^{ij}(x)u_{x_ix_j} + a^i(x)u_{x_i} + a(x)u = f(x), \quad x \in G \\ \mathcal{B}[u] \equiv \frac{\partial u}{\partial \vec{n}} + \chi(\omega)\frac{\partial u}{\partial r} + \frac{1}{|x|}\gamma(\omega)u = g(x), \quad x \in \partial G\backslash\mathcal{O}\end{gathered}, \tag{1.1}$$

where $\vec{n}$ denotes the unite exterior normal vector to $\partial G\backslash\mathcal{O}$.

We shall find an exact estimate of the type $u(x) = O(|x|^\alpha)$ for the strong solution to problem (1.1). Analogous estimates have been obtained in [5] for non-degenerate equations and in [3] for degenerate equations, but only with Dirichlet boundary conditions. We derive the Friedrichs-Wirtinger type inequality adapted to our problem, with an exact estimating constant, and establish some auxiliary integro-differential inequalities. We derive weighted estimates for local and global solutions, and find the best exponents of the continuity at the conical boundary point.

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We consider estimates for the solutions to equations with minimal smoothness on the coefficients; this is a principal feature of our work.

We introduce the following notation for a domain G which has a conical point at $\mathcal{O} \in \partial G$.

- $(r, \omega) = (r, \omega_1, \omega_2)$: the spherical coordinates in $\mathbb{R}^3$ with pole $\mathcal{O}$ defined by
$$x_1 = r\cos\omega_1, \quad x_2 = r\sin\omega_1\cos\omega_2, \quad x_3 = r\sin\omega_1\sin\omega_2;$$
- K: an open cone with vertex in $\mathcal{O}$, ∂K: the lateral surface of K;
- $\Omega := K \cap S^2$: a surface on sphere;
- $\partial\Omega$: a circle on the cone, $d\Omega$: the area element of Ω;
- $G_a^b := G \cap \{(r, \omega) : 0 \le a < r < b, \omega \in \Omega\}$: a layer in $\mathbb{R}^3$;
- $\Gamma_a^b := \partial G \cap \{(r, \omega) : 0 \le a < r < b, \omega \in \partial\Omega\}$: the lateral surface of the layer G_a^b;
- $G_d := G \backslash G_0^d$, $\Gamma_d := \partial G \backslash \Gamma_0^d$, $\Omega_\varrho := \overline{G_0^d} \cap \partial B_\varrho(0)$, $0 < \varrho \le d$, $d \in (0, 1)$;
- $G^{(k)} := G_{2^{-(k+1)}d}^{2^{-k}d}$, $k = 0, 1, 2, \dots$.

We recall some well known formulas related to spherical coordinates (r, ω_1, ω_2) centered at the conical point $\mathcal{O}$:
$$|\nabla u|^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}|\nabla_\omega u|^2,$$
where $|\nabla_\omega u|$ denotes the projection of the vector ∇u onto the tangent plane to the unit sphere at the point ω,
$$|\nabla_\omega u|^2 = \frac{1}{q_1}(\frac{\partial u}{\partial \omega_1})^2 + \frac{1}{q_2}(\frac{\partial u}{\partial \omega_2})^2,$$
$$\Delta_\omega u = \frac{1}{J(\omega)}\Big[\frac{\partial}{\partial \omega_1}(\frac{J(\omega)}{q_1} \cdot \frac{\partial u}{\partial \omega_1}) + \frac{\partial}{\partial \omega_2}(\frac{J(\omega)}{q_2} \cdot \frac{\partial u}{\partial \omega_2})\Big],$$
where $J(\omega) = \sin\omega_1$, $q_1 = 1$, $q_2 = \sin^2\omega_1$,
$$ds = r\,dr\,d\sigma$$
denotes the 2-dimensional area element of the lateral surface of the cone K and $d\sigma$ denotes the 1-dimensional length element on $\partial\Omega$ and $d\sigma = \sin\frac{\omega_0}{2}d\omega_2$.

Let us assume, without loss of generality, that there exists $d > 0$ such that G_0^d is a *rotational cone* with the vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (0, \pi)$. Thus
$$\Gamma_0^d = \{(r, \omega_1, \omega_2) : r \in (0, d), \omega_1 = \frac{\omega_0}{2}, \omega_2 \in (-\pi, \pi]\}.$$

We use the standard function spaces: $C^k(\overline{G})$; $C_0^k(G)$; the Lebesgue space $L^p(G)$, $p \ge 1$, with the norm $\|u\|_{L^p(G)} = (\int_G |u|^p dx)^{1/p}$; the Sobolev space $W^{k,p}(G)$ for integer $k \ge 0$, $1 \le p < \infty$, which is a set of all functions $u \in L_p(G)$ such that for every multi-index β with $|\beta| \le k$ the weak partial derivatives $D^\beta u$ belongs to $L_p(G)$, equipped with the finite norm $\|u\|_{W^{k,p}(G)} = (\int_G \sum_{|\beta| \le k} |D^\beta u|^p dx)^{1/p}$; the weighted Sobolev space $V_{p,\alpha}^k(G)$ for integer $k \ge 0$, $1 < p < \infty$ and $\alpha \in \mathbb{R}$, which is the space of distributions $u \in \mathcal{D}'(G)$ with the finite norm $\|u\|_{V_{p,\alpha}^k(G)} = (\int_G \sum_{|\beta| \le k} r^{\alpha + p(|\beta| - k)} |D^\beta u|^p dx)^{1/p}$ and $V_{p,\alpha}^{k-\frac{1}{p}}(\Gamma)$, which is the space of functions φ, given on ∂G, with the norm $\|\varphi\|_{V_{p,\alpha}^{k-\frac{1}{p}}(\partial G)} = \inf \|\Phi\|_{V_{p,\alpha}^k(G)}$, where the infimum is taken over all functions Φ such that $\Phi\Big|_{\partial G} = \varphi$ in the sense of traces.

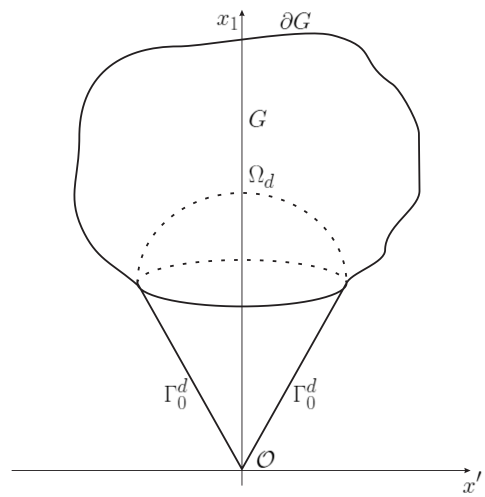

FIGURE 1. Three-dimensional bounded domain with the boundary conical point

For $p = 2$ we use the following notation

$$W^k(G) \equiv W^{k,2}(G), \quad \mathring{W}^k_\alpha(G) = V^k_{2,\alpha}(G), \quad \mathring{W}^{k-\frac{1}{2}}_\alpha(\Gamma) = V^{k-\frac{1}{2}}_{2,\alpha}(\Gamma).$$

Definition 1.1. A function $u(x)$ is called a strong solution of problem (1.1) provided that $u(x) \in W^{2,3}_{loc}(G) \cap W^2(G_\varepsilon) \cap C^0(\overline{G})$ for all $\varepsilon > 0$ and satisfies the equation $\mathcal{L}u = f$ for almost all $x \in G_\varepsilon$ as well as the boundary condition $\mathcal{B}u = g$ in the sense of traces on Γ_ε for all $\varepsilon > 0$.

We use the following assumptions:

(A1) the ellipticity condition

$$\nu|x|^\tau|\xi|^2 \le \sum_{i,j=1}^{3} a^{ij}(x)\xi_i\xi_j \le \mu|x|^\tau|\xi|^2, \quad \forall \xi \in \mathbb{R}^3, \ x \in \overline{G}$$

with $\tau \ge 0$ and the ellipticity constants $\nu, \mu > 0$; $a^{ij}(x) = a^{ji}(x)$, and $\lim_{|x|\to 0} |x|^{-\tau} a^{ij}(x) = \delta^j_i$;

(A2) $a^{ij}(x) \in C^0(\overline{G})$, $a^i(x) \in L^p(G)$, $p > 3$, $a(x) \in L^3(G)$, $f(x) \in L^3(G)$, $g(x) \in \mathring{W}^{1/2}_1(\partial G)$; there exists a monotonically increasing nonnegative function $\mathcal{A}$, continuous at zero, $\mathcal{A}(0) = 0$, such that for $x \in \overline{G}$

$$\Big(\sum_{i,j=1}^{3} \big||x|^{-\tau} a^{ij}(x) - \delta^j_i\big|^2\Big)^{1/2} + |x|^{1-\tau}\Big(\sum_{i=1}^{3} |a^i(x)|^2\Big)^{1/2} + |x|^{2-\tau}|a(x)| \le \mathcal{A}(|x|);$$

(A3) $a(x) \le 0$ in G;

(A4) $\gamma(\omega), \chi(\omega) \in C^1(\partial\Omega)$ and there exist numbers $\gamma_0 > \tan\frac{\omega_0}{2}$, $\chi_0 \ge 0$ such that $\gamma(\omega) \ge \gamma_0 > 0$, $0 \le \chi(\omega) \le \chi_0$;

(A5) there exist numbers $f_1 \ge 0$, $g_1 \ge 0$, $g_0 \ge 0$, $s > 1$ such that

$$|f(x)| \le f_1|x|^{s-2+\tau}, \quad |g(x)| \le g_1|x|^{s-1}, \quad \int_{G_0^\varrho} r|\nabla g|^2 dx \le g_0^2\varrho^{2s}, \ \varrho \in (0,1);$$

(A6) $M_0 = \max_{x\in\overline{G}} |u(x)|$ is known (see [12, 13]).

Remark 1.2. It is easy to verify that $f \in \mathring{W}^0_{1-2\tau}(G)$, by assumptions (A2) and (A5).

The following statement is our main result.

Theorem 1.3. *Let u be a strong solution of* (1.1) *and λ is the smallest positive eigenvalue of* (2.1) *(see subsection 2.1 and Appendix). Let assumptions* (A1)–(A6) *be satisfied with $\mathcal{A}(r)$ being Dini-continuous at zero. Then there are $d \in (0,1)$ and constant $C > 0$ depending only on ν, μ, s, λ, γ_0, χ_0,* $\operatorname{meas} G$, $\operatorname{diam} G$, $\|\chi\|_{C^1(\partial G)}$, $\|\gamma\|_{C^1(\partial G)}$, *on the modulus of continuity of leading coefficients and on the quantity $\int_0^1 \frac{\mathcal{A}(r)}{r}dr$, such that for all $x \in G_0^d$ holds the inequality*

$$\begin{aligned} |u(x)| \le C\Big(|u|_{0,G} + k_s + \|f\|_{\mathring{W}^0_{1-2\tau}(G)} + \|g\|_{\mathring{W}^{1/2}_1(\partial G)}\Big) \\ \times \begin{cases} |x|^\lambda, & \text{if } s > \lambda \\ |x|^\lambda \ln \frac{1}{|x|}, & \text{if } s = \lambda \\ |x|^s, & \text{if } s < \lambda \end{cases}, \end{aligned} \tag{1.2}$$

where

$$k_s = \Big(g_0^2 + \frac{1}{2s}(f_1^2 + g_1^2)\Big)^{1/2}. \tag{1.3}$$

Remark 1.4. For $s \le \lambda$ estimates (1.2) are valid for $\mathcal{A}(r)$ being continuous but not Dini-continuous at zero; see [2] and [5, Theorems 4.19, 4.20].

2. Preliminaries

2.1. **The eigenvalue problem.** Let $\chi(\omega) \ge 0$, $\gamma(\omega) > 0$ be $C^1(\partial\Omega)$-functions and $\vec{\nu}$ be the unite exterior normal vector to ∂K at the points of $\partial\Omega$. Let us consider the following eigenvalue problem for the Laplace-Beltrami operator Δ_ω on the unit sphere,

$$\begin{gathered} \Delta_\omega \psi + \lambda(\lambda+1)\psi(\omega) = 0, \quad \omega \in \Omega \\ \frac{\partial \psi}{\partial \vec{\nu}} + \langle \lambda\chi(\omega) + \gamma(\omega)\rangle \psi(\omega) = 0, \quad \omega \in \partial\Omega \end{gathered}, \tag{2.1}$$

which consists of the determination of all values $\lambda > 0$ (eigenvalues), for which (2.1) has a non-zero weak solutions $\psi(\omega)$ (eigenfunctions).

Remark 2.1. Since $\partial\Omega \subset \partial K$, on $\partial\Omega$ we have $\frac{\partial\psi}{\partial\vec{\nu}} = \frac{\partial\psi}{\partial\omega_1}$.

Definition 2.2. A function ψ is called a weak solution of problem (2.1) provided that $\psi \in W^1(\Omega)$ and satisfies the integral identity

$$\int_\Omega \Big(\frac{1}{q_i}\frac{\partial\psi}{\partial\omega_i}\frac{\partial\eta}{\partial\omega_i} - \lambda(\lambda+1)\psi\eta\Big) d\Omega + \int_{\partial\Omega} \langle \lambda\chi(\omega) + \gamma(\omega)\rangle \psi\eta \, d\sigma = 0$$

for all $\eta(x) \in W^1(\Omega)$.

2.2. **Friedrichs - Wirtinger type inequality.**

Theorem 2.3. *Let λ be the smallest positive eigenvalue of problem* (2.1) *and assumption* (A4) *is satisfied. For any $u \in W^1(\Omega)$ the inequality*

$$\int_\Omega u^2 d\Omega \le \frac{1}{\lambda(\lambda+1)}\Big[\int_\Omega |\nabla_\omega u|^2 d\Omega + \int_{\partial\Omega} \langle \lambda\chi(\omega) + \gamma(\omega)\rangle u^2 \, d\sigma\Big] \tag{2.2}$$

holds.

Proof. Let $u(\omega), \psi(\omega) \in C^1(\Omega)$, $\psi(\omega)$ be the eigenfunction corresponding to the eigenvalue λ. Let us define $v(\omega) \in C^1(\Omega)$ by $u(\omega) = \psi(\omega)v(\omega)$. Then

$$\begin{aligned}
&J|\nabla_\omega u|^2\\
&= \frac{J}{q_1}(\frac{\partial u}{\partial\omega_1})^2 + \frac{J}{q_2}(\frac{\partial u}{\partial\omega_2})^2\\
&\ge \frac{J}{q_1}v^2(\frac{\partial\psi}{\partial\omega_1})^2 + \frac{2J}{q_1}\psi v\frac{\partial\psi}{\partial\omega_1}\frac{\partial v}{\partial\omega_1} + \frac{J}{q_2}v^2(\frac{\partial\psi}{\partial\omega_2})^2 + \frac{2J}{q_2}\psi v\frac{\partial\psi}{\partial\omega_2}\frac{\partial v}{\partial\omega_2}\\
&= \frac{\partial}{\partial\omega_1}(\psi v^2\frac{J}{q_1}\frac{\partial\psi}{\partial\omega_1}) - \psi v^2\frac{\partial}{\partial\omega_1}(\frac{J}{q_1}\frac{\partial\psi}{\partial\omega_1}) + \frac{\partial}{\partial\omega_2}(\psi v^2\frac{J}{q_2}\frac{\partial\psi}{\partial\omega_2}) - \psi v^2\frac{\partial}{\partial\omega_2}(\frac{J}{q_2}\frac{\partial\psi}{\partial\omega_2}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_\Omega |\nabla_\omega u|^2 d\Omega\\
&\ge \int_\Omega \Big[\frac{\partial}{\partial\omega_1}(\psi v^2\frac{J}{q_1}\frac{\partial\psi}{\partial\omega_1}) + \frac{\partial}{\partial\omega_2}(\psi v^2\frac{J}{q_2}\frac{\partial\psi}{\partial\omega_2})\Big]d\omega\\
&\quad - \int_\Omega \psi v^2\Big[\frac{\partial}{\partial\omega_1}(\frac{J}{q_1}\frac{\partial\psi}{\partial\omega_1}) + \frac{\partial}{\partial\omega_2}(\frac{J}{q_2}\frac{\partial\psi}{\partial\omega_2})\Big]d\omega\\
&= \int_{\partial\Omega} \psi v^2\Big(\frac{1}{q_1}\frac{\partial\psi}{\partial\omega_1}\cos(\vec\nu,\omega_1) + \frac{1}{q_2}\frac{\partial\psi}{\partial\omega_2}\cos(\vec\nu,\omega_2)\Big)\, d\sigma - \int_\Omega \psi v^2\Delta_\omega\psi d\Omega.
\end{aligned}$$

Taking into account that $\cos(\vec\nu,\omega_1) = 1$, $\cos(\vec\nu,\omega_2) = 0$, $q_1 = 1$ and (2.1), we obtain

$$\int_\Omega |\nabla_\omega u|^2 d\Omega \ge \lambda(\lambda+1)\int_\Omega \psi^2 v^2 d\Omega - \int_{\partial\Omega}\langle\lambda\chi(\omega) + \gamma(\omega)\rangle\psi^2 v^2\, d\sigma.$$

Returning to $u = \psi v$, we obtain the desired inequality (2.2). The extension to $u \in W^1(\Omega)$ follows directly by the approximation arguments. $\square$

2.3. **Hardy - Friedrichs - Wirtinger type inequality.**

Theorem 2.4. *Let $v \in \mathring{W}^1_{-1}(G^d_0)$ and $\chi(\omega), \gamma(\omega) \in C^0(\partial G)$, $\gamma(\omega) \ge \gamma_0 > 0$, $\chi(\omega) \ge 0$ and $\lambda > 0$ be the smallest positive eigenvalue of* (2.1)*. Then*

$$\int_{G^d_0} r^{-3}v^2 dx \le \frac{1}{\lambda(\lambda+1)}\Big[\int_{G^d_0} r^{-1}|\nabla v|^2 dx + \int_{\Gamma^d_0}\langle\lambda\chi(\omega) + \gamma(\omega)\rangle r^{-2}v^2 ds\Big]. \tag{2.3}$$

Proof. We consider inequality (2.2) for $v(r,\omega)$. Multiplying it by r^{-1} and integrating for $r\in(0,d)$, we obtain

$$\begin{aligned}
&\lambda(\lambda+1)\int_{G_0^d} r^{-3}v^2dx\\
&=\lambda(\lambda+1)\int_0^d\int_\Omega r^{-1}v^2\,dr\,d\Omega\\
&\le\int_0^d\int_\Omega r^{-1}|\nabla_\omega v|^2\,dr\,d\Omega+\int_0^d\int_{\partial\Omega}\langle\lambda\chi(\omega)+\gamma(\omega)\rangle r^{-1}v^2\,dr\,d\sigma\\
&=\int_{G_0^d} r^{-3}|\nabla_\omega v|^2dx+\int_{\Gamma_0^d}\langle\lambda\chi(\omega)+\gamma(\omega)\rangle r^{-2}v^2ds.
\end{aligned}$$

Hence it follows the required inequality (2.3). □

Lemma 2.5. *Let G_0^d be a conical domain and $\nabla u(\varrho,\omega)\in L_2(\Omega)$ for almost everywhere $\varrho\in(0,d)$ and assumption* (A4) *is satisfied. Let $\lambda>0$ be the smallest positive eigenvalue of* (2.1) *and*

$$\widetilde{U}(\varrho)=\int_{G_0^\varrho} r^{-1}|\nabla u|^2dx+\int_{\Gamma_0^\varrho}\gamma(\omega)r^{-2}u^2ds. \tag{2.4}$$

Then

$$\int_\Omega\Big(\varrho u\frac{\partial u}{\partial r}\Big|_{r=\varrho}+\frac12u^2\Big|_{r=\varrho}\Big)d\Omega\le\frac{\varrho}{2\lambda}\widetilde{U}'(\varrho)+\frac12\int_{\partial\Omega}\chi(\omega)u^2d\Omega.$$

Proof. Writing $\widetilde{U}(\varrho)$ in spherical coordinates we have

$$\widetilde{U}(\varrho)=\int_0^\varrho r^{-1}r^2\int_\Omega\Big[(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}|\nabla_\omega u|^2\Big]\,d\Omega\,dr+\int_0^\varrho\frac1r\int_{\partial\Omega}\gamma(\omega)u^2\,d\sigma\,dr;$$

differentiating with respect to ϱ we obtain

$$\widetilde{U}'(\varrho)=\int_\Omega\Big[\varrho(\frac{\partial u}{\partial r})^2\Big|_{r=\varrho}+\frac1\varrho|\nabla_\omega u|^2\Big|_{r=\varrho}\Big]d\Omega+\frac1\varrho\int_{\partial\Omega}\gamma(\omega)u^2\Big|_{r=\varrho}\,d\sigma.$$

Furthermore, for any $\varepsilon>0$,

$$\varrho u\frac{\partial u}{\partial r}=u(\varrho\frac{\partial u}{\partial r})\le\frac\varepsilon2u^2+\frac{1}{2\varepsilon}\varrho^2(\frac{\partial u}{\partial r})^2,$$

by the Cauchy inequality. Choosing $\varepsilon=\lambda$ and applying the Friedrichs - Wirtinger type inequality (2.2), we obtain

$$\begin{aligned}
&\int_\Omega(\varrho u\frac{\partial u}{\partial r}\Big|_{r=\varrho}+\frac12u^2\Big|_{r=\varrho})d\Omega\\
&\le\int_\Omega\Big[\frac{\varepsilon+1}{2}u^2\Big|_{r=\varrho}+\frac{\varrho^2}{2\varepsilon}(\frac{\partial u}{\partial r})^2\Big|_{r=\varrho}\Big]d\Omega\\
&\le\int_\Omega\Big[\frac{\varepsilon+1}{2\lambda(\lambda+1)}|\nabla_\omega u|^2\Big|_{r=\varrho}+\frac{\varrho^2}{2\varepsilon}(\frac{\partial u}{\partial r})^2\Big|_{r=\varrho}\Big]d\Omega\\
&\quad+\frac{\varepsilon+1}{2\lambda(\lambda+1)}\int_{\partial\Omega}\langle\lambda\chi(\omega)+\gamma(\omega)\rangle u^2\Big|_{r=\varrho}\,d\sigma\\
&=\frac{\varrho}{2\lambda}\Big\{\int_\Omega\Big[\varrho(\frac{\partial u}{\partial r})^2\Big|_{r=\varrho}+\frac1\varrho|\nabla_\omega u|^2\Big|_{r=\varrho}\Big]d\Omega+\frac1\varrho\int_{\partial\Omega}\gamma(\omega)u^2\Big|_{r=\varrho}\,d\sigma\Big\}\\
&\quad+\frac12\int_{\partial\Omega}\chi(\omega)u^2\,d\sigma=\frac{\varrho}{2\lambda}\widetilde{U}'(\varrho)+\frac12\int_{\partial\Omega}\chi(\omega)u^2ds.
\end{aligned}$$

□

3. The barrier function

Let G_0^d be a convex rotational cone with a solid angle $\omega_0 \in (0,\pi)$ and the lateral surface Γ_0^d such that $G_0^d \subset \{x_1 \geq 0\}$. Let us define the following liner elliptic operator

$$\mathcal{L}_0 \equiv |x|^{-\tau} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \quad a^{ij}(x) = a^{ji}(x),\ x \in G_0^d,$$

where

$$\nu |x|^{\tau}\xi^2 \leq a^{ij}(x)\xi_i\xi_j \leq \mu |x|^{\tau}\xi^2, \quad \forall x \in G_0^d,\ \forall \xi \in \mathbb{R}^3,$$

where ν, μ are positive constants. Also define the boundary operator

$$\mathcal{B} \equiv \frac{\partial}{\partial \vec{n}} + \chi(\omega)\frac{\partial}{\partial r} + \frac{1}{|x|}\gamma(\omega), \quad \gamma(\omega) \geq \gamma_0 > 0,\ \chi_0 \geq \chi(\omega) \geq 0,\ x \in \Gamma_0^d \backslash \{\mathcal{O}\}.$$

Lemma 3.1 (Existence of the barrier function)**.** *Fix numbers $\gamma_0 > \tan\frac{\omega_0}{2}$, $g_1 \geq 0$, $d \in (0,1)$. There exist $h > 0$ depending only on ω_0, a number $B > 0$, a number $\varkappa_0 \in (0, \gamma_0 \cot\frac{\omega_0}{2} - 1)$, a function $w(x) \in C^1(\overline{G}_0^d) \cap C^2(G_0^d)$ that depends only on ω_0, the ellipticity constants ν and μ of operator $\mathcal{L}_0$, and quantities γ_0, g_1, $\varkappa_0$ such that for any $\varkappa \in (0,\varkappa_0]$ the following inequalities hold*

$$\mathcal{L}_0[w(x)] \leq -\nu h^2 |x|^{\varkappa-1}, \quad x \in G_0^d; \tag{3.1}$$

$$\mathcal{B}[w(x)] \geq g_1 |x|^{\varkappa}, \quad x \in \Gamma_0^d \backslash \mathcal{O}; \tag{3.2}$$

$$0 \leq w(x) \leq C_0(\varkappa_0, B, \omega_0)|x|^{\varkappa+1}, \quad x \in \overline{G_0^d}; \tag{3.3}$$

$$|\nabla w(x)| \leq C_1(\varkappa_0, B, \omega_0)|x|^{\varkappa}, \quad x \in \overline{G_0^d}. \tag{3.4}$$

Proof. We follow the proof in [3, Section 4.2.2] and [5, section 10.1.3]. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. In $\{x_1 \geq 0\}$ we consider the cone K with the vertex $\mathcal{O}$ such that $K \supset G_0^d$. Let ∂K be the lateral surface of K and let on $\partial K \cap x_2\mathcal{O}x_1 = \Gamma_\pm$ be $x_1 = \pm h x_2$, where $h = \cot\frac{\omega_0}{2}$, $0 < \omega_0 < \pi$ such that in the interior of K the inequality $x_1 > h|x_2|$ holds. We shall consider the function

$$w(x) = x_1^{\varkappa-1}(x_1^2 - h^2 x_2^2) + B x_1^{\varkappa+1}, \tag{3.5}$$

with some $\varkappa \in (0,1)$, $B > 0$.

Inequalities (3.1), (3.3) and (3.4) were proved in Lemma 10.18 [5]. Now we shall prove inequality (3.2). Using the spherical coordinates it is easy to derive that

$$\frac{\partial w}{\partial \vec{n}}\Big|_{\Gamma_\pm} = -r^{\varkappa}\frac{h^{\varkappa}}{(1+h^2)^{\frac{1+\varkappa}{2}}}[B(1+\varkappa) + 2(1+h^2)],$$

$$\frac{\partial w}{\partial r}\Big|_{\Gamma_\pm} = B(\varkappa+1)r^{\varkappa}\Big(\frac{h}{\sqrt{1+h^2}}\Big)^{\varkappa+1}.$$

Hence it follows that

$$\mathcal{B}[w]\Big|_{\Gamma_\pm} \geq r^{\varkappa}\frac{h^{\varkappa}}{(\sqrt{1+h^2})^{\varkappa+1}}[Bh\gamma_0 + Bh(\varkappa+1)\chi_0 - B(1+\varkappa) - 2(1+h^2)].$$

Since $0 < \varkappa \leq \varkappa_0 < h\gamma_0 - 1$, $h\gamma_0 > 1$, $\chi_0 \geq 0$, we obtain

$$\mathcal{B}[w]\Big|_{\Gamma_\pm} \geq \frac{h^{\varkappa_0} r^{\varkappa}}{(\sqrt{1+h^2})^{\varkappa_0+1}}\{B[(h\gamma_0 - 1 - \varkappa_0) + \chi_0 h] - 2(1+h^2)\} \geq g_1 r^{\varkappa},$$

for $0 < r < d < 1$, if we choose

$$B \geq \frac{1}{(h\gamma_0 - 1 - \varkappa_0) + h\chi_0}\Big[\frac{g_1(\sqrt{1+h^2})^{\varkappa_0+1}}{h^{\varkappa_0}} + 2(1+h^2)\Big]. \tag{3.6}$$

In this way, we show (3.2). □

Now we can estimate $|u(x)|$ for problem (1.1) in the neighborhood of a conical point.

Theorem 3.2. *Let $u(x)$ be a strong solution of problem* (1.1) *and satisfy assumptions* (A1)–(A6). *Then there exist numbers $d \in (0,1)$ and $\varkappa > 0$ depending only on ν, μ, $\varkappa_0$, f_1, γ_0, τ, s, g_1, M_0 and domain G, such that*

$$|u(x) - u(0)| \leq C_0|x|^{\varkappa+1}, \quad x \in G_0^d, \tag{3.7}$$

where the positive constant C_0 depends only on ν, μ, $\varkappa_0$, f_1, γ_0, s, g_1, M_0, and the domain G, and does not depend on $u(x)$.

Proof. We shall act similarly as in the proof of Theorem 10.19 [5]. We suppose, without loss of generality, that $u(0) \geq 0$. Let us take the barrier function $w(x)$ defined by (3.5) with $\varkappa \in (0, \varkappa_0)$ and the function $v(x) = u(x) - u(0)$. For them we shall show

$$\begin{gathered} \mathcal{L}(Aw(x)) \leq \mathcal{L}v(x), \quad x \in G_0^d, \\ \mathcal{B}[Aw(x)] \geq \mathcal{B}[v(x)], \quad x \in \Gamma_0^d, \\ Aw(x) \geq v(x), \quad x \in \Omega_d \cup \mathcal{O}, \end{gathered}$$

with some constant $A > 0$.

By assumptions (A3), (A5) and Lemma 3.1, calculating the operator $\mathcal{L}$ on function $v(x)$, we obtain

$$\mathcal{L}v(x) = \mathcal{L}[u(x) - u(0)] = \mathcal{L}u(x) - \mathcal{L}u(0) = f(x) - a(x)u(0) \geq f(x) \geq -f_1 r^{s-2+\tau}$$

and since $0 < \varkappa < \varkappa_0$,

$$\mathcal{L}w(x) \leq \mathcal{L}_0 w + a^i(x) w_{x_i} \leq -\nu h^2 r^{\varkappa-1} + \frac{\mathcal{A}(r)}{r} C_1 r^{\varkappa} \leq -\frac{1}{2}\nu h^2 r^{\varkappa_0-1}$$

if, by the continuity of $\mathcal{A}(r)$, we choose $d > 0$ so small that

$$C_1\mathcal{A}(r) \leq C_1\mathcal{A}(d) \leq \frac{1}{2}\nu h^2, \quad r \leq d. \tag{3.8}$$

Hence it follows that

$$\mathcal{L}[Aw(x)] \leq -\frac{1}{2}\nu A h^2 r^{\varkappa_0-1} \leq -f_1 r^{s-2} \leq \mathcal{L}v(x), \quad x \in G_0^d,$$

if we choose A as follows

$$\varkappa_0 \leq s - 1, \quad A \geq \frac{2f_1}{\nu h^2}. \tag{3.9}$$

From (3.2) we obtain

$$\mathcal{B}[Aw]\Big|_{\Gamma_\pm^d} \geq A g_1 r^{\varkappa}. \tag{3.10}$$

Now we calculate $\mathcal{B}[v]$ on $\Gamma_\pm^d$. If $A \geq 1$, from the boundary condition of (1.1) from (3.10) and because of $s > 1$, we obtain

$$\mathcal{B}[v(x)]\Big|_{\Gamma_\pm^d} = \frac{\partial u}{\partial \vec{n}} + \chi(\omega)\frac{\partial u}{\partial r} + \frac{1}{r}\gamma(\omega)[u(x) - u(0)]$$

$$= g(x) - \frac{1}{r}\gamma(\omega)u(0) \le g(x) \le g_1 r^{s-1} \le Ag_1 r^{\varkappa} \le \mathcal{B}[Aw], \quad x \in \Gamma_{\pm}^d.$$

Let us compare $u(x)$ and $w(x)$ on Ω_d. Since $x_1^2 \ge h^2 x_2^2$ in $\overline{K}$, from (3.5), we have

$$\begin{aligned} w(x)\Big|_{r=d} &= [x_1^{\varkappa-1}(x_1^2 - h^2x_2^2) + Bx_1^{\varkappa+1}]\Big|_{r=d} \\ &\ge Bx_1^{\varkappa+1}\Big|_{r=d} \ge Bd^{\varkappa+1}\cos^{\varkappa+1}\frac{\omega_0}{2}. \end{aligned} \tag{3.11}$$

On the other hand,

$$v(x)\Big|_{\Omega_d} = [u(x) - u(0)]\Big|_{\Omega_d} \le M_0. \tag{3.12}$$

By (3.11), (3.12), (3.6),

$$\begin{aligned} &Aw(x)\Big|_{\Omega_d} \\ &\ge ABd^{\varkappa+1}\cos^{\varkappa+1}\frac{\omega_0}{2} \\ &\ge Ad^{\varkappa_0+1}\Big(\frac{h}{\sqrt{1+h^2}}\Big)^{\varkappa_0+1}\Big[\frac{g_1(\sqrt{1+h^2})^{\varkappa_0+1}}{h^{\varkappa_0}} + 2(1+h^2)\Big]\frac{1}{(h\gamma_0 - 1 - \varkappa_0) + h\chi_0} \\ &\ge M_0 \ge v\Big|_{\Omega_d}, \end{aligned}$$

where A is made large enough to satisfy

$$A \ge M_0 \frac{(h\gamma_0 - 1 - \varkappa_0) + h\chi_0}{hd^{\varkappa_0+1}[g_1 + 2h^{\varkappa_0}(\sqrt{1+h^2})^{1-\varkappa_0}]}. \tag{3.13}$$

Choosing the small number $d > 0$ according to (3.8) and numbers $B > 0$, $A \ge 1$ according to (3.6), (3.9) and (3.13), we provide (3.7).

Therefore the functions $v(x)$ and $Aw(x)$ satisfy the comparison principle (see [5, Proposition 10.16]), and we have

$$v(x) = u(x) - u(0) \le w(x) \le Aw(x), \; x \in \overline{G_0^d}.$$

Considering an auxiliary function $v(x) = u(0) - u(x)$ we can derive the estimate

$$u(x) - u(0) \ge -Aw(x).$$

Thus, by (3.3), the theorem is proved. □

4. Global integral weighted estimates

Theorem 4.1. *Let u be a strong solution of problem* (1.1) *and assumptions* (A1)–(A5) *are satisfied. Then $u \in \mathring{W}_1^2(G)$ and*

$$\|u\|_{\mathring{W}_1^2(G)} + \Big(\int_{\partial G} r^{-2}\gamma(\omega)u^2 ds\Big)^{1/2} \le C\Big(|u|_{0,G} + \|f\|_{\mathring{W}^0_{1-2\tau}(G)} + \|g\|_{\mathring{W}_1^{1/2}(\partial G)}\Big), \tag{4.1}$$

where $C > 0$ depends on ν, μ, $\operatorname{diam} G$, $\|\chi\|_{C^1(\partial G)}$, $\|\gamma\|_{C^1(\partial G)}$ *and on the modulus of continuity of leading coefficients.*

Proof. Let us rewrite (1.1) in the following form

$$\Delta u = f(x)|x|^{-\tau} - |x|^{-\tau}[(a^{ij}(x) - \delta_i^j|x|^{\tau})u_{x_ix_j} + a^i(x)u_{x_i} + a(x)u]. \tag{4.2}$$

Integrating $r^{-1}u\Delta u$ over G_ε by parts and using the boundary condition, we have

$$\begin{aligned}
&\int_{G_\varepsilon} r^{-1}u\Delta u\,dx\\
&= \int_{G_\varepsilon} r^{-1}u\frac{\partial}{\partial x_i}\Big(\frac{\partial u}{\partial x_i}\Big)dx\\
&= \int_{\Gamma_\varepsilon} r^{-1}u\frac{\partial u}{\partial \vec{n}}ds - \varepsilon^{-1}\int_{\Omega_\varepsilon} u\frac{\partial u}{\partial r}d\Omega_\varepsilon - \int_{G_\varepsilon} u_{x_i}\Big(r^{-1}u_{x_i} - r^{-2}u\frac{\partial r}{\partial x_i}\Big)dx \qquad (4.3)\\
&= \int_{\Gamma_\varepsilon} r^{-1}u\Big(g(x) - \frac{1}{r}\gamma(\omega)u - \chi(\omega)\frac{\partial u}{\partial r}\Big)ds\\
&\quad - \varepsilon^{-1}\int_{\Omega_\varepsilon} u\frac{\partial u}{\partial r}d\Omega_\varepsilon - \int_{G_\varepsilon} r^{-1}|\nabla u|^2dx + \int_{G_\varepsilon} r^{-3}uu_{x_i}x_i\,dx.
\end{aligned}$$

We consider the last integral above,

$$\begin{aligned}
&\int_{G_\varepsilon} r^{-3}uu_{x_i}x_i\,dx\\
&= \frac{1}{2}\int_{G_\varepsilon} r^{-3}x_i\frac{\partial u^2}{\partial x_i}\,dx\\
&= \frac{1}{2}\int_{\Gamma_\varepsilon} r^{-3}u^2x_i\cos(\vec{n},x_i)\,ds \qquad (4.4)\\
&\quad - \frac{1}{2}\int_{\Omega_\varepsilon} r^{-3}u^2x_i\cos(\vec{n},x_i)d\Omega_\varepsilon - \frac{1}{2}\int_{G_\varepsilon} u^2\frac{\partial}{\partial x_i}(r^{-3}x_i)\,dx.
\end{aligned}$$

However,

$$\sum_{i=1}^{3}\frac{\partial}{\partial x_i}(r^{-3}x_i) = \sum_{i=1}^{3}\big(r^{-3} - 3r^{-4}x_i\frac{\partial r}{\partial x_i}\big) = 3r^{-3} - 3r^{-4}\frac{r^2}{r} = 0. \qquad (4.5)$$

Thus, because of

$$x_i\cos(\vec{n},x_i)\Big|_{\Omega_\varepsilon} = \varepsilon$$

equality (4.4) takes the form

$$\begin{aligned}
&\int_{G_\varepsilon} r^{-3}uu_{x_i}x_i\,dx\\
&= \frac{1}{2}\int_{\Gamma_\varepsilon} r^{-3}u^2x_i\cos(\vec{n},x_i)ds - \frac{\varepsilon^{-2}}{2}\int_{\Omega_\varepsilon} u^2d\Omega_\varepsilon \qquad (4.6)\\
&= \frac{1}{2}\int_{\Gamma_\varepsilon^d} r^{-3}u^2x_i\cos(\vec{n},x_i)ds + \frac{1}{2}\int_{\Gamma_d} r^{-3}u^2x_i\cos(\vec{n},x_i)ds - \frac{1}{2}\int_{\Omega} u^2d\Omega.
\end{aligned}$$

We know that (see [3, Lemma 1.3.2])

$$x_i\cos(\vec{n},x_i)\Big|_{\Gamma_0^d} = 0. \qquad (4.7)$$

By (4.7) and

$$\frac{\partial r}{\partial \vec{n}} = \frac{\partial r}{\partial x_i}\cos(\vec{n},x_i) = \frac{x_i}{r}\cos(\vec{n},x_i),$$

equation (4.6) takes the form

$$\int_{G_\varepsilon} r^{-3}uu_{x_i}x_i\,dx = -\frac{1}{2}\int_{\Omega} u^2d\Omega + \frac{1}{2}\int_{\Gamma_d} r^{-2}u^2\frac{\partial r}{\partial \vec{n}}ds.$$

Inserting it to equality (4.3) we obtain

$$\begin{aligned}\int_{G_\epsilon} r^{-1}u\Delta u dx &= \int_{\Gamma_\varepsilon} r^{-1}u(g-\frac{1}{r}\gamma(\omega)u-\chi(\omega)\frac{\partial u}{\partial r})ds-\varepsilon^{-1}\int_{\Omega_\varepsilon} u\frac{\partial u}{\partial r}d\Omega_\varepsilon \\ &\quad -\int_{G_\varepsilon} r^{-1}|\nabla u|^2dx-\frac{1}{2}\int_\Omega u^2d\Omega+\frac{1}{2}\int_{\Gamma_d} r^{-2}u^2\frac{\partial r}{\partial \vec{n}}ds.\end{aligned} \tag{4.8}$$

Let us multiply both sides of (4.2) by $r^{-1}u$ and integrate over G_ε

$$\begin{aligned}&\int_{G_\varepsilon} r^{-1}u\Delta u dx \\ &= \int_{G_\varepsilon} r^{-1-\tau}ufdx-\int_{G_\varepsilon} r^{-1-\tau}u[(a^{ij}(x)-\delta_i^j r^\tau)u_{x_ix_j}+a^i(x)u_{x_i}+a(x)u]dx.\end{aligned} \tag{4.9}$$

From (4.8) and (4.9) we have

$$\begin{aligned}&\int_{G_\varepsilon} r^{-1}|\nabla u|^2dx+\int_{\Gamma_\varepsilon}\gamma(\omega)r^{-2}u^2ds+\frac{1}{2}\int_\Omega u^2d\Omega \\ &= \int_{\Gamma_\varepsilon} r^{-1}ugds-\int_{\Gamma_\varepsilon}\chi(\omega)r^{-1}u\frac{\partial u}{\partial r}ds-\varepsilon^{-1}\int_{\Omega_\varepsilon} u\frac{\partial u}{\partial r}d\Omega_\varepsilon+\frac{1}{2}\int_{\Gamma_d} r^{-2}u^2\frac{\partial r}{\partial \vec{n}}ds \\ &\quad -\int_{G_\varepsilon} r^{-1-\tau}ufdx+\int_{G_\varepsilon} r^{-1-\tau}u[(a^{ij}(x)-\delta_i^j r^\tau)u_{x_ix_j}+a^i(x)u_{x_i}+a(x)u]dx.\end{aligned} \tag{4.10}$$

To estimate the integral over Ω_ε in the above equation we consider the function

$$M(\varepsilon)=\max_{x\in\overline{\Omega}_\varepsilon}|u(x)|. \tag{4.11}$$

Then, because of $u\in C^0(\overline{G})$,

$$\lim_{\varepsilon\to+0}M(\varepsilon)=|u(0)|. \tag{4.12}$$

□

Now we proof the following lemma.

Lemma 4.2. *There exists a positive constant c_0, which depends only on ν, μ, G, $\max_{x,y\in G}\mathcal{A}(|x-y|)$, $\|\chi\|_{C^1(\partial G)}$, $\|\gamma\|_{C^1(\partial G)}$ such that*

$$\lim_{\varepsilon\to+0}\varepsilon^{-1}\Big|\int_{\Omega_\varepsilon} u\frac{\partial u}{\partial r}d\Omega_\varepsilon\Big|\le c_0|u(0)|^2. \tag{4.13}$$

Proof. Considering the set $G_\varepsilon^{2\varepsilon}$ we have $\Omega_\varepsilon\subset\partial G_\varepsilon^{2\varepsilon}$. Using the following inequality (see [14, Lemma 6.36])

$$\int_{\Omega_\varepsilon}|w|d\Omega_\varepsilon\le c\int_{G_\varepsilon^{2\varepsilon}}(|w|+|\nabla w|)dx,$$

where c is dependent only on the domain G and putting $w=u\frac{\partial u}{\partial r}$ we obtain

$$|w|+|\nabla w|\le c_1(r^2u_{xx}^2+|\nabla u|^2+r^{-2}u^2).$$

Therefore,

$$\int_{\Omega_\varepsilon}|u\frac{\partial u}{\partial r}|d\Omega_\varepsilon\le c\int_{G_\varepsilon^{2\varepsilon}}(r^2u_{xx}^2+|\nabla u|^2+r^{-2}u^2)dx. \tag{4.14}$$

Let us consider new variable x', which is defined by $x = \varepsilon x'$ and the sets $G_{\varepsilon/2}^{5\varepsilon/2}$ and $G_{\varepsilon}^{2\varepsilon} \subset G_{\varepsilon/2}^{5\varepsilon/2}$. Then the function $w(x') = u(\varepsilon x')$ satisfies in $G_{1/2}^{5/2}$ the following problem for the uniformly elliptic equation

$$\begin{gathered} \varepsilon^{-\tau} a^{ij}(\varepsilon x') w_{x_i' x_j'} + \varepsilon^{1-\tau} a^i(\varepsilon x') w_{x_i'} + \varepsilon^{2-\tau} a(\varepsilon x') w = \varepsilon^{2-\tau} f(\varepsilon x'), \quad x' \in G_{1/2}^{5/2} \\ \frac{\partial w}{\partial \vec{n}'} + \chi(\omega) \frac{\partial w}{\partial r'} + \frac{1}{|x'|} \gamma(\omega) w = \varepsilon g(\varepsilon x'), \quad x' \in \Gamma_{1/2}^{5/2}. \end{gathered} \tag{4.15}$$

Because of L^2-estimate for the solution of problem (4.15) inside the domain and near a smooth portion of the boundary (see [1, Theorem 15.3]), we obtain

$$\int_{G_1^2} (w_{x'x'}^2 + |\nabla' w|^2 + w^2) dx' \le c_1 \int_{G_{1/2}^{5/2}} (\varepsilon^{4-2\tau} f^2 + w^2) dx' + c_2 \varepsilon^2 \|g\|^2_{W^{1/2}(\Gamma_{1/2}^{5/2})},$$

where $c_1, c_2 > 0$ depend only on ν, μ, G, $\max_{x',y' \in G_{1/2}^{5/2}} \mathcal{A}(|x'-y'|)$, $\|\chi\|_{C^1(\Gamma_{1/2}^{5/2})}$, $\|\gamma\|_{C^1(\Gamma_{1/2}^{5/2})}$.

Now, let us return to the variable x

$$\begin{aligned} &\int_{G_{\varepsilon}^{2\varepsilon}} (r^2 u_{xx}^2 + |\nabla u|^2 + r^{-2} u^2) dx \\ &\le \int_{G_{\varepsilon/2}^{5\varepsilon/2}} r^{-2} u^2 dx + \varepsilon C_1 \|f\|^2_{\overset{\circ}{W}{}^0_{1-2\tau}(G_{\varepsilon/2}^{5\varepsilon/2})} + \varepsilon C_2 \|g\|^2_{\overset{\circ}{W}{}_1^{1/2}(\Gamma_{\varepsilon/2}^{5\varepsilon/2})}. \end{aligned} \tag{4.16}$$

By the Mean Value Theorem (see [5, Theorem 1.58]) with regard to $u \in C^0(\overline{G})$ and (4.11), we have

$$\begin{aligned} \int_{G_{\varepsilon/2}^{5\varepsilon/2}} r^{-2} u^2 dx &= \int_{\varepsilon/2}^{5\varepsilon/2} \int_{\Omega} u^2(r,\omega) d\Omega dr \\ &\le 2\varepsilon \int_{\Omega} u^2(\theta_1 \varepsilon, \omega) d\Omega \le 2\varepsilon M^2(\theta_1 \varepsilon) \cdot \operatorname{meas} \Omega \end{aligned} \tag{4.17}$$

for some $\frac{1}{2} < \theta_1 < \frac{5}{2}$. From (4.14), (4.16)–(4.17) it follows that

$$\varepsilon^{-1} \int_{\Omega_\varepsilon} |u \frac{\partial u}{\partial r}| d\Omega_\varepsilon \le C_3 M^2(\varepsilon) + C_1 \|f\|^2_{\overset{\circ}{W}{}^0_{1-2\tau}(G_{\varepsilon/2}^{5\varepsilon/2})} + C_2 \|g\|^2_{\overset{\circ}{W}{}_1^{1/2}(\Gamma_{\varepsilon/2}^{5\varepsilon/2})}.$$

Hence, by assumptions about functions f, g and (4.12), we obtain (4.13). □

We get the following estimates of integrals from the right side of equality (4.10):

- by the Cauchy inequality, we obtain

$$\begin{aligned} \int_{G_\varepsilon} r^{-1-\tau} u f dx &\le \int_{G_\varepsilon} r^{-1-\tau} |u||f| dx = \int_{G_\varepsilon} (r^{-\frac{3}{2}} |u|)(r^{\frac{1}{2}-\tau} |f|) dx \\ &\le \frac{\delta}{2} \int_{G_\varepsilon} r^{-3} u^2 dx + \frac{1}{2\delta} \int_{G_\varepsilon} r^{1-2\tau} f^2 dx, \quad \forall \delta > 0; \end{aligned} \tag{4.18}$$

- because of $\gamma(\omega) \ge \gamma_0$, we have

$$\begin{aligned} \int_{\Gamma_\varepsilon} r^{-1} u g ds &\le \int_{\Gamma_\varepsilon} r^{-1} |u||g| ds = \int_{\Gamma_\varepsilon} \Big(r^{-1} \sqrt{\gamma(\omega)} |u| \Big) \Big(\frac{1}{\sqrt{\gamma(\omega)}} |g| \Big) ds \\ &\le \frac{\delta_1}{2} \int_{\Gamma_\varepsilon} \gamma(\omega) r^{-2} u^2 ds + \frac{1}{2\delta_1 \gamma_0} \int_{\Gamma_\varepsilon} g^2 ds, \ \forall \delta_1 > 0; \end{aligned} \tag{4.19}$$

• we have

$$\int_{\Gamma_d} r^{-2}u^2\frac{\partial r}{\partial \vec{n}}ds \le \int_{\Gamma_d} r^{-2}u^2 ds \le d^{-2}\int_{\Gamma_d} u^2 ds.$$

Further, we apply [14, Lemma 6.36]

$$d^{-2}\int_{\Gamma_d} u^2 \le \delta_2 d^{-2}\int_{G_d}|\nabla u|^2dx + c_{\delta_2}\int_{G_d}u^2dx, \quad \forall \delta_2>0; \tag{4.20}$$

• by assumption (A2) and the Cauchy inequality, we obtain

$$\begin{aligned}
&\int_{G_\varepsilon} r^{-1}|u|(|r^{-\tau}a^{ij}(x)-\delta_i^j||u_{x_ix_j}| + r^{-\tau}|a^i(x)||u_{x_i}| + r^{-\tau}|a(x)||u|)dx\\
&\le \int_{G_\varepsilon}\mathcal{A}(r)[(r^{1/2}|u_{xx}|)(r^{-\frac{3}{2}}|u|) + r^{-\frac{1}{2}}|\nabla u|(r^{-\frac{3}{2}}|u|) + r^{-3}u^2]dx \qquad (4.21)\\
&\le 2\int_{G_\varepsilon}\mathcal{A}(r)(ru_{xx}^2 + r^{-1}|\nabla u|^2 + r^{-3}u^2)dx;
\end{aligned}$$

• we have

$$-\int_{\Gamma_\varepsilon}\chi(\omega)r^{-1}u\frac{\partial u}{\partial r}ds = -\int_{\Gamma_\varepsilon^d}\chi(\omega)r^{-1}u\frac{\partial u}{\partial r}ds - \int_{\Gamma_d}\chi(\omega)r^{-1}u\frac{\partial u}{\partial r}ds.$$

Because $0\le \chi(\omega)\le\chi_0$,

$$-\int_{\Gamma_d}\chi(\omega)r^{-1}u\frac{\partial u}{\partial r}ds \le d^{-1}\chi_0\int_{\Gamma_d}|u\frac{\partial u}{\partial r}|ds \le C(d,\chi_0)\int_{G_d}(u_{xx}^2+|\nabla u|^2+u^2)dx, \tag{4.22}$$

by [14, Lemma 6.36]. Further

$$\begin{aligned}
-\int_{\Gamma_\varepsilon^d}\chi(\omega)r^{-1}u\frac{\partial u}{\partial r}ds &= -\frac{1}{2}\int_{\Gamma_\varepsilon^d}\chi(\omega)\frac{\partial u^2}{\partial r}\,dr\,d\sigma\\
&= -\frac{1}{2}\sin\frac{\omega_0}{2}\int_{-\pi}^{\pi}\chi(\frac{\omega_0}{2},w_2)\int_\varepsilon^d\frac{\partial u^2(r,\frac{\omega_0}{2},\omega_2)}{\partial r}drd\omega_2 \qquad (4.23)\\
&\le \frac{1}{2}\int_{-\pi}^{\pi}\chi(\frac{\omega_0}{2},\omega_2)u^2(\varepsilon,\frac{\omega_0}{2},\omega_2)d\omega_2,
\end{aligned}$$

by $\omega_0\in(0,\pi)$ and $\chi(\omega)\ge 0$. Hence and from (4.22) we obtain

$$\begin{aligned}
&-\int_{\Gamma_\varepsilon}\chi(\omega)r^{-1}u\frac{\partial u}{\partial r}ds\\
&\le C(\chi_0,d)\int_{G_d}(u_{xx}^2+|\nabla u|^2+u^2)dx + \frac{1}{2}\int_{-\pi}^{\pi}\chi(\frac{\omega_0}{2},\omega_2)u^2(\varepsilon,\frac{\omega_0}{2},\omega_2)d\omega_2.
\end{aligned}$$

Substituting (4.18)–(4.21) and (4.23) in inequality (4.10), we obtain

$$
\begin{aligned}
&\int_{G_\varepsilon} r^{-1}|\nabla u|^2 dx + \int_{\Gamma_\varepsilon} \gamma(\omega) r^{-2} u^2 ds \\
&\le \varepsilon^{-1} \int_{\Omega_\varepsilon} |u \frac{\partial u}{\partial r}| d\Omega_\varepsilon \\
&\quad + \frac{\delta_1}{2} \int_{\Gamma_\varepsilon} \gamma(\omega) r^{-2} u^2 ds + \int_{G_\varepsilon} \mathcal{A}(r)(r u_{xx}^2 + r^{-1}|\nabla u|^2 + r^{-3}u^2) dx \\
&\quad + \frac{\delta}{2} \int_{G_\varepsilon} r^{-3} u^2 dx + C(\chi_0, d) \int_{G_d} (u_{xx}^2 + |\nabla u|^2 + u^2) dx \\
&\quad + \frac{1}{2} \int_{-\pi}^{\pi} \chi(\frac{\omega_0}{2}, \omega_2) u^2(\varepsilon, \frac{\omega_0}{2}, \omega_2) d\omega_2 + \frac{1}{2\delta} \|f\|^2_{\mathring{W}^0_{1-2\tau}(G)} + \frac{1}{2\delta_1 \gamma_0} \|g\|^2_{L^2(\partial G)}.
\end{aligned} \tag{4.24}
$$

We have that $\mathcal{A}(r)$ is continuous in zero and $\mathcal{A}(0) = 0$, by assumption (A2). Thus for all $\delta > 0$ there exists $d > 0$ such that

$$
\mathcal{A}(r) < \delta \quad \text{for all } 0 < r < d.
$$

Assuming that $2\varepsilon < d$, by (4.16) and (4.17), we obtain

$$
\begin{aligned}
&\int_{G_\varepsilon} \mathcal{A}(r)(r u_{xx}^2 + r^{-1}|\nabla u|^2 + r^{-3}u^2) dx \qquad (4.25) \\
&= \int_{G_\varepsilon^{2\varepsilon}} \mathcal{A}(r)(r u_{xx}^2 + r^{-1}|\nabla u|^2 + r^{-3}u^2) dx \\
&\quad + \int_{G_{2\varepsilon}^d} \mathcal{A}(r)(r u_{xx}^2 + r^{-1}|\nabla u|^2 + r^{-3}u^2) dx \\
&\quad + \int_{G_d} \mathcal{A}(r)(r u_{xx}^2 + r^{-1}|\nabla u|^2 + r^{-3}u^2) dx \qquad (4.26) \\
&\le C\mathcal{A}(2\varepsilon) \Big\{ M^2(\varepsilon) + \|f\|^2_{\mathring{W}^0_{1-2\tau}(G_{\varepsilon/2}^{5\varepsilon/2})} + \|g\|^2_{\mathring{W}_1^{1/2}(\Gamma_{\varepsilon/2}^{5\varepsilon/2})} \Big\} \\
&\quad + \delta \int_{G_{2\varepsilon}^d} (r u_{xx}^2 + r^{-1}|\nabla u|^2 + r^{-3}u^2) dx \\
&\quad + C_1(d, \operatorname{diam} G) \int_{G_d} (u_{xx}^2 + |\nabla u|^2 + u^2) dx, \qquad (4.27)
\end{aligned}
$$

for all $\delta > 0$ and $0 < \varepsilon < d/2$. Setting $\varepsilon = 2^{-k-1}d$ to (4.16), we have

$$
\begin{aligned}
&\int_{G^{(k)}} (r u_{xx}^2 + r^{-1}|\nabla u|^2 + r^{-3}u^2) dx \\
&\le C_3 \int_{G^{(k-1)} \cup G^{(k)} \cup G^{(k+1)}} (r^{-3}u^2 + r^{1-2\tau} f^2) dx \\
&\quad + C_4 \inf \int_{G^{(k-1)} \cup G^{(k)} \cup G^{(k+1)}} (r|\nabla \mathcal{G}|^2 + r^{-1}\mathcal{G}^2) dx,
\end{aligned}
$$

where the infimum is taken over the set of all functions $\mathcal{G} \in \mathring{W}_1^1(G)$ such that $\mathcal{G} = g$ on ∂G. Summing these inequalities over $k = 0, 1, \dots, [\log_2(d/4\varepsilon)]$, for all

$\varepsilon \in (0, d/2)$ we obtain

$$\int_{G^d_{2\varepsilon}} (ru^2_{xx} + r^{-1}|\nabla u|^2 + r^{-3}u^2)dx \le C_3 \int_{G^{2d}_\varepsilon} (r^{-3}u^2 + r^{1-2\tau}f^2)dx + C_4\|g\|^2_{\mathring{W}^{1/2}_1(\Gamma^{2d}_\varepsilon)}. \tag{4.28}$$

By inequalities (4.27) and (4.28), we have

$$\begin{aligned}
&\int_{G_\varepsilon} \mathcal{A}(r)(ru^2_{xx} + r^{-1}|\nabla u|^2 + r^{-3}u^2)dx \\
&\le \mathcal{A}(2\varepsilon)\Big\{M^2(\varepsilon) + \|f\|^2_{\mathring{W}^0_{1-2\tau}(G^{5\varepsilon/2}_{\varepsilon/2})} + \|g\|^2_{\mathring{W}^{1/2}_1(\Gamma^{5\varepsilon/2}_{\varepsilon/2})}\Big\} + \delta \int_{G^d_\varepsilon} r^{-3}u^2 dx \\
&\quad + C_1(d, \operatorname{diam} G) \int_{G_d} (u^2_{xx} + |\nabla u|^2 + u^2) + c\Big(\|f\|^2_{\mathring{W}^0_{1-2\tau}(G)} + \|g\|^2_{\mathring{W}^{1/2}_1(\partial G)}\Big).
\end{aligned} \tag{4.29}$$

Thus, from (4.24) and (4.29), choosing $\delta_1 = 1$, we obtain

$$\begin{aligned}
&\int_{G_\varepsilon} r^{-1}|\nabla u|^2 dx + \int_{\Gamma_\varepsilon} \gamma(\omega) r^{-2}u^2 ds \\
&\le \varepsilon^{-1} \int_{\Omega_\varepsilon} |u\frac{\partial u}{\partial r}| d\Omega_\varepsilon + \int_{-\pi}^{\pi} \chi(\frac{\omega_0}{2}, \omega_2) u^2(\varepsilon, \frac{\omega_0}{2}, \omega_2) d\omega_2 \\
&\quad + \mathcal{A}(2\varepsilon)\Big\{M^2(\varepsilon) + \|f\|^2_{\mathring{W}^0_{1-2\tau}(G^{5\varepsilon/2}_{\varepsilon/2})} + \|g\|^2_{\mathring{W}^{1/2}_1(\Gamma^{5\varepsilon/2}_{\varepsilon/2})}\Big\} + \delta \int_{G_\varepsilon} r^{-3}u^2 dx \\
&\quad + C(\operatorname{diam} G, \chi_0, d) \int_{G_d} (u^2_{xx} + |\nabla u|^2 + u^2)dx + \widetilde{C}\Big(\|f\|^2_{\mathring{W}^0_{1-2\tau}(G)} + \|g\|^2_{\mathring{W}^{1/2}_1(\partial G)}\Big)
\end{aligned} \tag{4.30}$$

for any $\delta > 0$, where $\widetilde{C} > 0$ is dependent on γ_0 and is independent of ε. By Lemma 4.2 as well as $u \in C^0(\overline{G})$, we can pass in (4.30) to the limit $\varepsilon \to +0$, using the Fatou Theorem. In this way we obtain

$$\begin{aligned}
&\int_G r^{-1}|\nabla u|^2 dx + \int_{\partial G} \gamma(\omega) r^{-2}u^2 ds \\
&\le \delta \int_G r^{-3}u^2 dx + C \int_G (u^2_{xx} + |\nabla u|^2 + u^2)dx \\
&\quad + \widetilde{C}(|u|^2_{0,G} + \|f\|^2_{\mathring{W}^0_{1-2\tau}(G)} + \|g\|^2_{\mathring{W}^{1/2}_1(\partial G)}).
\end{aligned} \tag{4.31}$$

Now, we consider the first integral of the right side of (4.31). By the Hardy - Friedrichs - Wirtinger type inequality (2.3), we obtain

$$\begin{aligned}
&\int_G r^{-3}u^2 dx \\
&= \int_{G^d_0} r^{-3}u^2 dx + \int_{G_d} r^{-3}u^2 dx \\
&\le \frac{1}{\lambda(\lambda+1)}\Big\{\int_{G^d_0} r^{-1}|\nabla u|^2 dx + \int_{\Gamma^d_0} \langle \lambda\chi(\omega) + \gamma(\omega)\rangle r^{-2}u^2 ds\Big\} + C\int_G u^2 dx \\
&\le \frac{1}{\lambda(\lambda+1)}\Big\{\int_{G^d_0} r^{-1}|\nabla u|^2 dx + \lambda\chi_0 \int_{\Gamma^d_0} r^{-2}u^2 ds + \int_{\Gamma^d_0} \gamma(\omega) r^{-2}u^2 ds\Big\} \\
&\quad + C\int_G u^2 dx,
\end{aligned}$$

since $\chi(\omega) \le \chi_0$. Thus, because of $\gamma(\omega) \ge \gamma_0 > 0$,

$$
\begin{aligned}
&\delta \int_G r^{-3} u^2 dx \\
&\le \frac{\delta}{\lambda(\lambda+1)} \int_G r^{-1} |\nabla u|^2 dx + \frac{\delta}{\lambda+1}(1+\frac{\lambda\chi_0}{\gamma_0}) \int_{\partial G} \gamma(\omega) r^{-2} u^2 ds + C\delta \int_G u^2 dx.
\end{aligned} \tag{4.32}
$$

Choosing small number δ, from (4.31)–(4.32) it follows that

$$
\begin{aligned}
&\int_G r^{-1}|\nabla u|^2 dx + \int_{\partial G} \gamma(\omega) r^{-2} u^2 ds \\
&\le \widetilde{C}_2 \int_G (u_{xx}^2 + |\nabla u|^2 + u^2) dx + \widetilde{C}_1 (\|f\|^2_{\mathring{W}^0_{1-2\tau}(G)} + \|g\|^2_{\mathring{W}^{1/2}_1(\partial G)}).
\end{aligned} \tag{4.33}
$$

By L^2-estimate for solutions of problem (1.1) (see [1, Theorem 15.1]), we have

$$
\int_G (u_{xx}^2 + |\nabla u|^2 + u^2) dx \le c\Big(\|u\|^2_{L^2(G)} + \|f\|^2_{\mathring{W}^0_{1-2\tau}(G)} + \|g\|^2_{\mathring{W}^{1/2}_1(\partial G)}\Big), \tag{4.34}
$$

where the positive constant c dependents only on $\nu, \mu, \tau, d, G, \max_{x,y\in G} \mathcal{A}(|x-y|)$, $\|\chi\|_{C^1(\partial G)}$, $\|\gamma\|_{C^1(\partial G)}$. By (4.32)–(4.34), we have

$$
\begin{aligned}
&\int_G (r^{-1}|\nabla u|^2 + r^{-3}u^2) dx + \int_{\partial G} \gamma(\omega) r^{-2} u^2 ds \\
&\le \widetilde{C}_3 (|u|^2_{0,G} + \|f\|^2_{\mathring{W}^0_{1-2\tau}(G)} \|g\|^2_{\mathring{W}^{1/2}_1(\partial G)}).
\end{aligned} \tag{4.35}
$$

Let us pass in (4.28) to the limit $\varepsilon \to +0$. As a result we obtain

$$
\int_{G_0^d} r u_{xx}^2 dx \le C_3 \int_G r^{-3} u^2 dx + C_3 \|f\|^2_{\mathring{W}^0_{1-2\tau}(G)} + C_4 \|g\|^2_{\mathring{W}^{1/2}_1(\partial G)}. \tag{4.36}
$$

By (4.35)–(4.36) we obtain desired estimate (4.1).

Corollary 4.3. *Let u be a strong solution of problem* (1.1) *and assumptions* (A1)–(A6) *are satisfied. Then $u(0) = 0$.*

Proof. We have $\frac{1}{2}|u(0)|^2 \le |u(x)|^2 + |u(x) - u(0)|^2$, by the Cauchy inequality. Thus

$$
\frac{1}{2}|u(0)|^2 \int_{G_0^d} r^{-3} dx \le \int_{G_0^d} r^{-3}|u(x)|^2 dx + \int_{G_0^d} r^{-3}|u(x)-u(0)|^2 dx. \tag{4.37}
$$

The first integral from the right side is finite by Theorem 4.1. According to Theorem 3.2 we have for the second integral

$$
\begin{aligned}
\int_{G_0^d} r^{-3}|u(x)-u(0)|^2 dx &\le C_0^2 \int_{G_0^d} r^{2\varkappa-1} dx = C_0^2 \operatorname{meas}\Omega \int_0^d r^{2\varkappa+1} dr \\
&= C_0^2 \operatorname{meas}\Omega \frac{d^{2\varkappa+2}}{2\varkappa+2} < \infty.
\end{aligned}
$$

We see that the right side of inequality (4.37) is finite. But if $u(0) \ne 0$, the left side of this inequality is infinite, because of $\int_{G_0^d} r^{-3} dx \sim \int_0^d \frac{dr}{r} = \infty$. It leads to a contradiction. Therefore must be $u(0) = 0$. $\square$

5. Local integral weighted estimates

Theorem 5.1. *Let u be a strong solution of problem* (1.1) *and assumptions* (A1)–(A6) *are satisfied with $\mathcal{A}(r)$ being Dini-continuous at zero. Then there are $d \in (0,1)$ and a constant $C > 0$ depends only on $\nu, \mu, d, \mathcal{A}(d), s, \lambda, \gamma_0, g_1$,* $\operatorname{meas} G$, $\|\chi\|_{C^1(\partial G)}$, $\|\gamma\|_{C^1(\partial G)}$ *and on the quantity $\int_0^d \frac{\mathcal{A}(\tau)}{\tau} d\tau$, such that for all $\varrho \in (0,d)$*

$$\begin{aligned}\|u\|_{\mathring{W}_1^2(G_0^\varrho)} &\le C\Big(|u|_{0,G} + \|f\|_{\mathring{W}^0_{1-2\tau}(G)} + \|g\|_{\mathring{W}_1^{1/2}(\partial G)} + k_s\Big) \\ &\quad \times \begin{cases} \varrho^\lambda, & \text{if } s > \lambda \\ \varrho^\lambda \ln \frac{1}{\varrho}, & \text{if } s = \lambda\ , \\ \varrho^s, & \text{if } s < \lambda \end{cases}\end{aligned} \tag{5.1}$$

where k_s is defined by (1.3).

Proof. By Theorem 4.1, $u \in \mathring{W}_1^2(G)$. We consider the equation of the problem (1.1) in the form (4.2). We multiply both side of (4.2) by $r^{-1}u$ and integrate over the domain G_0^ϱ, $0 < \varrho < d$. As a result we obtain

$$\begin{aligned}\int_{G_0^\varrho} r^{-1} u \Delta u \, dx = \int_{G_0^\varrho} & r^{-1} u\{r^{-\tau} f - [(r^{-\tau} a^{ij}(x) - \delta_i^j) u_{x_i x_j} \\ & + r^{-\tau} a^i(x) u_{x_i} + r^{-\tau} a(x) u]\} dx.\end{aligned} \tag{5.2}$$

On the other hand

$$\int_{G_0^\varrho} r^{-1} u \Delta u \, dx = \int_{G_0^\varrho} r^{-1} u \frac{\partial}{\partial x_i}(u_{x_i}) dx = -\int_{G_0^\varrho} u_{x_i} \frac{\partial}{\partial x_i}(r^{-1} u) + \int_{\partial G_0^\varrho} r^{-1} u \frac{\partial u}{\partial \vec{n}} ds. \tag{5.3}$$

By direct calculations, we have

$$\int_{G_0^\varrho} r^{-1} u \Delta u \, dx = -\int_{G_0^\varrho} r^{-1} |\nabla u|^2 dx + \frac{1}{2} \int_{G_0^\varrho} r^{-3} x_i \frac{\partial u^2}{\partial x_i} dx + \int_{\partial G_0^\varrho} r^{-1} u \frac{\partial u}{\partial \vec{n}} ds. \tag{5.4}$$

Further,

$$\int_{G_0^\varrho} r^{-3} x_i \frac{\partial u^2}{\partial x_i} dx = -\int_{G_0^\varrho} u^2 \frac{\partial}{\partial x_i}(x_i r^{-3}) dx + \int_{\partial G_0^\varrho} r^{-3} u^2 x_i \cos(\vec{n}, x_i) ds.$$

Using the facts that $\partial G_0^\varrho = \Gamma_0^\varrho \cup \Omega_\varrho$, (4.5) and $x_i \cos(\vec{n}, x_i)\Big|_{\Omega_\varrho} = \varrho$, $x_i \cos(\vec{n}, x_i)\Big|_{\Gamma_0^\varrho} = 0$ we obtain

$$\int_{G_0^\varrho} r^{-3} x_i \frac{\partial u^2}{\partial x_i} dx = \varrho^{-2} \int_{\Omega_\varrho} u^2 d\Omega_\varrho = \int_\Omega u^2 d\Omega. \tag{5.5}$$

Now, we have

$$\begin{aligned}\int_{\partial G_0^\varrho} r^{-1} u \frac{\partial u}{\partial \vec{n}} ds &= \int_{\Gamma_0^\varrho} r^{-1} u \frac{\partial u}{\partial \vec{n}} ds + \int_{\Omega_\varrho} \varrho^{-1} u \frac{\partial u}{\partial r} d\Omega_\varrho \\ &= \int_{\Gamma_0^\varrho} r^{-1} u (g - \chi(\omega) \frac{\partial u}{\partial r} - \frac{1}{r} \gamma(\omega) u) ds + \varrho \int_\Omega u \frac{\partial u}{\partial r} d\Omega\,.\end{aligned} \tag{5.6}$$

by the boundary condition of (1.1). From equations (5.4)-(5.6) we obtain

$$\int_{G_0^\varrho} r^{-1} u \Delta u \, dx = -\int_{G_0^\varrho} r^{-1} |\nabla u|^2 dx + \int_\Omega (\varrho u \frac{\partial u}{\partial r} + \frac{1}{2} u^2) d\Omega$$

$$+\int_{\Gamma_0^\varrho} r^{-1}u(g-\chi(\omega)\frac{\partial u}{\partial r}-\frac{1}{r}\gamma(\omega)u)ds.$$

Hence from (5.2),

$$\begin{aligned}&\int_{G_0^\varrho} r^{-1}|\nabla u|^2dx+\int_{\Gamma_0^\varrho}\chi(\omega)r^{-1}u\frac{\partial u}{\partial r}ds+\int_{\Gamma_0^\varrho}\gamma(\omega)r^{-2}u^2ds\\&=\int_\Omega(\varrho u\frac{\partial u}{\partial r}+\frac{1}{2}u^2)d\Omega+\int_{\Gamma_0^\varrho}r^{-1}ugds-\int_{G_0^\varrho}r^{-1-\tau}ufdx\\&\quad+\int_{G_0^\varrho}r^{-1}u[(r^{-\tau}a^{ij}(x)-\delta_i^j)u_{x_ix_j}+r^{-\tau}a^i(x)u_{x_i}+r^{-\tau}a(x)u]dx.\end{aligned}\tag{5.7}$$

Now we estimate terms of the right side of (5.7):

- by the Cauchy inequality and assumption (A2),

$$\begin{aligned}\int_{G_0^\varrho} r^{-1}|u||r^{-\tau}a^{ij}(x)-\delta_i^j||u_{x_ix_j}|dx&\le\mathcal{A}(\varrho)\int_{G_0^\varrho}r^{-1}|u||u_{xx}|dx\\&=\mathcal{A}(\varrho)\int_{G_0^\varrho}(r^{1/2}|u_{xx}|)(r^{-\frac{3}{2}}|u|)dx\\&\le\frac{1}{2}\mathcal{A}(\varrho)\int_{G_0^\varrho}(ru_{xx}^2+r^{-3}u^2)dx;\end{aligned}$$

- similarly

$$\begin{aligned}\int_{G_0^\varrho} r^{-1-\tau}|u||a^i(x)||u_{x_i}|dx&\le\mathcal{A}(\varrho)\int_{G_0^\varrho}r^{-2}|u||\nabla u|dx\\&=\mathcal{A}(\varrho)\int_{G_0^\varrho}(r^{-\frac{3}{2}}|u|)(r^{-\frac{1}{2}}|\nabla u|)dx\\&\le\frac{1}{2}\mathcal{A}(\varrho)\int_{G_0^\varrho}(r^{-3}u^2+r^{-1}|\nabla u|^2)dx;\end{aligned}$$

- by assumption (A2),

$$\int_{G_0^\varrho} r^{-1-\tau}|a(x)|u^2dx\le\mathcal{A}(\varrho)\int_{G_0^\varrho}r^{-3}u^2dx;$$

-

$$\begin{aligned}\int_{G_0^\varrho} r^{-1-\tau}|u||f|dx&=\int_{G_0^\varrho}(r^{-\frac{3}{2}}|u|)(r^{\frac{1}{2}-\tau}|f|)dx\\&\le\frac{\delta}{2}\int_{G_0^\varrho}r^{-3}u^2dx+\frac{1}{2\delta}\|f\|^2_{\mathring{W}^0_{1-2\tau}(G_0^\varrho)},\quad\forall\delta>0;\end{aligned}$$

-

$$\int_{\Gamma_0^\varrho} r^{-1}|u||g|ds\le\frac{\delta_1}{2}\int_{\Gamma_0^\varrho}r^{-2}u^2ds+\frac{1}{2\delta_1}\int_{\Gamma_0^\varrho}g^2ds;$$

- analogously to (4.28) we have

$$\int_{G_0^\varrho} ru_{xx}^2dx\le C_3\int_{G_0^{2\varrho}}(r^{-3}u^2+r^{1-2\tau}f^2)dx+C_4\|g\|^2_{\mathring{W}_1^{1/2}(\Gamma_0^{2\varrho})}.\tag{5.8}$$

- further,

$$\int_{\Gamma_0^\varrho}\chi(\omega)r^{-1}u\frac{\partial u}{\partial r}ds=\frac{1}{2}\sin\frac{\omega_0}{2}\int_0^\varrho\int_{-\pi}^{\pi}\chi(\frac{\omega_0}{2},\omega_2)\frac{\partial u^2}{\partial r}drd\omega_2,\quad\omega_0\in(0,\pi);$$

since $\int_0^\varrho \frac{\partial u^2}{\partial r} dr = u^2(\varrho,\omega) - u^2(0) = u^2(\varrho,\omega)$, from the above,

$$\int_{\Gamma_0^\varrho} \chi(\omega) r^{-1} u \frac{\partial u}{\partial r} ds = \frac{1}{2}\int_{\partial\Omega} \langle \chi(\omega) u^2(\varrho,\omega)\rangle\Big|_{\omega_1=\frac{\omega_0}{2}} d\sigma \geq 0;$$

- because of $\gamma(\omega) \geq \gamma_0$, $0 \leq \chi(\omega) \leq \chi_0$, by (2.4), we have

$$\int_{\Gamma_0^\varrho} \chi(\omega) r^{-2} u^2 ds = \int_{\Gamma_0^\varrho} \frac{\chi(\omega)}{\gamma(\omega)} \gamma(\omega) r^{-2} u^2 ds \leq \frac{\chi_0}{\gamma_0} \int_{\Gamma_0^\varrho} \gamma(\omega) r^{-2} u^2 ds \leq \frac{\chi_0}{\gamma_0} \widetilde{U}(\varrho) \tag{5.9}$$

also

$$\int_{\Gamma_0^\varrho} r^{-2} u^2 ds \leq \frac{1}{\gamma_0} \int_{\Gamma_0^\varrho} \gamma(\omega) r^{-2} u^2 ds \leq \frac{1}{\gamma_0} \widetilde{U}(\varrho);$$

- applying the Hardy - Friedrichs - Wirtinger type inequality (2.3), by (2.4) and (5.9),

$$\begin{aligned} \int_{G_0^\varrho} r^{-3} u^2 dx &\leq \frac{1}{\lambda(\lambda-1)} \Big[\int_{G_0^\varrho} r^{-1} |\nabla u|^2 dx + \int_{\Gamma_0^\varrho} \langle \lambda\chi(\omega) + \gamma(\omega)\rangle r^{-2} u^2 ds\Big] \\ &= \frac{1}{\lambda(\lambda-1)} \widetilde{U}(\varrho) + \frac{1}{\lambda-1} \int_{\Gamma_0^\varrho} \chi(\omega) r^{-2} u^2 ds \\ &\leq C(\lambda,\chi_0,\gamma_0) \widetilde{U}(\varrho). \end{aligned} \tag{5.10}$$

From inequality (5.7), by the above estimates and Lemma 2.5 we obtain

$$\begin{aligned} &\langle 1 - (\mathcal{A}(\varrho) + \delta)\rangle \widetilde{U}(\varrho) \\ &\leq \frac{\varrho}{2\lambda} \widetilde{U}'(\varrho) + \mathcal{A}(\varrho)\widetilde{U}(2\varrho) + c_1\delta^{-1}(\|f\|^2_{\mathring{W}^0_{1-2\tau}(G_0^{2\varrho})} + \|g\|^2_{\mathring{W}^{1/2}_1(\Gamma_0^{2\varrho})}), \quad \forall \delta > 0 \end{aligned} \tag{5.11}$$

where the positive constant c_1 is dependent on $\gamma_0, \chi_0, \lambda$. Using assumption (A5), the last inequality (5.11) takes the form

$$\langle 1 - (\mathcal{A}(\varrho) + \delta)\rangle \widetilde{U}(\varrho) \leq \frac{\varrho}{2\lambda} \widetilde{U}'(\varrho) + \mathcal{A}(\varrho)\widetilde{U}(2\varrho) + c_2 k_s^2 \delta^{-1} \varrho^{2s}, \quad \forall \delta > 0. \tag{5.12}$$

We have

$$\widetilde{U}(d) \leq C\Big(|u|^2_{0,G} + \|f\|^2_{\mathring{W}^0_{1-2\tau}(G)} + \|g\|^2_{\mathring{W}^{1/2}_1(\partial G)}\Big) \equiv U_0, \tag{5.13}$$

by Theorem 4.1. Inequalities (5.12) and (5.13) are the Cauchy problem (CP) (see [5, Theorem 1.57]) with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho}\langle 1 - (\mathcal{A}(\varrho) + \delta)\rangle, \quad \mathcal{N}(\varrho) = \frac{2\lambda}{\varrho}\mathcal{A}(\varrho), \quad \mathcal{Q}(\varrho) = 2\lambda c_2 k_s^2 \delta^{-1} \varrho^{2s-1}, \quad \forall \delta > 0. \tag{5.14}$$

The solution of this problem satisfies

$$\begin{aligned} \widetilde{U}(\varrho) &\leq \Big[U_0 \exp\Big(-\int_\varrho^d \mathcal{P}(\varsigma) d\varsigma\Big) + \int_\varrho^d \mathcal{Q}(\varsigma) \exp\Big(-\int_\varrho^\varsigma \mathcal{P}(\sigma)\, d\sigma\Big) d\varsigma\Big] \\ &\quad \times \exp\Big(\int_\varrho^d \mathcal{B}(\varsigma) d\varsigma\Big), \quad \mathcal{B}(\varrho) = \mathcal{N}(\varrho) \exp\Big(\int_\varrho^{2\varrho} \mathcal{P}(\sigma)\, d\sigma\Big), \end{aligned} \tag{5.15}$$

by [5, Theorem 1.57].

There are three possible cases: $s > \lambda$, $s = \lambda$ and $s < \lambda$.

Case $s > \lambda$. Let us choose $\delta = \varrho^\varepsilon$, for any $\varepsilon > 0$. From (5.14) it follows

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} - 2\lambda \frac{\mathcal{A}(\varrho)}{\varrho} - 2\lambda\varrho^{\varepsilon-1}, \quad \mathcal{N}(\varrho) = 2\lambda \frac{\mathcal{A}(\varrho)}{\varrho}, \quad \mathcal{Q}(\varrho) = 2\lambda c_2 k_s^2 \varrho^{2s-1-\varepsilon}.$$

We calculate

$$-\int_{\varrho}^{\varsigma}\mathcal{P}(\sigma)\,d\sigma=\ln\big(\frac{\varrho}{\varsigma}\big)^{2\lambda}+2\lambda\frac{\varsigma^{\varepsilon}-\varrho^{\varepsilon}}{\varepsilon}+2\lambda\int_{\varrho}^{\varsigma}\frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma,\quad \varsigma\in(\varrho,d).$$

Thus

$$\exp\Big(-\int_{\varrho}^{\varsigma}\mathcal{P}(\sigma)\,d\sigma\Big)\le C_1\big(\frac{\varrho}{\varsigma}\big)^{2\lambda},\quad C_1=\exp(\frac{2\lambda}{\varepsilon}d^{\varepsilon})\exp\Big(2\lambda\int_0^d\frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma\Big)$$

and

$$\int_{\varrho}^{2\varrho}\mathcal{P}(\sigma)\,d\sigma=\ln 2^{2\lambda}-2\lambda\frac{(2\varrho)^{\varepsilon}-\varrho^{\varepsilon}}{\varepsilon}-2\lambda\int_{\varrho}^{2\varrho}\frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma\le\ln 2^{2\lambda}.$$

Therefore,

$$\int_{\varrho}^{d}\mathcal{B}(\varsigma)d\varsigma=\int_{\varrho}^{d}\mathcal{N}(\varsigma)\exp\Big(\int_{\varsigma}^{2\varsigma}\mathcal{P}(\sigma)\,d\sigma\Big)d\varsigma\le\lambda 2^{2\lambda+1}\int_{\varrho}^{d}\frac{\mathcal{A}(\varsigma)}{\varsigma}d\varsigma\le C_2,$$

$$C_2=\lambda 2^{2\lambda+1}\int_0^d\frac{\mathcal{A}(\varsigma)}{\varsigma}d\varsigma$$

and further

$$\begin{aligned}\int_{\varrho}^{d}\mathcal{Q}(\varsigma)\exp\Big(-\int_{\varrho}^{\varsigma}\mathcal{P}(\sigma)\,d\sigma\Big)d\varsigma&\le 2\lambda C_1C_2k_s^2\varrho^{2\lambda}\int_{\varrho}^{d}\varsigma^{2s-1-\varepsilon-2\lambda}d\varsigma\\&=2\lambda C_1C_2k_s^2\varrho^{2\lambda}\frac{d^{2(s-\lambda)-\varepsilon}-\varrho^{2(s-\lambda)-\varepsilon}}{2(s-\lambda)-\varepsilon}\\&\le C_3k_s^2(\frac{\varrho}{d})^{2\lambda},\end{aligned}$$

if we choose $\varepsilon=s-\lambda>0$. By (5.15) and from the above inequalities, we obtain

$$\widetilde{U}(\varrho)\le\widetilde{C}_1(U_0+k_s^2)\varrho^{2\lambda},\tag{5.16}$$

where the positive constant $\widetilde{C}_1$ depends only on λ, d, s and on $\int_0^d\frac{\mathcal{A}(\varsigma)}{\varsigma}d\varsigma$.

Case $s=\lambda$. Now, we can take in (5.14) any function $\delta(\varrho)>0$ instead of $\delta>0$. In this way we obtain the Cauchy problem (5.15) with

$$\mathcal{P}(\varrho)=\frac{2\lambda}{\varrho}\langle 1-(\mathcal{A}(\varrho)+\delta(\varrho)\rangle,\quad\mathcal{N}(\varrho)=2\lambda\frac{\mathcal{A}(\varrho)}{\varrho},\quad\mathcal{Q}(\varrho)=2\lambda c_2k_s^2\delta^{-1}(\varrho)\varrho^{2\lambda-1}.$$

Let us choose $\delta(\varrho)=\frac{1}{2\lambda\ln\frac{ed}{\varrho}}$, $\varrho\in(0,d)$, where e denotes the Euler number. We calculate

$$\begin{aligned}-\int_{\varrho}^{\varsigma}\mathcal{P}(\sigma)\,d\sigma&\le\ln\big(\frac{\varrho}{\varsigma}\big)^{2\lambda}+\int_{\varrho}^{\varsigma}\frac{d\sigma}{\sigma\ln\frac{ed}{\sigma}}+2\lambda\int_0^d\frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma\\&=\ln\big(\frac{\varrho}{\varsigma}\big)^{2\lambda}+\ln(\frac{\ln\frac{ed}{\varrho}}{\ln\frac{ed}{\varsigma}})+2\lambda\int_0^d\frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma;\end{aligned}$$

$$\exp\Big(-\int_{\varrho}^{\varsigma}\mathcal{P}(\sigma)\,d\sigma\Big)\le\big(\frac{\varrho}{\varsigma}\big)^{2\lambda}\frac{\ln\frac{ed}{\varrho}}{\ln\frac{ed}{\varsigma}}\exp\Big(2\lambda\int_0^d\frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma\Big),\quad\varsigma\in(\varrho,d);$$

$$\int_{\varrho}^{2\varrho}\mathcal{P}(\sigma)\,d\sigma\le\ln 2^{2\lambda}+\ln(\frac{\ln(\frac{ed}{2\varrho})}{\ln(\frac{ed}{\varrho})})\le\ln 2^{2\lambda},$$

because of $\ln(\frac{ed}{2\varrho}) < \ln(\frac{ed}{\varrho})$. Therefore,

$$\int_\varrho^d \mathcal{B}(\varsigma)d\varsigma = \int_\varrho^d \mathcal{N}(\varsigma)\exp\Big(\int_\varsigma^{2\varsigma}\mathcal{P}(\sigma)\,d\sigma\Big)d\varsigma \le \lambda 2^{2\lambda+1}\int_\varrho^d \frac{\mathcal{A}(\varsigma)}{\varsigma}d\varsigma \le C_2$$

with constant C_2 as above in the case 1. Moreover

$$\begin{aligned}
&\int_\varrho^d \mathcal{Q}(\varsigma)\exp\Big(-\int_\varrho^\varsigma \mathcal{P}(\sigma)\,d\sigma\Big)d\varsigma \\
&= 4\lambda^2 c_2 k_s^2 \varrho^{2\lambda}\ln\frac{ed}{\varrho}\exp(2\lambda\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma)\int_\varrho^d \frac{d\varsigma}{\varsigma} \\
&\le C_4 k_s^2 \varrho^{2\lambda}\ln^2\frac{ed}{\varrho}, \quad C_4 = 4\lambda^2 c_2\exp(2\lambda\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma).
\end{aligned}$$

By (5.15), from the above inequalities, we obtain

$$\widetilde{U}(\varrho) \le \widetilde{C}_2(U_0 + k_s^2)\varrho^{2\lambda}\ln^2\frac{ed}{\varrho}, \tag{5.17}$$

where the positive constant $\widetilde{C}_2$ depends on λ, d, s and on $\int_0^d \frac{\mathcal{A}(\varsigma)}{\varsigma}d\varsigma$.

Case $s < \lambda$**.** In this case from (5.14) with any $\delta > 0$ we obtain the Cauchy problem (5.15). We calculate

$$-\int_\varrho^\varsigma \mathcal{P}(\sigma)\,d\sigma \le \ln\big(\frac{\varrho}{\varsigma}\big)^{2\lambda(1-\delta)} + 2\lambda\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma;$$

$$\exp\Big(-\int_\varrho^\varsigma \mathcal{P}(\sigma)\,d\sigma\Big) \le \big(\frac{\varrho}{\varsigma}\big)^{2\lambda(1-\delta)}\exp\Big(2\lambda\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma\Big), \quad \varsigma \in (\varrho, d).$$

Therefore, $\int_\varrho^d \mathcal{B}(\varsigma)d\varsigma \le C_2$ with constant C_2 as above in case 1. Moreover,

$$\begin{aligned}
&\int_\varrho^d \mathcal{Q}(\varsigma)\exp\Big(-\int_\varrho^\varsigma \mathcal{P}(\sigma)\,d\sigma\Big)d\varsigma \\
&\le 2\lambda c_2 k_s^2 \delta^{-1}\varrho^{2\lambda(1-\delta)}\exp\Big(2\lambda\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma\Big)\int_\varrho^d \varsigma^{2(s-\lambda+\lambda\delta)-1}d\varsigma \\
&= 2\lambda c_2 k_s^2 \delta^{-1}\varrho^{2\lambda(1-\delta)}\exp\Big(2\lambda\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma\Big)\frac{d^{2s-2\lambda+2\lambda\delta} - \varrho^{2s-2\lambda+2\lambda\delta}}{2s-2\lambda+2\lambda\delta} \le C_5 k_s^2 \varrho^{2s},
\end{aligned}$$

if we choose $\delta = \frac{\lambda - s}{2\lambda} > 0$. By (5.15), from the above inequalities, we obtain

$$\widetilde{U}(\varrho) \le \widetilde{C}_3(U_0 + k_s^2)\varrho^{2s}, \tag{5.18}$$

where the positive constant $\widetilde{C}_3 > 0$ depends only on λ, d, s and on $\int_0^d \frac{\mathcal{A}(\varsigma)}{\varsigma}d\varsigma$. Finally, by (5.16) - (5.18), taking into account of (5.8), (5.10), (5.13), we obtain the desired estimate (5.1). □

6. The power modulus of continuity

Proof of Theorem 1.3. Let us define the function

$$\psi(\varrho) = \begin{cases} \varrho^\lambda, & s > \lambda \\ \varrho^\lambda \ln\frac{1}{\varrho}, & s = \lambda \\ \varrho^s, & s < \lambda \end{cases}$$

for $0<\varrho<d$ and consider two sets $G^{2\varrho}_{\varrho/4}$ and $G^{\varrho}_{\varrho/2}\subset G^{2\varrho}_{\varrho/4}$, $\varrho>0$. We make transformation $x=\varrho x'$, $u(\varrho x')=\psi(\varrho)w(x')$. The function $w(x')$ satisfies the problem

$$\varrho^{-\tau}a^{ij}(\varrho x')w_{x'_ix'_j}+\varrho^{1-\tau}a^{i}(\varrho x')w_{x'_i}+\varrho^{2-\tau}a(\varrho x')w=\frac{\varrho^{2-\tau}}{\psi(\varrho)}f(\varrho x'),\quad x'\in G^2_{1/4}$$
$$\frac{\partial w}{\partial\vec{n}'}+\frac{1}{|x'|}\gamma(\omega)w+\chi(\omega)\frac{\partial w}{\partial r'}=\frac{\varrho}{\psi(\varrho)}g(\varrho x'),\quad x'\in\Gamma^2_{1/4}.$$

Applying the local maximum principle (see [12, Theorem 3.3], [14, Corollary 7.34]) we obtain

$$\begin{aligned}&\sup_{G^1_{1/2}}|w(x')|\\&\le C\Big[\Big(\int_{G^2_{1/4}}w^2dx'\Big)^{1/2}+\frac{\varrho}{\psi(\varrho)}\sup_{G^2_{1/4}}|g(\varrho x')|+\frac{\varrho^{2-\tau}}{\psi(\varrho)}\Big(\int_{G^2_{1/4}}|f(\varrho x')|^3dx'\Big)^{1/3}\Big],\end{aligned}\tag{6.1}$$

where the positive constant C depends only on $\max_{\omega\in\partial G}\gamma(\omega)$, χ_0, $\int_0^1\frac{\mathcal{A}(t)}{t}dt$. Let us return to variable x and to function $u(x)$. As a result we obtain

$$\int_{G^2_{1/4}}w^2dx'=\frac{1}{\psi^2(\varrho)}\int_{G^2_{1/4}}u^2(\varrho x')dx'\le\frac{2^3}{\psi^2(\varrho)}\int_{G^{2\varrho}_{\varrho/4}}r^{-3}u^2dx.$$

By Theorem 5.1, we have

$$\int_{G^2_{1/4}}w^2dx'\le C\Big(|u(x)|_{0,G}+\|f(x)\|_{\mathring{W}^0_{1-2\tau}(G)}+\|g(x)\|_{\mathring{W}^{1/2}_1(\partial G)}+k_s\Big)^2,\tag{6.2}$$

where k_s is defined by (1.3). According to assumption (A5),

$$\frac{\varrho}{\psi(\varrho)}\sup_{G^{2\varrho}_{\varrho/4}}|g(x)|\le\frac{\varrho}{\psi(\varrho)}g_1\varrho^{s-1}=g_1\begin{cases}\varrho^{s-\lambda}<1, & s>\lambda\\ \frac{1}{\ln\frac{1}{\varrho}}<1, & s=\lambda\\ 1, & s<\lambda\end{cases}\tag{6.3}$$

which implies

$$\frac{\varrho}{\psi(\varrho)}|g(x)|\le g_1.$$

In the same way,

$$\begin{aligned}\frac{\varrho^{2-\tau}}{\psi(\varrho)}\Big(\int_{G^2_{1/4}}|f(\varrho x')|^3dx'\Big)^{1/3}&\le\frac{\varrho^{1-\tau}}{\psi(\varrho)}\Big(\int_{G^{2\varrho}_{\varrho/4}}|f(x)|^3dx\Big)^{1/3}\\&\le\frac{\varrho^{1-\tau}}{\psi(\varrho)}f_1\Big(\int_{\varrho/4}^{2\varrho}r^{3(s-2+\tau)}r^2dr\cdot\operatorname{meas}\Omega\Big)^{1/3}\\&\le\widetilde{f}_1\frac{\varrho^{1-\tau}}{\psi(\varrho)}\varrho^{s-1+\tau}=\widetilde{f}_1\frac{\varrho^s}{\psi(\varrho)}\\&=\widetilde{f}_1\begin{cases}\varrho^{s-\lambda}<1, & s>\lambda\\ \frac{1}{\ln\frac{1}{\varrho}}<1, & s=\lambda\\ 1, & s<\lambda\end{cases}\end{aligned}\tag{6.4}$$

which implies

$$\frac{\varrho^{2-\tau}}{\psi(\varrho)}\Big(\int_{G_{\varrho/4}^{2\varrho}}|f(x)|^3dx\Big)^{1/3}\le \widetilde{f}_1.$$

For all $|x|\in(\frac{\varrho}{2},\varrho)$, we have

$$\sup_{G_{\varrho/2}^{\varrho}}|u|\le C_1(|u|_{0,G}+\|f\|_{\mathring{W}^0_{1-2\tau}(G)}+\|g\|_{\mathring{W}_1^{1/2}(\partial G)}+k_s)\psi(\varrho),$$

by (6.1)–(6.4). Putting $|x|=\frac{3}{2}\varrho$ we obtain the required estimate (1.2). □

7. Examples

Let us present some examples that demonstrate that the assumptions on the coefficients of the operator $\mathcal{L}$ are essential for validity of Theorem 1.3. We assume that the domain G lies inside the cone

$$G_0=\{(r,\omega_1,\omega_2):r>0,\,\omega_1\in(0,\frac{\omega_0}{2}),\,\omega_2\in(-\pi,\pi];\ \omega_0\in(0,\pi)\},$$

where $\mathcal{O}\in\partial G$ and in a neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the lateral surface of the cone G_0. Let us denote

$$\Gamma_0=\{(r,\omega_1,\omega_2):r>0,\,\omega_1=\frac{\omega_0}{2},\,\omega_2\in(-\pi,\pi];\,\omega_0\in(0,\pi)\}.$$

Let χ_0 is a nonnegative constant and γ_0 is positive constant.

As a first example, we consider the problem

$$\begin{gathered}\Delta u=0,\quad x\in G_0,\\ \frac{\partial u}{\partial\vec{n}}|_{\Gamma_0}+\chi_0\frac{\partial u}{\partial r}|_{\Gamma_0}+\frac{1}{r}\gamma_0u|_{\Gamma_0}=0.\end{gathered}\tag{7.1}$$

The solution to this problem is the function

$$u(r,\omega_1,\omega_2)=r^{\lambda^*}\mathcal{P}_{\lambda^*}(\cos\omega_1),quad\forall\omega_2\in(-\pi,\pi],$$

where $\mathcal{P}_{\lambda^*}(\cos\omega_1)$ is the Legendre spherical harmonic (see [11, section 7.3]), λ^* is the smallest positive solution of (8.6) and is estimated by (8.15).

As a second example, we consider the problem

$$\begin{gathered}\Delta u=-(2\lambda+1)r^{\lambda-2}\psi(\omega_1),\quad x\in G_0\\ (\frac{\partial u}{\partial\vec{n}}+\chi_0\frac{\partial u}{\partial r}+\frac{1}{r}\gamma_0u)\Big|_{\omega_1=\frac{\omega_0}{2}}=-\chi_0r^{\lambda-1}\psi(\frac{\omega_0}{2}).\end{gathered}$$

The solution of this problem is the function

$$u(r,\omega_1,\omega_2)=r^{\lambda}\ln\frac{1}{r}\psi(\omega_1),$$

where $\lambda>0$ and $\psi(\omega_1)$ are defined by (8.15) and (8.5). From here

$$f(x)=O(|x|^{\lambda-2}),\quad g(x)=O(|x|^{\lambda-1}).$$

In this case $s=\lambda$, $\tau=0$. Thus, this example confirms the validity (1.2) of Theorem 1.3 for $s=\lambda$.

8. Appendix: Eigenvalue problem (2.1)

We want to prove the existence of the smallest positive eigenvalue of problem (2.1). Let us consider the equation of problem (2.1). Calculating the Beltrami-Laplace operator we obtain

$$\frac{\partial^2 v}{\partial \omega_1^2} + \frac{\partial v}{\partial \omega_1}\cot\omega_1 + \frac{1}{\sin^2\omega_1}\frac{\partial^2 v}{\partial \omega_2^2} + \lambda(\lambda+1) = 0,$$
$$\omega_1 \in (0, \frac{\omega_0}{2}), \quad \omega_2 \in (-\pi, \pi], \quad \omega_0 \in (0, \pi).$$

We use the method of separation of variables: $v(\omega_1, \omega_2) = \psi(\omega_1)\varphi(\omega_2)$. From above equation it follows

$$\sin^2\omega_1 \cdot [\frac{\psi''}{\psi} + \frac{\psi'}{\psi}\cot\omega_1 + \lambda(\lambda+1)] = -\frac{\varphi''}{\varphi} = \mu^2$$
$$\Longrightarrow \varphi(\omega_2) = A\sin(\mu\omega_2) + B\cos(\mu\omega_2), \quad \forall A, B$$

and with regard to the boundary condition of (2.1),

$$\begin{gathered} \psi''(\omega_1) + \psi'(\omega_1)\cot\omega_1 + \langle\lambda(\lambda+1) - \frac{\mu^2}{\sin^2\omega_1}\rangle\psi(\omega_1) = 0, \quad \omega_1 \in (0, \frac{\omega_0}{2}), \\ \psi'(\frac{\omega_0}{2}) + (\lambda\chi_0 + \gamma_0)\psi(\frac{\omega_0}{2}) = 0, \end{gathered} \tag{8.1}$$

where $\omega_0 \in (0, \pi)$, $\chi_0 = \chi(\frac{\omega_0}{2}) \geq 0$, $\gamma_0 = \gamma(\frac{\omega_0}{2}) > 0$. We multiply equation of (8.1) by $\sin\omega_1$ and write it in the form

$$(p\psi')' - q\psi + \varrho\lambda(\lambda+1)\psi = 0, \tag{8.2}$$

where

$$p \equiv \sin\omega_1 > 0, \quad q \equiv \mu^2\sin^{-1}\omega_1, \quad \varrho \equiv \sin\omega_1, \quad \omega_1 \in (0, \omega_0/2).$$

By [7, Theorem 7, Chapter VI], we know that if the coefficient q changes everywhere in the same sense, every eigenvalue of (8.2) changes in this same sense. Thus, if $\mu = 0$ we obtain the problem for the smallest positive eigenvalue

$$\begin{gathered} \psi''(\omega_1) + \cot\omega_1 \cdot \psi'(\omega_1) + \lambda(\lambda+1)\psi(\omega_1) = 0, \quad \omega_1 \in (0, \frac{\omega_0}{2}), \\ \psi'(\frac{\omega_0}{2}) + (\lambda\chi_0 + \gamma_0)\psi(\frac{\omega_0}{2}) = 0. \end{gathered} \tag{8.3}$$

Now we want to solve this problem. For this we set

$$\psi(\omega_1) = \eta(\xi), \quad \xi = \cos\omega_1. \tag{8.4}$$

Let us denote $\xi_0 \equiv \cos\frac{\omega_0}{2}$. Then our problem takes the form

$$(1-\xi^2)\eta''_{\xi\xi} - 2\xi\eta'_{\xi} + \lambda(\lambda+1)\eta = 0, \quad \xi \in (\cos\frac{\omega_0}{2}, 1)$$
$$-\sqrt{1-\xi_0^2}\eta'(\xi_0) + (\lambda\chi + \gamma)\eta(\xi_0) = 0.$$

Solutions of this equation are the Legendre spherical harmonics (see [11, section 7.3]) $\eta(\xi) = \mathcal{P}_\lambda(\xi)$ or by (8.4),

$$\psi(\omega_1) = \mathcal{P}_\lambda(\cos\omega_1). \tag{8.5}$$

Using the boundary condition, we obtain the following equation for λ,

$$\lambda\mathcal{P}_{\lambda-1}(\cos\frac{\omega_0}{2}) - \lambda\cos\frac{\omega_0}{2}\mathcal{P}_\lambda(\cos\frac{\omega_0}{2}) = (\lambda\chi + \gamma)\sin\frac{\omega_0}{2}\mathcal{P}_\lambda(\cos\frac{\omega_0}{2}). \tag{8.6}$$

Now we define the function

$$\mathcal{F}(\lambda)=\frac{\lambda}{\sin\frac{\omega_0}{2}}\mathcal{P}_{\lambda-1}(\cos\frac{\omega_0}{2})-(\frac{\lambda\cos\frac{\omega_0}{2}}{\sin\frac{\omega_0}{2}}+\lambda\chi_0+\gamma_0)\mathcal{P}_\lambda(\cos\frac{\omega_0}{2}), \tag{8.7}$$

where $\omega_0\in(0,\pi)$. According to [11, (7.3.13), (7.3.14)], $\mathcal{P}_{-\lambda-1}(\xi_0)=\mathcal{P}_\lambda(\xi_0)$ and $\mathcal{P}_0(\xi_0)=\mathcal{P}_{-1}(\xi_0)=1$, we obtain

$$\mathcal{F}(0)=-\gamma_0<0. \tag{8.8}$$

Now, we use the asymptotic representation of $\mathcal{P}_\lambda(\cos\frac{\omega_0}{2})$ (see [11, (7.11.12)]). We have for $\lambda\to+\infty$

$$\mathcal{P}_{\lambda-1}(\cos\frac{\omega_0}{2})=\sqrt{\frac{2}{\pi(\lambda-1)\sin\frac{\omega_0}{2}}}\sin[(\lambda-\frac12)\frac{\omega_0}{2}+\frac{\pi}{4}][1+O(\frac{1}{\lambda-1})],$$

$$\mathcal{P}_{\lambda}(\cos\frac{\omega_0}{2})=\sqrt{\frac{2}{\pi\lambda\sin\frac{\omega_0}{2}}}\sin[(\lambda+\frac12)\frac{\omega_0}{2}+\frac{\pi}{4}][1+O(\frac{1}{\lambda})].$$

Choosing

$$\lambda=\frac{3\pi}{2\omega_0}+\frac{4k\pi}{\omega_0},\quad k\in\mathbb{N},\ k\gg1,$$

we obtain $\sin[(\frac{3\pi}{2\omega_0}+\frac{4k\pi}{\omega_0}-\frac12)\frac{\omega_0}{2}+\frac{\pi}{4}]>0$ and $\sin[(\frac{3\pi}{2\omega_0}+\frac{4k\pi}{\omega_0}+\frac12)\frac{\omega_0}{2}+\frac{\pi}{4}]<0$. Thus we have

$$\mathcal{F}(\frac{3\pi}{2\omega_0}+\frac{4k\pi}{\omega_0})>0 \tag{8.9}$$

for $k\gg1$, $\omega_0\in(0,\pi)$, because of $\gamma_0>0$, $\chi_0\ge0$. Finally, from (8.8), (8.9) and continuity of function $\mathcal{F}(\lambda)$ (see [11]), it follows that there is the smallest positive solution of (8.7). Indeed, the continuous function $\mathcal{F}(\lambda)$ at the ends of the interval $[0,+\infty)$ takes different signs and therefore it must have the first positive zero. Thus there exists the smallest positive eigenvalue of problem (2.1).

Let us estimate the value of λ. Putting $\frac{\psi'}{\psi}=y(\omega_1)$ in (8.3), we obtain

$$\begin{gathered}y'+y^2+y\cot\omega_1+\lambda(\lambda+1)=0,\quad \omega_1\in(0,\frac{\omega_0}{2}),\\ y(\frac{\omega_0}{2})=-\lambda\chi_0-\gamma_0,\quad \gamma_0>0,\ \chi_0\ge0.\end{gathered} \tag{8.10}$$

By (8.5) and $\mathcal{P}'_\lambda(\xi)=-\frac{1}{\sqrt{1-\xi^2}}\mathcal{P}^1_\lambda(\xi)$, $\xi\in(-1,1)$ (see [11, (7.12.5)]) we have

$$y(\omega_1)=-\sin\omega_1\cdot\frac{\mathcal{P}'_\lambda(\cos\omega_1)}{\mathcal{P}_\lambda(\cos\omega_1)}=\frac{\mathcal{P}^1_\lambda(\cos\omega_1)}{\mathcal{P}_\lambda(\cos\omega_1)}. \tag{8.11}$$

Using formula [11, (7.12.28)] we obtain

$$\mathcal{P}^1_\lambda(\xi)=-\frac{\Gamma(\lambda+2)}{2\Gamma(2)\Gamma(\lambda)}\sqrt{1-\xi^2}\cdot F(1-\lambda,2+\lambda,2,\frac{1-\xi}{2}), \tag{8.12}$$

where $F(a,b,c,x)$ denotes the hypergeometric function. From (8.11), (8.12) we find

$$y(0)=\frac{\mathcal{P}^1_\lambda(1)}{\mathcal{P}_\lambda(1)}=0,$$

by $\mathcal{P}_\lambda(1)=1$ (see [11, (7.3.13)]) and $F(a,b,c,0)=1$ by the definition. From the equation of problem (8.10) we have

$$\begin{gathered}y'+y\cot\omega_1<0,\\ y(0)=0.\end{gathered}$$

Considering the Cauchy problem

$$\begin{gathered}\widetilde{y}' + \widetilde{y}\cot\omega_1 = 0, \quad \omega_1 \in (0, \frac{\omega_0}{2})\\ \widetilde{y}(0) = 0,\end{gathered}$$

it implies $\widetilde{y}(\omega_1) \equiv 0$. Using the Chaplygin comparison principle [6], we obtain that $y(\omega_1) \le 0$. Hence from (8.10) it follows that

$$\begin{gathered}y' \ge -y^2 - \lambda(\lambda+1), \quad \omega_1 \in (0, \frac{\omega_0}{2}),\\ y(0) = 0.\end{gathered}$$

Now, we consider the Cauchy problem

$$\begin{gathered}z' = -z^2 - \lambda(\lambda+1), \quad \omega_1 \in (0, \frac{\omega_0}{2}),\\ z(0) = 0.\end{gathered}$$

Solving this problem we have

$$z(\omega_1) = -\sqrt{\lambda(\lambda+1)}\tan(\omega_1\sqrt{\lambda(\lambda+1)}).$$

Thus, using again the Chaplygin comparison principle we finally obtain

$$-\sqrt{\lambda(\lambda+1)}\tan(\omega_1\sqrt{\lambda(\lambda+1)}) \le y(\omega_1) \le 0,\ \omega_1 \in [0, \frac{\omega_0}{2}].$$

Let

$$\varkappa = \frac{\omega_0}{2}\sqrt{\lambda(\lambda+1)}, \quad 0 < \omega_0 < \pi. \tag{8.13}$$

From the boundary condition

$$\tan\varkappa \ge \frac{\lambda\chi_0 + \gamma_0}{\sqrt{\lambda(\lambda+1)}}.$$

Determining the value $\lambda > 0$ from (8.13), we obtain $\lambda = \sqrt{\frac{1}{4} + \frac{4\varkappa^2}{\omega_0^2}} - \frac{1}{2}$. Therefore,

$$\tan\varkappa \ge \frac{\omega_0}{2\varkappa}\Big[\Big(\sqrt{\frac{1}{4} + \frac{4\varkappa^2}{\omega_0^2}} - \frac{1}{2}\Big)\chi_0 + \gamma_0\Big] \tag{8.14}$$

where $\gamma_0 > 0$, $\chi_0 \ge 0$, $\omega_0 \in (0, \pi)$.

By the graphic method (see Figure 2), we obtain that $0 < \varkappa^* < \frac{\pi}{2}$, where $\varkappa^*$ is the smallest positive solution of (8.14). Because of (8.13), we obtain

$$0 < \lambda^* < \sqrt{\frac{1}{4} + \frac{\pi^2}{\omega_0^2}} - \frac{1}{2} \tag{8.15}$$

for $0 < \omega_0 < \pi$, where λ^* is the smallest positive solution of (8.6).

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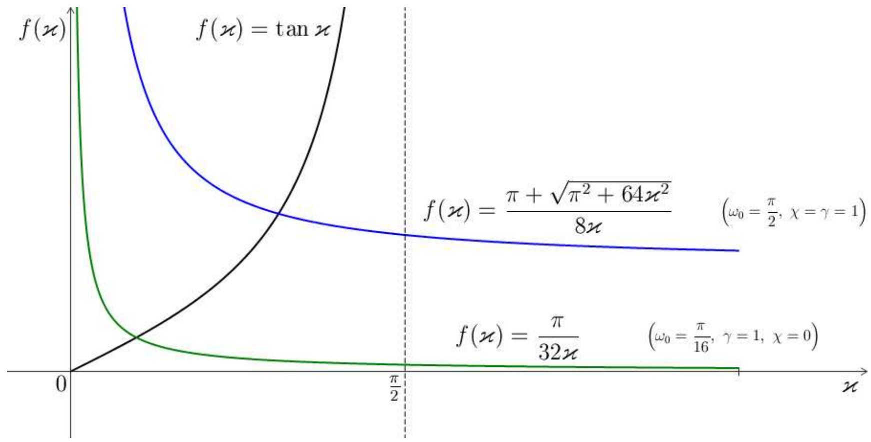

FIGURE 2. Smallest positive solution of (8.14)

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