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GROWTH OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

HUI HU, XIU-MIN ZHENG

Abstract. In this article, we study the growth of solutions of linear differential equations with some dominant entire coefficients. Especially, we obtain some results on the iterated p-lower order of these solutions, which extend previous results. Moreover, we investigate the iterated exponent of convergence of distinct zeros of $f^{(j)}(z) - \varphi(z)$.

1. Introduction

We shall assume that readers are familiar with the fundamental results and the standard notations of Nevanlinna's theory; see e.g. [5, 8, 13]. Let us define inductively for $r \in [0, +\infty)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp(\exp_p r)$, $p \in \mathbb{N}$. For all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{p+1} r = \log(\log_p r)$, $p \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. We recall the following definitions of finite iterated order; see e.g. [2, 3, 8, 9, 10, 12].

Definition 1.1. The iterated p-order $\sigma_p(f)$ of a meromorphic function $f(z)$ is defined as

$$\sigma_p(f) = \limsup_{r\to\infty} \frac{\log_p T(r,f)}{\log r} \quad (p \in \mathbb{N}).$$

Remark 1.2. If $f(z)$ is an entire function, then

$$\sigma_p(f) = \limsup_{r\to\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log_{p+1} M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log_p \nu_f(r)}{\log r},$$

where $p \in \mathbb{N}$, $\nu_f(r)$ is the central index of $f(z)$.

Definition 1.3. The iterated p-lower order $\mu_p(f)$ of a meromorphic function $f(z)$ is defined by

$$\mu_p(f) = \liminf_{r\to\infty} \frac{\log_p T(r,f)}{\log r} \quad (p \in \mathbb{N}).$$

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Remark 1.4. The iterated p-lower order $\mu_p(f)$ of an entire function $f(z)$ is defined by

$$\mu_p(f) = \liminf_{r\to\infty} \frac{\log_p T(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log_{p+1} M(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log_p \nu_f(r)}{\log r} \quad (p \in \mathbb{N}).$$

Definition 1.5. The finiteness degree of the order of a meromorphic function $f(z)$ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is rational;} \\ \min\{j \in \mathbb{N} : \sigma_j(f) < \infty\}, & \text{if } f \text{ is transcendental with} \\ & \sigma_j(f) < \infty \text{ for some } j \in \mathbb{N}; \\ \infty, & \text{if } \sigma_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Definition 1.6. The iterated convergence exponent of the sequence of a-points of a meromorphic function $f(z)$ is defined by

$$\lambda_p(f-a) = \lambda_p(f,a) = \limsup_{r\to\infty} \frac{\log_p N(r, \frac{1}{f-a})}{\log r} \quad (p \in \mathbb{N}),$$

and the iterated convergence exponent of the sequence of distinct a-points of a meromorphic function $f(z)$ is defined by

$$\overline{\lambda}_p(f-a) = \overline{\lambda}_p(f,a) = \limsup_{r\to\infty} \frac{\log_p \overline{N}(r, \frac{1}{f-a})}{\log r} \quad (p \in \mathbb{N}).$$

If $a = 0$, the iterated convergence exponent of the zeros or the iterated convergence exponent of the distinct zeros is defined respectively by

$$\lambda_p(f) = \lambda_p(f,0) = \limsup_{r\to\infty} \frac{\log_p N(r, \frac{1}{f})}{\log r} \ (p \in \mathbb{N}),$$

or

$$\overline{\lambda}_p(f) = \overline{\lambda}_p(f,0) = \limsup_{r\to\infty} \frac{\log_p \overline{N}(r, \frac{1}{f})}{\log r} \quad (p \in \mathbb{N}).$$

If $a = \infty$, the iterated convergence exponent of the poles or the iterated convergence exponent of the distinct poles is defined respectively by

$$\lambda_p(\frac{1}{f}) = \limsup_{r\to\infty} \frac{\log_p N(r,f)}{\log r} \quad (p \in \mathbb{N}),$$

or

$$\overline{\lambda}_p(\frac{1}{f}) = \limsup_{r\to\infty} \frac{\log_p \overline{N}(r,f)}{\log r} \quad (p \in \mathbb{N}).$$

Furthermore, we can get the definitions of $\lambda_p(f-\varphi)$ and $\overline{\lambda}_p(f-\varphi)$, when a is replaced by a meromorphic function φ.

Definition 1.7. Let $f(z)$ be an entire function. Then the iterated p-type of an entire function $f(z)$, with iterated p-order $0 < \sigma_p(f) < \infty$ is defined by

$$\tau_p(f) = \limsup_{r\to\infty} \frac{\log_{p-1} T(r,f)}{r^{\sigma_p(f)}} = \limsup_{r\to\infty} \frac{\log_p M(r,f)}{r^{\sigma_p(f)}} \quad (p \in \mathbb{N}\backslash\{1\}).$$

We definite the iterated p-lower type of $f(z)$ as follows.

Definition 1.8. Let $f(z)$ be an entire function. Then the iterated p-lower type of an entire function $f(z)$, with iterated p-lower order $0 < \mu_p(f) < \infty$, is defined by

$$\underline{\tau}_p(f) = \liminf_{r\to\infty} \frac{\log_{p-1} T(r,f)}{r^{\mu_p(f)}} = \liminf_{r\to\infty} \frac{\log_p M(r,f)}{r^{\mu_p(f)}} \quad (p \in \mathbb{N}\backslash\{1\}).$$

Remark 1.9. If $p = 1$, then the equalities

$$\limsup_{r\to\infty} \frac{\log_{p-1} T(r,f)}{r^{\sigma_p(f)}} = \limsup_{r\to\infty} \frac{\log_p M(r,f)}{r^{\sigma_p(f)}},$$
$$\liminf_{r\to\infty} \frac{\log_{p-1} T(r,f)}{r^{\mu_p(f)}} = \liminf_{r\to\infty} \frac{\log_p M(r,f)}{r^{\mu_p(f)}}$$

in Definitions 1.7 and 1.8 respectively fail to hold. For example, for the function $f(z) = e^z$, we have $\lim_{r\to\infty} \frac{T(r,f)}{r} = \frac{1}{\pi} \neq 1 = \lim_{r\to\infty} \frac{\log M(r,f)}{r}$. Therefore, we assume $p \in \mathbb{N}\backslash\{1\}$ in the following.

We denote the linear measure and the logarithmic measure of a set $E \subset [0,+\infty)$ by $mE = \int_E dt$ and $m_l E = \int_E dt/t$ respectively (see e.g. [6]).

2. Main Results

In 1998, Kinnunen investigated complex oscillation properties of the solutions of the higher order linear differential equations

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0 \tag{2.1}$$

and

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{2.2}$$

with entire coefficients of finite iterated order and obtained the following result in [9].

Theorem 2.1. *Let $A_0(z), A_1(z), \dots, A_{n-1}(z)$ be entire functions and let $i(A_0) = p$, $0 < p < \infty$. If $i(A_j) < p$ or $\sigma_p(A_j) < \sigma_p(A_0) = \kappa$ for all $j = 1, 2, \dots, n-1$, then $i(f) = p + 1$ and $\sigma_{p+1}(f) = \kappa$ hold for all non-trivial solutions of (2.1).*

Note that there is some coefficient $A_0(z)$ strictly dominating other coefficients in Theorem 2.1. Thus, a natural question arises: If there are some coefficients have the same iterated order as $A_0(z)$, can the similar result hold? B. Belaïdi in [1] considered the question and obtained next result.

Theorem 2.2. *Let $A_0(z), A_1(z), \dots, A_{n-1}(z)$ be entire functions, and let $i(A_0) = p$. Assume that $\max\{\sigma_p(A_j) : j \neq 0\} \leq \sigma_p(A_0)(> 0)$ and $\max\{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_0)\} < \tau_p(A_0) = \tau(0 < \tau < \infty)$. Then every solution $f(z) \not\equiv 0$ of (2.1) satisfies $i(f) = p + 1$ and $\sigma_{p+1}(f) = \sigma_p(A_0)$.*

Theorems 2.1 and 2.2 investigated the iterated order of solutions of (2.1), when there is some dominating coefficient with iterated order. Another question is: If there is some dominating coefficient with iterated lower order, what can we say about the growth of solutions of (2.1). For the special case $p = 2$, Zhang-Tu in [14] discussed it and obtained the following result.

Theorem 2.3. *Let $A_0(z), \dots, A_{n-1}(z)$ be entire functions satisfying $\max\{\sigma(A_j), j = 1, \dots, n-1\} < \mu(A_0) \leq \sigma(A_0) < \infty$, then every solution $f(z) \not\equiv 0$ of (2.1) satisfies*

$$\mu(A_0) = \mu_2(f) \leq \sigma_2(f) = \sigma(A_0).$$

In this paper, we investigate the above problems. Moreover, we investigate the iterated exponent of convergence of distinct zeros of $f^{(j)}(z) - \varphi(z)$. Firstly, we extend Theorem 2.3 into a general case and obtain the same result.

Theorem 2.4. *Let $A_0(z), A_1(z), \dots, A_{n-1}(z)$ be entire functions of finite iterated order satisfying $\max\{\sigma_p(A_j), j = 1, \dots, n-1\} < \mu_p(A_0) \le \sigma_p(A_0) < \infty$, then every solution $f(z) \not\equiv 0$ of (2.1) satisfies*

$$\mu_p(A_0) = \mu_{p+1}(f) \le \sigma_{p+1}(f) = \sigma_p(A_0). \tag{2.3}$$

Secondly, when there are some coefficients with iterated order equal to $\mu_p(A_0)$, we obtain the following two results.

Theorem 2.5. *Let $A_0(z), A_1(z), \dots, A_{n-1}(z)$ be entire functions, and let $i(A_0) = p$. Assume that $\max\{\sigma_p(A_j) : j \ne 0\} \le \mu_p(A_0) \le \sigma_p(A_0)$ and $\tau_1 = \max\{\tau_p(A_j) : \sigma_p(A_j) = \mu_p(A_0)\} < \underline{\tau}_p(A_0) = \tau (0 < \tau < \infty)$. Then every solution $f(z) \not\equiv 0$ of (2.1) satisfies*

$$\mu_{p+1}(f) = \mu_p(A_0) \le \sigma_p(A_0) = \sigma_{p+1}(f) = \lambda_{p+1}(f - \varphi) = \overline{\lambda}_{p+1}(f - \varphi), \tag{2.4}$$

where $\varphi(z) \not\equiv 0$ is an entire function satisfying $\sigma_{p+1}(\varphi) < \mu_p(A_0)$.

Theorem 2.6. *Let $A_0(z), A_1(z), \dots, A_{n-1}(z)$ be entire functions of finite iterated order satisfying $\max\{\sigma_p(A_j), j \ne 0\} \le \mu_p(A_0) = \mu$ and*

$$\limsup_{r\to\infty} \sum_{j=1}^{n-1} m(r, A_j)/m(r, A_0) < 1.$$

Then every non-trivial solution $f(z)$ of (2.1) satisfies

$$\mu_{p+1}(f) = \mu_p(A_0) \le \sigma_p(A_0) = \sigma_{p+1}(f) = \lambda_{p+1}(f - \varphi) = \overline{\lambda}_{p+1}(f - \varphi), \tag{2.5}$$

where $\varphi(z) \not\equiv 0$ is an entire function satisfying $\sigma_{p+1}(\varphi) < \mu_p(A_0)$.

Remark 2.7. All solutions of (2.1) in Theorems 2.4, 2.5, 2.6 are of regular growth $\mu_{p+1}(f) = \sigma_{p+1}(f)$, when the coefficient $A_0(z)$ is of regular growth $\mu_p(A_0) = \sigma_p(A_0)$.

Theorem 2.8. *Let $A_0(z), A_1(z), \dots, A_{n-1}(z)$ be meromorphic functions of finite iterated order satisfying*

$$\max\{\lambda_p(\frac{1}{A_0}), \sigma_p(A_j), j = 1, \dots, n-1\} < \mu_p(A_0) \le \sigma_p(A_0) < \infty,$$

if $f(z) \not\equiv 0$ is a meromorphic solution of (2.1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p-1}\{r^b\}$, $(b < \mu_p(A_0))$, then we have

$$\sigma_p(A_0) = \sigma_{p+1}(f) = \lambda_{p+1}(f^{(j)} - \varphi) = \overline{\lambda}_{p+1}(f^{(j)} - \varphi), \ (j = 0, 1, \dots), \tag{2.6}$$

where $\varphi(z) \not\equiv 0$ is a meromorphic function satisfying $\sigma_{p+1}(\varphi) < \sigma_p(A_0)$.

Corollary 2.9. *Let $A_0(z), A_1(z), \dots, A_{n-1}(z)$ satisfying the hypotheses of Theorem 2.4, then every solution $f(z) \not\equiv 0$ of (2.1) satisfies*

$$\mu_{p+1}(f) = \mu_p(A_0) \le \sigma_p(A_0) = \sigma_{p+1}(f) = \lambda_{p+1}(f^{(j)} - \varphi) = \overline{\lambda}_{p+1}(f^{(j)} - \varphi),$$

where $\varphi(z) \not\equiv 0$ is an entire function satisfying $\sigma_{p+1}(\varphi) < \sigma_p(A_0)$.

3. Lemmas for the proofs of main results

Lemma 3.1. *Let $f(z)$ be a transcendental entire function. There exists a set E_1 of r of finite logarithmic measure, such that for all z satisfying $|z| = r \notin E_1$ and $|f(z)| = M(r,f)$, we have*

$$\frac{f^{(k)}(z)}{f(z)} = \big(\frac{\nu_f(r)}{z}\big)^k(1+o(1)), \quad (k \in \mathbb{N}, r \notin E_1),$$

where $\nu_f(r)$ is the central index of $f(z)$.

Lemma 3.2 ([6, 8])**.** *Let $g : [0,+\infty) \to \mathbb{R}$ and $h : [0,+\infty) \to \mathbb{R}$ be monotone increasing functions. If (i) $g(r) \le h(r)$ outside of an exceptional set of finite linear measure, or (ii) $g(r) \le h(r)$, $r \notin E_2 \cup (0,1]$, where $E_2 \subset [1,\infty)$ is a set of finite logarithmic measure, then for any constant $\alpha > 1$, there exists $r_0 = r_0(\alpha) > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

Lemma 3.3 ([11])**.** *Let $A_0(z), A_1(z), \dots, A_{n-1}(z)$ be meromorphic functions of finite iterated order satisfying $\max\{\sigma_p(A_j), j = 1,\dots,n-1\} < \mu_p(A_0) \le \sigma_p(A_0) < \infty$, if $f(z) \not\equiv 0$ is a meromorphic solution of (2.1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p-1}\{r^b\}$, $(b < \mu_p(A_0))$, then $\sigma_{p+1}(f) = \sigma_p(A_0)$.*

Lemma 3.4. *Let $f(z)$ be an entire function with $\mu_p(f) < \infty$, then for any given $\varepsilon > 0$, there exists a set $E_4 \subset (1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_4$, we have*

$$\mu_p(f) = \lim_{r\to\infty,\, r\in E_4} \frac{\log_p T(r,f)}{\log r} = \lim_{r\to\infty,\, r\in E_4} \frac{\log_{p+1} M(r,f)}{\log r} = \lim_{r\to\infty,\, r\in E_4} \frac{\log_p \nu_f(r)}{\log r},$$

and

$$M(r,f) < \exp_p\{r^{\mu_p(f)+\varepsilon}\}.$$

Proof. We use a similar proof as [11, Lemma 3.8]. By the definition of iterated p-lower order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1+\frac{1}{n})r_n < r_{n+1}$, and

$$\lim_{r_n\to\infty} \frac{\log_{p+1} M(r_n,f)}{\log r_n} = \mu_p(f).$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \ge n_1$ and any $r \in [r_n, (1+\frac{1}{n})r_n]$, we have

$$\frac{\log_{p+1} M(r_n,f)}{\log(1+\frac{1}{n})r_n} \le \frac{\log_{p+1} M(r,f)}{\log r} \le \frac{\log_{p+1} M((1+\frac{1}{n})r_n,f)}{\log r_n}.$$

Let $E_4 = \cup_{n=n_1}^{\infty}[r_n, (1+\frac{1}{n})r_n]$, then for any $r \in E_4$, we have

$$\lim_{r\to\infty,\, r\in E_4} \frac{\log_{p+1} M(r,f)}{\log r} = \lim_{r_n\to\infty} \frac{\log_{p+1} M(r_n,f)}{\log r_n} = \mu_p(f),$$

and

$$m_l E = \sum_{n=n_1}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^{\infty} \log(1+\frac{1}{n}) = \infty.$$

It is easy to see

$$\lim_{r\to\infty,\, r\in E_4} \frac{\log_p T(r,f)}{\log r} = \lim_{r\to\infty,\, r\in E_4} \frac{\log_{p+1} M(r,f)}{\log r} = \lim_{r\to\infty,\, r\in E_4} \frac{\log_p \nu_f(r)}{\log r}.$$

The proof is complete. $\square$

Lemma 3.5 ([11]). *Let $A_0(z), A_1(z), \dots, A_{n-1}(z)$ be entire functions of finite iterated order satisfying $i(A_0) = p$, $\sigma_p(A_0) = \sigma$ and*

$$\limsup_{r\to\infty} \sum_{j=1}^{n-1} m(r, A_j)/m(r, A_0) < 1,$$

then every non-trivial solution $f(z)$ of (2.1) satisfies $\sigma_{p+1}(f) = \sigma_p(A_0) = \sigma$.

Lemma 3.6 ([4]). *Let $f(z)$ be a transcendental meromorphic function. Let $\alpha > 1$ be a constant, and k and j be integers satisfying $k > j \geq 0$. Then the following two statements hold:*

(a) There exists a set $E_6 \subset [1, \infty)$ which has finite logarithmic measure, and a constant $C > 0$, such that for all z satisfying $|z| = r \notin E_6 \cup [0, 1]$, we have

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq C\Big[\frac{T(\alpha r, f)}{r}(\log r)^{\alpha} \log T(\alpha r, f)\Big]^{k-j}. \tag{3.1}$$

(b) There exists a set $E_6' \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) - E_6'$, then there is a constant $R = R(\theta) > 0$ such that (3.1) holds for all z satisfying $\arg z = \theta$ and $|z| \geq R$.

Lemma 3.7 ([10]). *Let $A_0(z), A_1(z), \dots, A_{n-1}(z), F(z) \not\equiv 0$ be meromorphic functions and let $f(z)$ be a meromorphic solution of (2.2) satisfying one of the following two conditions*

(i) $\max\{i(F) = q, i(A_j), j = 0, 1, \dots, n-1\} < i(f) = p + 1$, $(0 < p < \infty)$;
(ii) $b = \max\{\sigma_{p+1}(F), \sigma_{p+1}(A_j), j = 0, 1, \dots, n-1\} < \sigma_{p+1}(f) = \sigma$;

then $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$.

Lemma 3.8. *Let $B_j(z)$, $(j = 0, 1, \dots, n-1)$ be meromorphic functions of finite iterated orders. Assume that $\max\{\sigma_p(B_j) : j \neq 0\} \leq \mu_p(B_0) \leq \sigma_p(B_0)$, $\lambda_p(\frac{1}{B_0}) < \mu_p(B_0)$ and $\tau_1 = \max\{\tau_p(B_j) : \sigma_p(B_j) = \mu_p(B_0), j \neq 0\} < \underline{\tau}_p(B_0) = \tau(0 < \tau < \infty)$. Then every meromorphic solution $f(z) \not\equiv 0$ of the equation*

$$f^{(n)} + B_{n-1}(z)f^{(n-1)} + \dots + B_1(z)f' + B_0(z)f = 0, \tag{3.2}$$

satisfies $\sigma_{p+1}(f) \geq \mu_p(B_0)$.

Proof. By (3.2), we obtain

$$-B_0(z) = \frac{f^{(n)}(z)}{f(z)} + B_{n-1}(z)\frac{f^{(n-1)}(z)}{f(z)} + \dots + B_1(z)\frac{f'(z)}{f(z)}. \tag{3.3}$$

By the logarithmic derivative lemma and (3.3), we have

$$m(r, B_0) \leq \sum_{j=1}^{n-1} m(r, B_j) + O\left(\log(rT(r, f))\right), \quad (r \notin E), \tag{3.4}$$

where E is a set of r of finite linear measure.

Noting the assumption that $\lambda_p(\frac{1}{B_0}) < \mu_p(B_0)$, we have

$$N(r, B_0) = o(T(r, B_0)), \quad r \to \infty. \tag{3.5}$$

Therefore, by (3.5) we have

$$\mu_p(B_0) = \liminf_{r\to\infty} \frac{\log_p m(r, B_0)}{\log r} \quad \text{and} \quad \tau = \underline{\tau}_p(B_0) = \liminf_{r\to\infty} \frac{\log_{p-1} m(r, B_0)}{r^{\mu_p(B_0)}}. \tag{3.6}$$

By (3.6), for sufficiently large r, we have

$$m(r, B_0) \geq \exp_{p-1}\{(\tau - \varepsilon) r^{\mu_p(B_0)}\}. \tag{3.7}$$

Set $b = \max\{\sigma_p(B_j) : \sigma_p(B_j) < \mu_p(B_0)\}$. If $\sigma_p(B_j) < \mu_p(B_0)$, then for any given $\varepsilon (0 < 2\varepsilon < \min\{\mu_p(B_0) - b, \tau - \tau_1\})$, we have

$$\limsup_{r\to\infty} \frac{\log_p m(r, B_j)}{\log r} \leq b < \mu_p(B_0). \tag{3.8}$$

By (3.8), for sufficiently large r, we have

$$m(r, B_j) \leq \exp_{p-1}\{r^{b+\varepsilon}\}. \tag{3.9}$$

If $\sigma_p(B_j) = \mu_p(B_0)$, $j \neq 0$, then we have

$$\limsup_{r\to\infty} \frac{\log_{p-1} m(r, B_j)}{r^{\mu_p(B_0)}} \leq \tau_1 < \tau. \tag{3.10}$$

By (3.10), for sufficiently large r, we have

$$m(r, B_j) < \exp_{p-1}\{(\tau_1 + \varepsilon) r^{\mu_p(B_0)}\}. \tag{3.11}$$

By (3.4),(3.7),(3.9) and (3.11), we obtain

$$\exp_{p-1}\{(\tau - \varepsilon) r^{\mu_p(B_0)}\} \leq (n-1) \exp_{p-1}\{(\tau_1 + \varepsilon) r^{\mu_p(B_0)}\} + O(\log(rT(r, f))), \tag{3.12}$$

where $r \notin E$, E is a set of r of finite linear measure. By Lemma 3.2 and (3.12), we have $\sigma_{p+1}(f) \geq \mu_p(B_0)$. □

Lemma 3.9 ([11])**.** *Let $f(z)$ be a meromorphic function of finite iterated order satisfying $i(f) = p$, then there exists a set $E_8 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_8$, we have*

$$\lim_{r\to\infty,\, r\in E_8} \frac{\log_p T(r, f)}{\log r} = \sigma_p(f).$$

Lemma 3.10. *Let $B_j(z)$, $(j = 0, 1, \dots, n-1)$ be meromorphic functions of finite iterated orders. If*

$$\beta_1 = \max\Big\{\limsup_{r\to\infty} \frac{\log_p m(r, B_j)}{\log r}, j \neq 0\Big\} < \beta_0 = \lim_{r\to\infty} \frac{\log_p m(r, B_0)}{\log r},\ r \in E_9, \tag{3.13}$$

where E_9 is a subset of r of infinite logarithmic measure. Then every meromorphic solution $f(z) \not\equiv 0$ of (3.2) satisfies $\sigma_{p+1}(f) \geq \beta_0$.

Proof. By (3.13), we have

$$m(r, B_j) < \exp_{p-1}\{r^{\beta_1 + \varepsilon}\}, \tag{3.14}$$

for any given $\varepsilon > 0$ and sufficiently large r. By the hypotheses of Lemma 3.10, there exists a set E_9 having infinite logarithmic measure such that for all $|z| = r \in E_9$, we have

$$m(r, B_0) > \exp_{p-1}\{r^{\beta_0 - \varepsilon}\}. \tag{3.15}$$

By (3.4),(3.14) and (3.15), we have

$$\exp_{p-1}\{r^{\beta_0 - \varepsilon}\} \leq O\left(\log(rT(r, f))\right) + (n-1) \exp_{p-1}\{r^{\beta_1 + \varepsilon}\}, \tag{3.16}$$

for any given $\varepsilon (0 < 2\varepsilon < \beta_0 - \beta_1)$, where $r \in E_9 \backslash E, r \to \infty$, and E is a set of r of finite linear measure. By (3.16), we have $\sigma_{p+1}(f) \geq \beta_0$. □

4. Proofs of main theorems

Proof of Theorem 2.4. By Theorem 2.1, we know that every solution $f(z) \not\equiv 0$ of (2.1) satisfies $\sigma_{p+1}(f) = \sigma_p(A_0)$. Then we only need to prove that every solution $f(z)$ of (2.1) satisfies $\mu_{p+1}(f) = \mu_p(A_0)$.

We rewrite (2.1) as

$$|A_0(z)| \le \Big|\frac{f^{(n)}(z)}{f(z)}\Big| + |A_{n-1}(z)|\Big|\frac{f^{(n-1)}(z)}{f(z)}\Big| + \dots + |A_1(z)|\Big|\frac{f'(z)}{f(z)}\Big|. \tag{4.1}$$

Set $\max\{\sigma_p(A_j) : j \neq 0\} = c$, then for any given $\varepsilon (0 < 2\varepsilon < \mu_p(A_0) - c)$ and for sufficiently large r, we have

$$M(r, A_0) \ge \exp_p\{r^{\mu_p(A_0)-\varepsilon}\}, \tag{4.2}$$

and

$$M(r, A_j) \le \exp_p\{r^{c+\varepsilon}\}, \quad (j = 1, 2, \dots, n-1). \tag{4.3}$$

By Lemma 3.6, there exists a set E_6 having finite logarithmic measure and a constant $C > 0$ such that for all z satisfying $|z| = r \notin E_6 \cup [0, 1]$, we have

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \le C(T(2r, f))^{k+1}, \ (k \ge 1). \tag{4.4}$$

Substituting (4.2)-(4.4) into (4.1), for the above $\varepsilon > 0$, we have

$$\exp_p\{r^{\mu_p(A_0)-\varepsilon}\} \le Cn \exp_p\{r^{c+\varepsilon}\} \left(T(2r, f)\right)^{n+1}, \tag{4.5}$$

for all z satisfying $|z| = r \notin E_6 \cup [0, 1]$, $r \to \infty$ and $|A_0(z)| = M(r, A_0)$. By Lemma 3.2 and (4.5), we have $\mu_{p+1}(f) \ge \mu_p(A_0) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain

$$\mu_{p+1}(f) \ge \mu_p(A_0). \tag{4.6}$$

By (2.1), we have

$$\Big|\frac{f^{(n)}(z)}{f(z)}\Big| \le |A_{n-1}(z)|\Big|\frac{f^{(n-1)}(z)}{f(z)}\Big| + \dots + |A_1(z)|\Big|\frac{f'(z)}{f(z)}\Big| + |A_0(z)|. \tag{4.7}$$

By Lemma 3.1, there exists a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1$, and $|f(z)| = M(r, f)$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| = \Big|\frac{\nu_f(r)}{z}\Big|^j |1 + o(1)|, \quad (j = 1, \dots, n). \tag{4.8}$$

By Lemma 3.4, there exists a set $E_4 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_4 \backslash E_1$, we have

$$|A_0(z)| \le M(r, A_0) \le \exp_p\{r^{\mu_p(A_0)+\varepsilon}\}. \tag{4.9}$$

Hence, by (4.3),(4.7)-(4.9), we have

$$|\nu_f(r)|^n |1 + o(1)| \le n \exp_p\{r^{\mu_p(A_0)+\varepsilon}\} r^n |\nu_f(r)|^{n-1} |1 + o(1)|,$$

then we obtain

$$|\nu_f(r)||1 + o(1)| \le nr^n \exp_p\{r^{\mu_p(A_0)+\varepsilon}\}, \quad (r \in E_4 \backslash E_1). \tag{4.10}$$

By the definition of iterated p-lower order and (4.10), we have $\mu_{p+1}(f) \le \mu_p(A_0) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have

$$\mu_{p+1}(f) \le \mu_p(A_0). \tag{4.11}$$

By (4.6) and (4.11), we obtain $\mu_{p+1}(f) = \mu_p(A_0)$. The proof is complete. $\square$

Proof of Theorem 2.5. By Theorem 2.2, we have $\sigma_{p+1}(f)=\sigma_p(A_0)$. Now we need to prove (1) $\mu_{p+1}(f)=\mu_p(A_0)$ and (2) $\sigma_{p+1}(f)=\overline{\lambda}_{p+1}(f-\varphi)$.

(1) On the one hand, we set $b=\max\{\sigma_p(A_j),\sigma_p(A_j)<\mu_p(A_0)\}$. If $\sigma_p(A_j)<\mu_p(A_0)$, then for any given $\varepsilon(0<2\varepsilon<\min\{\mu_p(A_0)-b,\tau-\tau_1\})$ and for sufficiently large r, we have

$$M(r,A_j)\le \exp_p\{r^{b+\varepsilon}\}\le \exp_p\{r^{\mu_p(A_0)-\varepsilon}\}. \tag{4.12}$$

If $\sigma_p(A_j)=\mu_p(A_0)$, $\tau_p(A_j)\le\tau_1<\tau=\underline{\tau}_p(A_0)$, then for sufficiently large r, we have

$$M(r,A_j)\le \exp_p\{(\tau_1+\varepsilon)r^{\mu_p(A_0)}\}, \tag{4.13}$$

$$M(r,A_0)\ge \exp_p\{(\tau-\varepsilon)r^{\mu_p(A_0)}\}. \tag{4.14}$$

By (4.12), (4.13), (4.14), (4.1) and (4.4), we obtain

$$\exp_p\{(\tau-\varepsilon)r^{\mu_p(A_0)}\}\le n\exp_p\{(\tau_1+\varepsilon)r^{\mu_p(A_0)}\}CT(r,f)^{n+1}, \tag{4.15}$$

where $C>0$ is a constant, for all z satisfying $|z|=r\notin E_6\cup[0,1]$, $r\to\infty$ and $|A_0(z)|=M(r,A_0)$. By Lemma 3.2 and (4.15), we have $\mu_{p+1}(f)\ge\mu_p(A_0)$.

On the other hand, by Lemma 3.4, there exists a set E_4 having infinite logarithmic measure such that for all $r\in E_4$, we have

$$|A_0(z)|\le M(r,A_0)\le \exp_p\{(\tau+\varepsilon)r^{\mu_p(A_0)}\}. \tag{4.16}$$

By (4.7), (4.8), (4.12), (4.13) and (4.16), we have

$$|\nu_f(r)|^n|1+o(1)|\le n\exp_p\{(\tau+\varepsilon)r^{\mu_p(A_0)}\}r^n|\nu_f(r)|^{n-1}|1+o(1)|, \tag{4.17}$$

where $r\in E_4\backslash E_1$, $r\to\infty$. By the definition of iterated p-lower order and (4.17), we obtain $\mu_{p+1}(f)\le\mu_p(A_0)$. Thus, we have $\mu_{p+1}(f)=\mu_p(A_0)$.

(2) We prove that $\overline{\lambda}_{p+1}(f-\varphi)=\sigma_{p+1}(f)$. Assume that $f(z)\not\equiv 0$ is a solution of (2.1), then $\sigma_{p+1}(f)=\sigma_p(A_0)$. Set $g=f-\varphi$, since $\sigma_{p+1}(\varphi)<\mu_p(A_0)\le\sigma_p(A_0)$, then $\sigma_{p+1}(g)=\sigma_{p+1}(f)=\sigma_p(A_0)$, $\overline{\lambda}_{p+1}(g)=\overline{\lambda}_{p+1}(f-\varphi)$. Substituting $f=g+\varphi, f'=g'+\varphi',\dots,f^{(n)}=g^{(n)}+\varphi^{(n)}$, into (2.1), we obtain

$$g^{(n)}+A_{n-1}(z)g^{(n-1)}+\dots+A_0(z)g=-[\varphi^{(n)}+A_{n-1}(z)\varphi^{(n-1)}+\dots+A_0(z)\varphi]. \tag{4.18}$$

If $F(z)=\varphi^{(n)}+A_{n-1}(z)\varphi^{(n-1)}+\dots+A_0(z)\varphi\equiv 0$, then by Lemma 3.8, we have $\sigma_{p+1}(\varphi)\ge\mu_p(A_0)$, which is a contradiction. Since $F(z)\not\equiv 0$ and $\sigma_{p+1}(F)<\sigma_{p+1}(f)=\sigma_{p+1}(g)$. By Lemma 3.7 and (4.18), we have $\overline{\lambda}_{p+1}(g)=\lambda_{p+1}(g)=\sigma_{p+1}(g)=\sigma_p(A_0)$. Therefore, $\overline{\lambda}_{p+1}(f-\varphi)=\lambda_{p+1}(f-\varphi)=\sigma_{p+1}(f)=\sigma_p(A_0)$. The proof is complete. □

Proof of Theorem 2.6. By Lemma 3.5, we have $\sigma_{p+1}(f)=\sigma_p(A_0)$. Now we need to prove (1) $\mu_{p+1}(f)=\mu_p(A_0)$ and (2) $\sigma_{p+1}(f)=\overline{\lambda}_{p+1}(f-\varphi)$.

(1) On the one hand, by (4.1) and the logarithmic derivative lemma, we have

$$m(r,A_0)\le\sum_{j=1}^{n-1}m(r,A_j)+O(\log(rT(r,f))),\quad (r\notin E), \tag{4.19}$$

where E is a set of r of finite linear measure.

Setting $\limsup_{r\to\infty}\sum_{j=1}^{n-1} m(r,A_j)/m(r,A_0) < \beta < 1$, for sufficiently large r, we have

$$\sum_{j=1}^{n-1} m(r,A_j) < \beta m(r,A_0). \tag{4.20}$$

By (4.19) and (4.20), we have

$$(1-\beta)m(r,A_0) \le O(\log(rT(r,f))), \quad (r \notin E). \tag{4.21}$$

By $\mu_p(A_0) = \mu$, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$m(r,A_0) \ge \exp_{p-1}\{r^{\mu-\varepsilon}\}. \tag{4.22}$$

By (4.21) and (4.22), for the above $\varepsilon > 0, r \notin E, r \to \infty$, we have

$$(1-\beta)\exp_{p-1}\{r^{\mu-\varepsilon}\} \le O\big(\log(rT(r,f))\big). \tag{4.23}$$

By Lemma 3.2 and (4.23), we have $\mu - \varepsilon \le \mu_{p+1}(f)$. Since $\varepsilon > 0$ is arbitrary, we have $\mu_p(A_0) = \mu \le \mu_{p+1}(f)$.

On the other hand, since $\max\{\sigma_p(A_j), j \ne 0\} \le \mu_p(A_0) = \mu$, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$|A_j(z)| \le \exp_p\{r^{\mu+\varepsilon}\}, \quad (j = 1, \dots, n-1). \tag{4.24}$$

By Lemma 3.4, there exists a set of E_2 having infinite logarithmic measure such that for all $r \in E_2$, we have

$$|A_0(z)| \le \exp_p\{r^{\mu+\varepsilon}\}. \tag{4.25}$$

By (4.7), (4.8), (4.24) and (4.25), we have

$$|\nu_f(r)|^n|1+o(1)| \le n\exp_p\{r^{\mu+\varepsilon}\}r^n|\nu_f(r)|^{n-1}|1+o(1)|. \tag{4.26}$$

By (4.26), for the above $\varepsilon > 0$, we obtain

$$|\nu_f(r)||1+o(1)| \le nr^n\exp_p\{r^{\mu+\varepsilon}\}, \tag{4.27}$$

where $|z| = r \in E_2\backslash E_1, r \to \infty, |f(z)| = M(r,f)$. By (4.27), we obtain $\mu_{p+1}(f) \le \mu + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\mu_{p+1}(f) \le \mu$. Thus, we have $\mu_{p+1}(f) = \mu_p(A_0)$.

(2) We prove that $\overline{\lambda}_{p+1}(f-\varphi) = \sigma_{p+1}(f)$. Setting $g = f - \varphi$, since $\sigma_{p+1}(\varphi) < \mu_p(A_0)$, we have $\sigma_{p+1}(g) = \sigma_{p+1}(f) = \sigma_p(A_0)$, $\overline{\lambda}_{p+1}(g) = \overline{\lambda}_{p+1}(f-\varphi)$. Substituting $f = g+\varphi, f' = g'+\varphi', \dots, f^{(n)} = g^{(n)} + \varphi^{(n)}$ into (2.1), we obtain

$$g^{(n)}+A_{n-1}(z)g^{(n-1)}+\dots+A_0(z)g = -[\varphi^{(n)}+A_{n-1}(z)\varphi^{(n-1)}+\dots+A_0(z)\varphi]. \tag{4.28}$$

If $F(z) = \varphi^{(n)} + A_{n-1}(z)\varphi^{(n-1)} + \dots + A_0(z)\varphi \equiv 0$, then by part (1), we have $\sigma_{p+1}(\varphi) \ge \mu_p(A_0)$, which is a contradiction. Since $F(z) \not\equiv 0$ and $\sigma_{p+1}(F) < \sigma_{p+1}(f) = \sigma_{p+1}(g)$. By Lemma 3.7 and (4.28), we have $\overline{\lambda}_{p+1}(g) = \lambda_{p+1}(g) = \sigma_{p+1}(g) = \sigma_p(A_0)$. Therefore, $\mu_p(A_0) = \mu_{p+1}(f) \le \sigma_{p+1}(f) = \sigma_p(A_0) = \overline{\lambda}_{p+1}(f-\varphi) = \lambda_{p+1}(f-\varphi)$. The proof is complete. $\square$

5. Proof of Theorem 2.8

By Lemma 3.3, we have $\sigma_{p+1}(f) = \sigma_p(A_0)$. Now we prove that $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f)$.

(1) We prove the $\overline{\lambda}_{p+1}(f - \varphi) = \sigma_{p+1}(f)$. Setting $g = f - \varphi$, since $\sigma_{p+1}(\varphi) < \sigma_p(A_0)$, we have $\sigma_{p+1}(g) = \sigma_{p+1}(f) = \sigma_p(A_0)$, $\overline{\lambda}_{p+1}(g) = \overline{\lambda}_{p+1}(f-\varphi)$. Substituting $f = g + \varphi, f' = g' + \varphi', \dots, f^{(n)} = g^{(n)} + \varphi^{(n)}$ into (2.1), we obtain

$$g^{(n)} + A_{n-1}(z)g^{(n-1)} + \dots + A_0(z)g = -[\varphi^{(n)} + A_{n-1}(z)\varphi^{(n-1)} + \dots + A_0(z)\varphi]. \tag{5.1}$$

Since $\lambda_p(\frac{1}{A_0}) < \mu_p(A_0)$, we have $N(r, A_0) = o(T(r, A_0)), r \to \infty$. Therefore, by Lemma 3.9, we have

$$\begin{aligned} \sigma_p(A_0) &= \limsup_{r\to\infty} \frac{\log_p T(r, A_0)}{\log r} = \lim_{r\to\infty,\, r\in E_8} \frac{\log_p T(r, A_0)}{\log r} \\ &= \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, A_0)}{\log r}, \end{aligned} \tag{5.2}$$

where E_8 is a subset of r of infinite logarithmic measure. Combining the assumption and (5.2), we have

$$\limsup_{r\to\infty} \frac{\log_p m(r, A_j)}{\log r} < \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, A_0)}{\log r} = \sigma_p(A_0),\ j = 1, \dots, n. \tag{5.3}$$

If $F(z) = \varphi^{(n)} + A_{n-1}(z)\varphi^{(n-1)} + \dots + A_0(z)\varphi \equiv 0$, then by Lemma 3.10, we have $\sigma_{p+1}(\varphi) \geq \sigma_p(A_0)$, which is a contradiction. Since $F(z) \not\equiv 0$ and $\sigma_{p+1}(F) < \sigma_{p+1}(f) = \sigma_{p+1}(g)$, by Lemma 3.7 and (5.1), we have $\overline{\lambda}_{p+1}(g) = \lambda_{p+1}(g) = \sigma_{p+1}(g) = \sigma_p(A_0)$. Therefore, $\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \sigma_{p+1}(f) = \sigma_p(A_0)$.

(2) We prove that $\overline{\lambda}_{p+1}(f' - \varphi) = \sigma_{p+1}(f)$. Setting $g_1 = f' - \varphi$, we have $\sigma_{p+1}(g_1) = \sigma_{p+1}(f) = \sigma_p(A_0)$ and

$$f' = g_1 + \varphi, \dots, f^{(n+1)} = g_1^{(n)} + \varphi^{(n)}. \tag{5.4}$$

By (2.1), we have

$$f(z) = -\frac{1}{A_0(z)} \left(f^{(n)} + \dots + A_1(z)f' \right). \tag{5.5}$$

The derivative of (2.1) is

$$f^{(n+1)} + A_{n-1}f^{(n)} + (A'_{n-1} + A_{n-2})f^{(n-1)} + \dots + (A'_1 + A_0)f' + A'_0 f = 0. \tag{5.6}$$

Substituting (5.4) and (5.5) into (5.6), we obtain

$$\begin{aligned} & g_1^{(n)} + (A_{n-1} - \frac{A'_0}{A_0})g_1^{(n-1)} + (A_{n-2} + A'_{n-1} - \frac{A_{n-1}A'_0}{A_0})g_1^{(n-2)} + \dots \\ & + (A_0 + A'_1 - \frac{A_1A'_0}{A_0})g_1 \\ & = -\big[\varphi^{(n)} + (A_{n-1} - \frac{A'_0}{A_0})\varphi^{(n-1)} + \dots + (A_0 + A'_1 - \frac{A_1A'_0}{A_0})\varphi\big]. \end{aligned}$$

Setting

$$\begin{gathered} B_{n-1} = A_{n-1} - \frac{A'_0}{A_0}, \quad B_{n-2} = A_{n-2} + A'_{n-1} - \frac{A_{n-1}A'_0}{A_0}, \\ \dots, \quad B_0 = A_0 + A'_1 - \frac{A_1A'_0}{A_0}, \end{gathered} \tag{5.7}$$

we have

$$g_1^{(n)}+B_{n-1}g_1^{(n-1)}+B_{n-2}g_1^{(n-2)}+\dots+B_0g_1=-\big[\varphi^{(n)}+B_{n-1}\varphi^{(n-1)}+\dots+B_0\varphi\big]. \tag{5.8}$$

By (5.3) and (5.7), we have

$$\limsup_{r\to\infty}\frac{\log_p m(r,B_j)}{\log r}<\lim_{r\to\infty,\,r\in E_8}\frac{\log_p m(r,A_0)}{\log r}=\sigma_p(A_0),\quad (j\neq 0), \tag{5.9}$$

and

$$\sigma_p(A_0)=\lim_{r\to\infty,\,r\in E_8}\frac{\log_p m(r,A_0)}{\log r}=\lim_{r\to\infty,\,r\in E_8}\frac{\log_p m(r,B_0)}{\log r}, \tag{5.10}$$

where E_8 is a subset of infinite logarithmic measure r. Let $F_1(z)=\varphi^{(n)}+B_{n-1}\varphi^{(n-1)}+\dots+B_0\varphi$. We affirm $F_1(z)\not\equiv 0$. If $F_1(z)\equiv 0$, then by (5.9), (5.10) and Lemma 3.10, we obtain $\sigma_{p+1}(\varphi)\geq\sigma_p(A_0)$, which is a contradiction. Since $F_1(z)\not\equiv 0$, and $\sigma_{p+1}(F_1)<\sigma_{p+1}(g_1)=\sigma_p(A_0)$. By Lemma 3.7 and (5.8), we obtain

$$\overline{\lambda}_{p+1}(f'-\varphi)=\lambda_{p+1}(f'-\varphi)=\sigma_{p+1}(f).$$

(3) We prove that $\overline{\lambda}_{p+1}(f''-\varphi)=\sigma_{p+1}(f)$. Setting $g_2=f''-\varphi$, we have $\sigma_{p+1}(g_2)=\sigma_{p+1}(f)=\sigma_p(A_0)$ and

$$f''=g_2+\varphi,\dots,f^{(n+2)}=g_2^{(n)}+\varphi^{(n)}. \tag{5.11}$$

Substituting (5.5) into (5.6), we have

$$\begin{aligned}&f^{(n+1)}+(A_{n-1}-\frac{A_0'}{A_0})f^{(n)}+(A_{n-2}+A_{n-1}'-\frac{A_{n-1}A_0'}{A_0})f^{(n-1)}+\dots\\&+(A_0+A_1'-\frac{A_1A_0'}{A_0})f'=0.\end{aligned} \tag{5.12}$$

The derivative of (5.12) is

$$\begin{aligned}&f^{(n+2)}+(A_{n-1}-\frac{A_0'}{A_0})f^{(n+1)}+\big[(A_{n-1}-\frac{A_0'}{A_0})'+(A_{n-2}+A_{n-1}'-\frac{A_{n-1}A_0'}{A_0})\big]f^{(n)}\\&+\dots+(A_0+A_1'-\frac{A_1A_0'}{A_0})'f'=0.\end{aligned} \tag{5.13}$$

By (5.12), we have

$$f'=-\big[\frac{1}{A_0+A_1'-\frac{A_1A_0'}{A_0}}f^{(n+1)}+\frac{A_{n-1}-\frac{A_0'}{A_0}}{A_0+A_1'-\frac{A_1A_0'}{A_0}}f^{(n)}+\dots+\frac{A_1+A_2'-\frac{A_2A_0'}{A_0}}{A_0+A_1'-\frac{A_1A_0'}{A_0}}f''\big]. \tag{5.14}$$

Substituting (5.14) into (5.13), we have

$$\begin{aligned}&f^{(n+2)}+\big[(A_{n-1}-\frac{A_0'}{A_0})-\frac{(A_0+A_1'-\frac{A_1A_0'}{A_0})'}{A_0+A_1'-\frac{A_1A_0'}{A_0}}\big]f^{(n+1)}+\dots\\&+\big[(A_0+A_1'-\frac{A_1A_0'}{A_0})+(A_1+A_2'-\frac{A_2A_0'}{A_0})'\\&-\frac{(A_1+A_2'-\frac{A_2A_0'}{A_0})(A_0+A_1'-\frac{A_1A_0'}{A_0})'}{A_0+A_1'-\frac{A_1A_0'}{A_0}}\big]f''=0.\end{aligned}$$

Setting

$$C_{n-1} = B_{n-1} - \frac{B_0'}{B_0}, \quad C_{n-2} = B_{n-2} + B_{n-1}' - \frac{B_{n-1}B_0'}{B_0}, \\ \dots, \quad C_0 = B_0 + B_1' - \frac{B_1 B_0'}{B_0}, \tag{5.15}$$

we obtain

$$f^{(n+2)} + C_{n-1}(z)f^{(n+1)} + \dots + C_0(z)f'' = 0. \tag{5.16}$$

Substituting (5.11) into (5.16), we obtain

$$g_2^{(n)} + C_{n-1}(z)g_2^{(n-1)} + \dots + C_0(z)g_2 = -\big[\varphi^{(n)} + C_{n-1}(z)\varphi^{(n-1)} + \dots + C_0(z)\varphi\big]. \tag{5.17}$$

By (5.2), (5.9), (5.10) and (5.15), we have

$$\limsup_{r\to\infty} \frac{\log_p m(r, C_j)}{\log r} < \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, A_0)}{\log r} = \sigma_p(A_0), (j \neq 0), \tag{5.18}$$

and

$$\sigma_p(A_0) = \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, A_0)}{\log r} = \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, C_0)}{\log r}, \tag{5.19}$$

where E_8 is a subset of r of infinite logarithmic measure. If $F_2(z) \equiv \varphi^{(n)} + C_{n-1}(z)\varphi^{(n-1)} + \dots + C_0(z)\varphi \equiv 0$, then by (5.18), (5.19) and Lemma 3.10, we have $\sigma_{p+1}(\varphi) \geq \sigma_p(A_0)$, which is a contradiction. Therefore, $F_2(z) \not\equiv 0$. Since $\sigma_{p+1}(F_2) < \sigma_{p+1}(g_2) = \sigma_p(A_0)$, by Lemma 3.7 and (5.17), we have

$$\overline{\lambda}_{p+1}(f'' - \varphi) = \lambda_{p+1}(f'' - \varphi) = \sigma_{p+1}(f).$$

(4) We prove that $\overline{\lambda}_{p+1}(f''' - \varphi) = \sigma_{p+1}(f)$. Setting $g_3 = f''' - \varphi$, then $\sigma_{p+1}(g_3) = \sigma_{p+1}(f) = \sigma_p(A_0)$ and

$$f''' = g_3 + \varphi, \quad \dots, \quad f^{(n+3)} = g_3^{(n)} + \varphi^{(n)}. \tag{5.20}$$

The derivative of (5.16) is

$$f^{(n+3)} + C_{n-1}f^{(n+2)} + (C_{n-1}' + C_{n-2})f^{(n+1)} + \dots + (C_1' + C_0)f''' + C_0' f'' = 0. \tag{5.21}$$

By (5.16), we have

$$f'' = -\Big[\frac{1}{C_0}f^{(n+2)} + \frac{C_{n-1}}{C_0}f^{(n+1)} + \dots + \frac{C_1}{C_0}f'''\Big]. \tag{5.22}$$

Substituting (5.22) into (5.21), we have

$$f^{(n+3)} + \Big(C_{n-1} - \frac{C_0'}{C_0}\Big)f^{(n+2)} + \Big(C_{n-2} + C_{n-1}' - \frac{C_{n-1}C_0'}{C_0}\Big)f^{(n+1)} \\ + \dots + \Big(C_0 + C_1' - \frac{C_1 C_0'}{C_0}\Big)f''' = 0. \tag{5.23}$$

Setting

$$D_{n-1} = C_{n-1} - \frac{C_0'}{C_0}, \quad D_{n-2} = C_{n-2} + C_{n-1}' - \frac{C_{n-1}C_0'}{C_0}, \\ \dots, \quad D_0 = C_0 + C_1' - \frac{C_1 C_0'}{C_0}, \tag{5.24}$$

we have

$$f^{(n+3)} + D_{n-1}(z)f^{(n+2)} + \dots + D_0(z)f''' = 0. \tag{5.25}$$

Substituting (5.20) into (5.25), we obtain

$$g_3^{(n)} + D_{n-1}(z)g_3^{(n-1)} + \cdots + D_0(z)g_3 = -[\varphi^{(n)} + D_{n-1}(z)\varphi^{(n-1)} + \cdots + D_0(z)\varphi]. \tag{5.26}$$

By (5.18), (5.19) and (5.24), we have

$$\limsup_{r\to\infty} \frac{\log_p m(r, D_j)}{\log r} < \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, A_0)}{\log r} = \sigma_p(A_0), \ (j \neq 0), \tag{5.27}$$

and

$$\sigma_p(A_0) = \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, A_0)}{\log r} = \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, D_0)}{\log r}, \tag{5.28}$$

where E_8 is a subset of r of infinite logarithmic measure. Let $F_3(z) = \varphi^{(n)} + D_{n-1}(z)\varphi^{(n-1)} + \cdots + D_0(z)\varphi \equiv 0$, by (5.27), (5.28) and Lemma 3.10, we have $F_3(z) \not\equiv 0$. Since $\sigma_{p+1}(F_3) < \sigma_{p+1}(g_3) = \sigma_p(A_0)$, by Lemma 3.7 and (5.26), we have

$$\overline{\lambda}_{p+1}(f''' - \varphi) = \lambda_{p+1}(f''' - \varphi) = \sigma_{p+1}(f).$$

(5) We prove that $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f)$, $(j > 3)$. Setting $g_j = f^{(j)} - \varphi$, $(j > 3)$, then $\sigma_{p+1}(g_j) = \sigma_{p+1}(f^{(j)}) = \sigma_p(A_0)$ and

$$f^{(j+1)} = g_j' + \varphi', \quad \dots, f^{(n)} = g_j^{(n-j)} + \varphi^{(n-j)}, \quad (j > 3). \tag{5.29}$$

By successive derivation on (5.25), we also get an equation which has similar form with (5.23). Furthermore, combining (5.29), we can get

$$\begin{aligned} &g_j^{(n)} + (H_{n-1} - \frac{H_0'}{H_0})g_j^{(n-1)} + \cdots + (H_0 + H_1' - \frac{H_1 H_0'}{H_0})g_j \\ &= -\big[\varphi^{(n)} + \cdots + (H_0 + H_1' - \frac{H_1 H_0'}{H_0})\varphi\big], \end{aligned} \tag{5.30}$$

where $H_j(z)$, $(j = 0, 1, \dots, n-1)$ are meromorphic functions which have the same form as $D_j(z), (j = 1, \dots, n-1)$. Setting $G_{n-1} = H_{n-1} - \frac{H_0'}{H_0}, \dots, G_0 = H_0 + H_1' - \frac{H_1 H_0'}{H_0}$, we have

$$\limsup_{r\to\infty} \frac{\log_p m(r, G_j)}{\log r} < \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, A_0)}{\log r} = \sigma_p(A_0), \ (j \neq 0),$$

and

$$\sigma_p(A_0) = \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, A_0)}{\log r} = \lim_{r\to\infty,\, r\in E_8} \frac{\log_p m(r, G_0)}{\log r},$$

where E_8 is a subset of r of infinite logarithmic measure. By Lemmas 3.7 and 3.10, we can get $\overline{\lambda}_{p+1}(g_j) = \lambda_{p+1}(g_j) = \sigma_{p+1}(g_j)$; i.e., $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f)$. the proof of Theorem 2.8 is complete.

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Hui Hu
Institute of Mathematics and Information Science, Jiangxi Normal University, 330022, China
E-mail address: h_h87_6@hotmail.com

Xiu-Min Zheng (Corresponding author)
Institute of Mathematics and Information Science, Jiangxi Normal University, 330022, China
E-mail address: zhengxiumin2008@sina.com