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FRONT TRACKING FOR A 2×2 SYSTEM OF CONSERVATION LAWS

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Abstract. This article studies a front-tracking algorithm for 2×2 systems of conservation laws. After revisiting the classical results of DiPerna [12] and Bressan [8], we address the case of a 2×2 system arising in the study of granular flows [2]. For the latter we prove the well-definiteness of a simplified front-tracking algorithm and its convergence to a weak entropic solution of the system, in the case of large BV initial data.

1. Introduction

Let us consider a Cauchy problem associated with a 2×2 system of conservation laws

$$u_t + f(u)_x = 0, \quad (x,t) \in \mathbb{R} \times [0, \infty[, \tag{1.1}$$

and initial conditions

$$u(x,0) = \overline{u}(x). \tag{1.2}$$

Assume that the system is strictly hyperbolic, with smooth coefficients and defined on an open set $\Omega \subseteq \mathbb{R}^2$. Moreover, suppose that each characteristic field is either genuinely nonlinear or linearly degenerate. Given a function $\overline{u}$ with sufficiently small total variation, one can prove the global in time existence of a weak, entropy-admissible solution. The first proof of this result was due to Glimm and his celebrated random choice algorithm [10]. Nowadays, one of the most largely used methods to achieve the same result is front-tracking, described in [8], that consists in constructing a sequence of piecewise constant approximate solutions, a subsequence of which converges to a weak solution of the Cauchy problem (1.1)-(1.2).

The basic ideas involved were introduced by Dafermos in [11] for scalar equations and DiPerna in [12] for 2×2 systems, then extended by Bressan in [7, 8] to general $n \times n$ systems with genuinely nonlinear or linearly degenerate characteristic fields. The construction of the approximate solutions starts at time $t = 0$ by taking a piecewise constant approximation $\tilde{u}(x)$ of the initial data $\overline{u}(x)$. At each point of discontinuity a piecewise constant approximate solution of the corresponding Riemann problem is chosen so that it coincides with the exact one if it contains only shocks or contact discontinuities. Otherwise, if centred rarefaction waves are present, they are replaced with rarefaction fans containing several small jumps

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traveling with speed close to the characteristic one. The approximate solution $\tilde{u}$ is then prolonged until the first time two wave-fronts interact. Since at this time $\tilde{u}$ is still a piecewise constant function, the corresponding Riemann problems can be approximately solved again within the class of piecewise constant functions and so on.

For $n \times n$ systems the main source of technical difficulty derives from the fact that the number of lines of discontinuity may approach infinity in finite time, in which case the construction would break down. This is due mostly to the fact that at each interaction point there are two incoming fronts, while the number of outgoing ones is n or even larger if rarefaction waves are involved. To overcome this difficulty, the algorithm in [8, 6] adopts two different procedures to approximately solve an emerging Riemann problem: an *Accurate Riemann Solver* which introduces several new fronts, and a *Simplified Riemann Solver*, which involves at most two physical outgoing fronts and collect the remaining new waves into a single "non-physical" front traveling with a speed strictly larger than all characteristic speeds. In [8, 6] the algorithm is proved to converge to a weak solution of (1.1)-(1.2) at least in the case of small BV data. Afterwards these results were extended to systems of conservation laws whose characteristic fields are neither genuinely nonlinear nor linearly degenerate [4, 3, 5].

Nevertheless, when dealing with a 2×2 system satisfying the previous assumptions it is possible to avoid non-physical fronts and always use an accurate solver to construct approximate solutions. This was initially proved for the front-tracking introduced by DiPerna in [12], in the case of small BV data. However his construction is quite tricky and less used than the one proposed by Bressan in [7] and refined in [6], a slight modification of which can avoid the introduction of non-physical fronts in the 2×2 case, too. In a few words, in the $n = 2$ case the only problem comes from the fact that rarefaction waves can be partitioned generating several fronts and this is the unique way the total number of waves could increase. In the first part of the paper we will revisit Bressan's construction and propose a further slight modification of the algorithm, still avoiding the non-physical fronts, which will be used in the second part.

The second part of the paper will be devoted to the case study of a 2×2 system of balance laws modeling granular flows, discussed by Amadori and Shen [2]. In their paper they prove the existence of a weak solution for the initial value problem associated to that system, in the case of possibly large BV data. Unfortunately the front-tracking algorithm they refer to leads to the introduction of another kind of non-physical fronts; since the subsequent part of their paper strongly depends on the analysis along characteristics, that algorithm seems not well suited for their purposes. Moreover, the previous results [6, 8, 5] are not applicable since the data in [2] may have large total variation. The second goal of this paper is to prove that the simplified version, without non-physical fronts, of the front-tracking algorithm for 2×2 conservation laws works also for the system proposed in [2] and in the presence of large BV data; this will be accomplished in Theorem 3.4.

2. Front-tracking for 2×2 systems

In this section we briefly recall the front-tracking algorithm, for full details the reader is referred to [8]. In general we can consider a strictly hyperbolic $n \times n$ system of conservation laws (1.1) with $u \in \Omega \subseteq \mathbb{R}^n$ in which each characteristic family is either genuinely nonlinear or linearly degenerate, and where the flux f is $C^2(\Omega)$. Given two states u^-, u^+ sufficiently close, the corresponding Riemann problem, that is the Cauchy problem with initial data given by

$$\overline{u}(x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0, \end{cases} \tag{2.1}$$

admits a self-similar solution given by at most $n+1$ constant states separated by shocks, contact discontinuities or rarefaction waves [8]. More precisely, there exist C^2 curves $\sigma \mapsto \psi_i(\sigma)(u^-)$, $i = 1, \dots, n$, parametrized by arclength, such that

$$u^+ = \psi_n(\sigma_n) \circ \cdots \circ \psi_1(\sigma_1)(u^-), \tag{2.2}$$

for some $\sigma_1, \dots, \sigma_n$. We define $u_0 \doteq u^-$ and $u_i \doteq \psi_i(\sigma_i) \circ \cdots \circ \psi_1(\sigma_1)(u_0)$. When σ_i is positive (negative) and the i-th characteristic family is genuinely nonlinear, the states u_{i-1} and u_i are separated by an i-rarefaction (i-shock) wave. If the i-th characteristic family is linearly degenerate the states u_{i-1} and u_i are separated by a contact discontinuity. The *strength* of the i-wave is defined as $|\sigma_i|$. Therefore, the solution of (1.1),(2.1) is given by the concatenation of at most n waves, which are discontinuous functions in the case of shocks or contact discontinuities, or Lipschitz continuous functions in the case of rarefactions.

The construction of approximate solutions by front-tracking proceeds as follows: at time $t = 0$ we approximate the initial data $\overline{u}$ with a piecewise constant function. At each point of discontinuity the resulting Riemann problems are solved as above in terms of elementary waves. If a rarefaction is present in the solution, we replace it with a rarefaction fan containing several small jumps of strength less than a fixed size η and traveling with speed close to the characteristic one. Piecing together the solution of all the Riemann problems, we obtain an approximate solution of (1.1)-(1.2) defined on a small time interval $[0, \bar{t}]$. Every front is then prolonged until the first time two wave-fronts interact: at this time we approximately solve the emerging Riemann problem within the class of piecewise constant functions and so on. As recalled in the introduction, to show that this procedure can be carried on for all times, thus generating an approximate solution globally defined on $\mathbb{R} \times [0, +\infty[$, the main problem is to provide a uniform (in time) bound on the number of fronts and interactions that can be generated at each time. This bound would prevent the number to approach infinity in finite time. For $n \times n$ systems, in [8, 6] to achieve this goal two different procedures are used to approximately solve a Riemann problem emerging at time $t > 0$: an *Accurate Riemann Solver* which introduces several new fronts, and a *Simplified Riemann Solver* which, roughly speaking, lets the incoming waves pass through each other, if of different characteristic family, or stick together, if of the same family, and collects all the remaining newly born waves belonging to the other characteristic families in one single front traveling with speed strictly greater than all characteristic speeds; for this reason this front is in some sense non-physical, since it is not directly related to elementary waves/speeds of the system. The first procedure is used when the product of the strengths of the incoming waves is greater than a fixed threshold, the other one when it is smaller. The main difficulties in proving that the approximate solutions converge (up to subsequences) to a weak solution of (1.1)-(1.2) come from controlling the overall error generated by the introduction of non-physical fronts. However, if $n = 2$ the second procedure can be avoided; this was first observed by DiPerna for his algorithm in [12]. Here we want to verify that this is still true for a slight modification of the algorithm proposed in [8], in the case of small BV data. The simplified algorithm $\mathcal{ALG}$ can be simply described as follows:

- at time $t = 0$ the initial datum is approximated by a piecewise constant function having a finite number of jumps. Then we solve the Riemann problems arising at each discontinuity point. If a newly generated rarefaction front has strength $|\sigma|$ greater than a small parameter $\eta > 0$, fixed at the beginning, then it is partitioned in a number of $\lfloor |\sigma|/\eta \rfloor + 1$ (entropy violating) discontinuities;
- the fronts are prolonged until two of them interact, the emerging Riemann problem is approximately solved and so on. By slightly perturbing the speed of just one front we can assume that at each time a single interaction occurs, involving two incoming waves only. For interaction times $t > 0$ we always partition an outgoing rarefaction, except when a rarefaction of the same characteristic family is present also before the interaction. In that case the outgoing rarefaction will not be split but substituted by a single jump with the same strength.

In contrast to [8], here we prefer to distinguish cases according to the size of the outgoing rarefactions instead of the size of the interaction potential between interacting waves. Indeed, we will shortly verify that every time a newly generated rarefaction has strength greater than η the potential decreases by a fixed amount. Therefore this algorithm can differ from that in [8] only for interacting waves with sufficiently large potential and producing small rarefactions.

In order to ensure that the total number of wave-fronts and interactions remains finite even in this case, it suffices to verify that the number of interactions creating possibly large rarefactions is finite. Indeed, this is what is required in the following lemma (for a proof see [1, Lemma 2.3]).

Lemma 2.1. *Let a wave front-tracking pattern be given in $[0, T[\times\mathbb{R}$, made of segments of the two families. Assume that the velocities of the fronts of the first family lay between two constants $a_1 < a_2$ and the velocities of the fronts of the second family lay between $b_1 < b_2$, with $a_2 < b_1$. Assume that the wave front-tracking pattern has also the following properties:*

(i) *at $t = 0$ there is a finite number N_0 of waves;*
(ii) *the interactions occur only between two wave-fronts at any single time;*
(iii) *except a finite number of interactions, there is at most one outgoing wave of each family for each interaction.*

Then the number of interactions in the region $\mathbb{R} \times [0, T[$ is finite.

The algorithm $\mathcal{ALG}$ satisfies the first two requirements of this lemma, while the assumption on the velocities can be achieved by choosing the data with sufficiently small total variation. Therefore we only need to verify iii). The interactions giving rise to rarefactions that need to be split are among those that involve two waves of the same family. Indeed, when two waves of different families interact with each other, the outgoing fronts maintain the same sign of the incoming ones (because of the smallness of the data, this is an easy consequence of the Glimm's estimates). Moreover when a i-rarefaction is produced by the interaction of two i-waves, one of the incoming waves must be a rarefaction since the interactions of two i-shocks give rise to i-shocks (again by the smallness of the data), hence the outgoing i-rarefaction will not be partitioned, even if its strength is larger than η. This implies that the breakings can occur only when two waves of the same family interact and the outgoing wave of the other family, which will be the only one partitioned, is a rarefaction of strength larger than η.

Given a front-tracking approximate solution u defined at least on a strip $\mathbb{R}\times[0,T[$, let $\sigma_\alpha = \sigma_\alpha(t)$, $\alpha = 1,\dots,N$, be the sizes of the waves in $u(\cdot,t)$, and introduce the classical functionals

$$V(t) := \sum_\alpha |\sigma_\alpha|, \quad Q(t) := \sum_{(\alpha,\beta)\in A} |\sigma_\alpha \sigma_\beta|,$$

measuring the *total wave strength* in $u(\cdot,t)$ and the *interaction potential*, respectively, where the summation in Q ranges over all couples of approaching wave-fronts at time t. We recall that a front σ_α located at x_α and belonging to the family i_α is approaching the front σ_β located at $x_\beta > x_\alpha$ and belonging to the family i_β if either $i_\alpha > i_\beta$ or $i_\alpha = i_\beta$ and at least one of them is a shock; moreover, for every couple of interacting waves σ_i', σ_i'' belonging to the same i-th family, the strength $|\sigma_k|$ of the outgoing wave of family $k \neq i$ satisfies [8, Lemma 7.2]

$$|\sigma_k| \le \mathcal{O}(1)|\sigma_i'\sigma_i''|\big(|\sigma_i'| + |\sigma_i''|\big). \tag{2.3}$$

Suppose that at a time $\tau > 0$ an interaction takes place between two fronts of the same family and sizes σ_α and σ_β. It is well-known that as long as a piecewise constant approximate front-tracking solution is defined, if the initial data have sufficiently small total variation the interaction potential Q is decreasing in time, more precisely $\Delta Q(\tau) \le -|\sigma_\alpha\sigma_\beta|/2$ (see [8, (7.56) and (7.57)]). In the case where $|\sigma| > \eta$ is the strength of the outgoing rarefaction wave of the other family, the interaction estimate (2.3) yields

$$\eta < |\sigma| \le \mathcal{O}(1)|\sigma_\alpha\sigma_\beta|(|\sigma_\alpha| + |\sigma_\beta|) \le C_1|\sigma_\alpha\sigma_\beta| \le -2C_1\Delta Q(\tau),$$

for a suitable constant C_1, as long as the total variation remains bounded (and small). Then $\Delta Q(\tau) < -\eta/(2C_1)$ and this means that whenever such interactions occur, the potential Q decreases by a fixed positive amount, and this can happen only finitely many times (since $Q(0)$ is bounded and $Q(t)$ is decreasing). Therefore, applying Lemma 2.1 we obtain that the total number of wave-fronts and interactions remains finite in time and the algorithm is well-defined. In conclusion we have verified the following result.

Proposition 2.2. *There exists $\delta > 0$ such that for all initial data $\overline{u}$ with* $\mathrm{T.V.}(\overline{u}) \le \delta$ *and for every $\eta > 0$ the algorithm $\mathcal{ALG}$ is well-defined and provides a piecewise constant approximate solution u_η defined for all $t > 0$. Choosing $\eta = \eta_\nu \to 0$, the corresponding sequence u_{η_ν} converges in L^1_{loc}, up to subsequences, to a weak, entropy-admissible solution of* (1.1)-(1.2).

Proof. The well-definiteness of the algorithm was proved above. The convergence of the sequence is standard, since the approximate solutions here defined are ε-approximate front-tracking solutions in the sense of [8, Definition 7.1]. Indeed, the only thing still to prove is that the maximum size of the rarefactions can be bounded by a small quantity converging to zero as $\eta \to 0$. This can be done as in [8, pp. 138–139]. In addition, each limit point coincides with the corresponding trajectory of the Standard Riemann semigroup solution in the sense of [8, Definition 9.1], hence the whole sequence converges to that trajectory. $\square$

Notice that the main ingredient in the previous analysis was the existence of a decreasing functional Q which controls the strength of the newly generated rarefactions. However, in general Q is proved to be decreasing only for small BV data. With the aim of extending this result to possibly large BV data, we will not always be able to rely on Q, and this also justifies our choice to distinguish cases according only to the size of the outgoing rarefactions.

3. Front-tracking for a 2×2 system of balance laws modeling granular flows

In this section we show an example of a 2×2 system, arising in modeling granular flows and proposed by Amadori and Shen in [2], for which the simplified algorithm $\mathcal{ALG}$ works even in the presence of large BV data. In [2], one is concerned with the construction of global BV solutions for a Cauchy problem associated with the system of balance laws

$$\begin{aligned} h_t - (hp)_x &= (p-1)h, \\ p_t + ((p-1)h)_x &= 0, \end{aligned} \tag{3.1}$$

(where $h = h(x,t) \geq 0$ and $p = p(x,t) > 0$) and initial data

$$h(x,0) = \overline{h}(x), \quad p(x,0) = \overline{p}(x), \tag{3.2}$$

having arbitrarily large total variation and a small L^∞ bound only on the h coordinate. Unlike what has been seen in the previous section, the system (3.1) contains source terms and the first characteristic field is neither genuinely nonlinear nor linearly degenerate in the domain. In general, this last property may lead to the appearance of composite waves of the first family, but this is not the case. Indeed, the line $p = 1$ separates the domain into two invariant regions from the point of view of the Riemann problem, where both characteristic fields are genuinely nonlinear, and it is verified that an interaction of two waves generates at most two simple waves, one from each family.

In [2] approximate solutions to the Cauchy problem (3.1)-(3.2) are defined by an operator splitting method. Consider the sequence of times $t_k = k\Delta t$, where $\Delta t \geq 0$ is a fixed time step. On each time interval $[t_{k-1}, t_k[$ an approximation (h,p) of the system of conservation laws

$$\begin{aligned} h_t - (hp)_x &= 0, \\ p_t + ((p-1)h)_x &= 0, \end{aligned} \tag{3.3}$$

is constructed by means of a front-tracking algorithm; while at each time t_k the functions (h,p) are redefined in order to account for the source term. In [2] the authors suggest the use of an algorithm similar to [4]; the latter extends previous results of Bressan and Colombo [9] to non genuinely nonlinear systems with small BV data. However these results could not be applied directly to system (3.3), since (3.3) does not satisfy the assumptions in [4] and allows for large BV and L^∞ data. In addition, the study of the full balance system (3.1) in [2] depends on the analysis along the characteristics of the related homogeneous system (3.3). Hence it is important that all the waves in the approximate solutions be physical, i.e. shocks or rarefactions. In fact the algorithm proposed in [9, 4] does not introduce non-physical fronts but relies on a suitable interpolation between shock and rarefaction curves. This leads to the generation of a sort of new interpolated waves (which are neither shocks nor rarefactions), and these should have been taken into account in the analysis of interactions patterns/estimates, but actually are not.

In the following, we prove that the front-tracking algorithm $\mathcal{ALG}$ applied to system (3.3) between instants t_{k-1} and t_k (indeed on all $[0,+\infty[$) is well-defined, even allowing for large data, as in the assumptions in [2]. Towards this goal, we need to control the number of waves and interactions, and to understand how the strength of a rarefaction front varies over the time interval $[t_{k-1}, t_k]$.

3.1. **The number of interactions and wave-fronts is finite.** To claim that the front-tracking algorithm used is well-defined, one must demonstrate that the total number of waves and interactions remains finite. Recalling Lemma 2.1, it suffices to show that within the interval considered the number of interactions that generate more than one outgoing wave of the same family is finite, since the other requests of the lemma are easily satisfiable. As already pointed out, these are the interactions that give rise to a new rarefaction front of strength $|\sigma|$ larger than a small parameter $\eta > 0$ and that should be split into a number $\lfloor |\sigma|/\eta \rfloor + 1$ of fronts, each of strength at most η. Since incoming rarefactions are never partitioned, the cases when the above situation may occur are the following:

- when two waves of the first family interact with each other, they may generate a rarefaction of the second family of strength larger than η;
- similarly, when two waves of the second family interact with each other, they may give rise to a rarefaction of the first family of strength larger than η;
- when a wave of the second family that crosses the line $p = 1$ interacts with a shock of the first family, it gives rise to a rarefaction of the first family that possibly has strength larger than η.

Observe that for an interaction involving a wave of the first family and one of the second family that does not cross $p = 1$ (i.e. for an interaction that takes place entirely in $\{p < 1\}$ or $\{p > 1\}$), the outgoing waves have the same sign of the incoming ones (w.r.t. the characteristic family), hence they are of the same type (see [2]). Notice that in general this may not be true when the initial data are large; in this case it holds thanks to the special structure of system (3.3) and to the fact that the first characteristic field is genuinely nonlinear in each of the two invariant regions of the domain separated by the line $p = 1$. Thus, in these interactions there can not be any partitioning of rarefactions. The same holds for any outgoing rarefaction of the same family of the two incoming ones.

As before, to control the number of the new waves created we will try to find a non-increasing functional that decreases by a fixed amount at each interaction where more than two waves appear. In contrast to the previous section, the interaction potential $\mathcal{Q}$ used in [2] will not be useful, since when two waves of the first family and of different sign interact one has $\Delta\mathcal{Q} = \mathcal{O}(1)\|h\|_{L^\infty}|\sigma_\alpha\sigma_\beta|$ and $\mathcal{Q}$ may increase [2, p. 1024]. In fact our functional will be based on the functional $\mathcal{S}$ also introduced in [2] and which will be briefly recalled below. As it will be clear in Lemma 3.1, $\mathcal{S}$ is a decreasing functional which, by Lemma 3.3, is used to control the change in wave-strengths after interactions. Unfortunately, $\mathcal{S}$ does not always control the strength of the newly generated rarefactions since there exist interactions between (arbitrarily) large shocks of the first family and (arbitrarily) small waves of the second family for which the interaction potential is very small but still a big 1-rarefaction is created. Hence $\mathcal{S}$ will not decrease by a *fixed* amount after those interactions, and we will have to modify it. First we need some definitions: given a piecewise constant approximate solution $u = (h,p) : \mathbb{R} \times [t_{k-1}, t_k[\to \mathbb{R}^2$, let $\sigma_\alpha = \sigma_\alpha(t)$ be the size of the waves of $u(\cdot,t)$ at time t, of family i_α ($i_\alpha \in \{1,2\}$) and positioned at x_α. Let

$$V(t) = \sum_\alpha |\sigma_\alpha|$$

be the total strength of waves in $u(\cdot,t)$ and, with the notations of [2],

$$\mathcal{Q}(t) = \mathcal{Q}_{hh}(t) + \mathcal{Q}_{pp}(t) + \mathcal{Q}_{ph}(t) \tag{3.4}$$

be the interaction potential between couples of approaching waves. The strength of the 1-waves and 2-waves (which will also be called h-waves and p-waves, respectively) are measured in suitable Riemann coordinates (H,P). The functionals $\mathcal{Q}_{hh}$, $\mathcal{Q}_{pp}$ and $\mathcal{Q}_{ph}$ are the interaction potentials between pairs of h-h waves, p-p waves and p-h waves, respectively. More precisely

$$\mathcal{Q}_{hh}(t) = \sum_{\substack{i_\alpha = i_\beta = 1\\ x_\alpha < x_\beta}} w_{\alpha,\beta} |\sigma_\alpha \sigma_\beta|,$$

where $w_{\alpha,\beta} = \delta_0 \min\{|P_l^\alpha - 1|, |P_l^\beta - 1|\}$, for some δ_0, when σ_α and σ_β are both shocks located on the same side with respect to the line $p = 1$, and P_l^α, P_l^β are the left traces of the Riemann coordinate $P(x,t)$ for $x \to x_\alpha^-$, $x \to x_\beta^-$, respectively. In all other cases $w_{\alpha,\beta} = 0$. As for the other two potentials one defines

$$\mathcal{Q}_{pp}(t) = \sum_{\substack{i_\alpha = i_\beta = 2\\ x_\alpha < x_\beta}} |\sigma_\alpha \sigma_\beta|, \quad \mathcal{Q}_{ph}(t) = \sum_{\substack{i_\alpha = 2, i_\beta = 1\\ x_\alpha < x_\beta}} |\sigma_\alpha \sigma_\beta|.$$

The main results concerning functionals V and $\mathcal{Q}$ in [2] are summarized in the following lemma.

Lemma 3.1. *Given $M, p_0 > 0$, there exist $\delta > 0$ and $c, \delta_0 > 0$ such that for every initial data $(\overline{h}, \overline{p})$ with*

$$\mathrm{T.V.}(\overline{h}) \le M, \quad \mathrm{T.V.}(\overline{p}) \le M,$$
$$\|\overline{h}\|_{L^1} \le M, \quad \|\overline{p} - 1\|_{L^1} \le M, \quad \overline{p} \ge p_0 > 0,$$

if $\|\overline{h}\|_{L^\infty} \le \delta$ and as long as the front-tracking approximation of (3.3)-(3.2) is defined, the function

$$\mathcal{S}(t) := V(t) + c\mathcal{Q}(t)$$

is non-increasing in time. More precisely, given two waves σ_α and σ_β interacting at time t the following hold:

(i) *if they are both h-shocks then $\Delta\mathcal{S}(t) \le -w_{\alpha,\beta} c |\sigma_\alpha \sigma_\beta|/4$;*
(ii) *if they are a h-shock and a h-rarefaction then $\Delta\mathcal{S}(t) \le -\min\{|\sigma_\alpha|, |\sigma_\beta|\}$;*
(iii) *if they are a p- and a h-wave then $\Delta\mathcal{S}(t) \le -c|\sigma_\alpha \sigma_\beta|/4$;*
(iv) *if they are both p-waves then $\Delta\mathcal{S}(t) \le -c|\sigma_\alpha \sigma_\beta|/4$.*

For a proof of the above lemma, see [2, Section 4.4].

A *wave crossing* $p = 1$ is a wave connecting states (h_l, p_l) and (h_r, p_r) such that $(p_l - 1)(p_r - 1) \le 0$ and at least one of p_l and p_r is different from 1; it is easy to see that these waves must be p-waves.

Now, let u be an approximate solution of (3.3)-(3.2) obtained by $\mathcal{ALG}$ and denote by $y_i = y_i(t)$ the wave-fronts of the second family crossing $p = 1$ at time $t \in [t_{k-1}, t_k[$, which are finite in number. Introduce the functional $\mathcal{M}$

$$\mathcal{M}(t) := \mathcal{S}(t) + b\,\mathcal{N}(t), \tag{3.5}$$

where b is a constant to be determined and $\mathcal{N}$ is defined as follows:

$$\mathcal{N}(t) := \sum_{\alpha} N^{2k_\alpha}, \tag{3.6}$$

where the summation ranges over the wave-fronts x_α at time t, N is the constant

$$N := \left\lfloor \frac{\sup_\tau \|(H,P)(\cdot,\tau)\|_{L^\infty}}{\eta} \right\rfloor + 1 \tag{3.7}$$

and k_α is the cardinality of the finite set of fronts crossing $p = 1$ and to the left of x_α, i.e. the set

$$\{y_i : y_i \text{ is a front crossing } p = 1 \text{ at time } t \text{ and } y_i < x_\alpha\}\,.$$

Lemma 3.2. *If $\mathcal{M}$ is the functional defined in* (3.5)*, then the following hold:*

1. *the number of fronts crossing $p = 1$ is non-increasing in time between t_{k-1} and t_k, therefore it is bounded by $\widetilde{k} \in \mathbb{N}$, which is the number of these fronts at time t_{k-1};*
2. *there exists a constant $b > 0$ such that $\Delta\mathcal{M}(t) \le 0$ for every $t \in [t_{k-1}, t_k[$, that is, $\mathcal{M}$ is non-increasing;*
3. *there exists a constant $\mu > 0$ such that whenever a rarefaction of strength $|\sigma| > \eta$ is created at $t > t_{k-1}$, then*
$$\Delta\mathcal{M}(t) \le -\mu < 0.$$

Proof. The first statement is easy to check. Indeed, the number of fronts crossing $p = 1$ can not increase, as it remains constant when there is an interaction between one of those and a front of the first family or one of the second family which does not cross the line, while it decreases by one when two of those fronts interact with each other leading to a cancellation.

To prove the second statement, we must consider several cases. In particular, suppose that at a time t an interaction between two fronts not crossing $p = 1$ takes place. In the following, we denote by σ_α and σ_β the sizes of the incoming waves and by σ_h and σ_p those of the outgoing ones, respectively of the first and second characteristic family. First, note that, since no front y_i is involved in the interaction, one has

$$k_h = k_p = k_\alpha = k_\beta. \tag{3.8}$$

Suppose that the interaction is between two shocks of the first family and that the outgoing wave of the second family is either a shock or a rarefaction of strength smaller than η (i.e. it does not need to be partitioned). In this case, by Lemma 3.1 one has $\Delta\mathcal{S} < 0$, while for $\mathcal{N}$ it holds $\Delta\mathcal{N} = 0$, so that $\Delta\mathcal{M} < 0$. On the other hand, if the outgoing wave of the second family is a rarefaction of strength $|\sigma_p| > \eta$, then it is partitioned into at most N fronts. Thus,

$$\Delta\mathcal{N} \le N^{2k_h} + NN^{2k_p} - N^{2k_\alpha} - N^{2k_\beta} = N^{2k_\alpha}(N-1) \le N^{2\widetilde{k}}(N-1). \tag{3.9}$$

Moreover, from (i) of Lemma 3.1, one has

$$\Delta\mathcal{S} \le -\frac{c}{4} w_{\alpha,\beta} |\sigma_\alpha \sigma_\beta|,$$

and, thanks to the interaction estimates (3.16),

$$\eta < |\sigma_p| \le O(1)\|h\|_{L^\infty} w_{\alpha,\beta}|\sigma_\alpha\sigma_\beta| \le w_{\alpha,\beta}|\sigma_\alpha\sigma_\beta| \le -\frac{4}{c}\Delta\mathcal{S}$$

if $\|h\|_{L^\infty}$ is small enough. Consequently, $\Delta\mathcal{S} < -c\eta/4$ and

$$\Delta\mathcal{M} = \Delta\mathcal{S} + b\,\Delta\mathcal{N} < -\frac{c}{4}\eta + b\,N^{2\widetilde{k}}(N-1).$$

If the constant b is chosen so that

$$b \le \frac{c\eta}{8N^{2\widetilde{k}}(N-1)}, \tag{3.10}$$

one finally gets

$$\Delta\mathcal{M} < -\frac{c}{8}\eta < 0. \tag{3.11}$$

When the interaction involves two waves of the first family but of different sign (with $|\sigma_\alpha| < |\sigma_\beta|$), the functional $\mathcal{M}$ decreases trivially if the outgoing wave of the second family is a shock or a rarefaction of strength $|\sigma_p| \le \eta$. On the other hand, if it is a rarefaction of strength larger than η, it is again partitioned into at most N fronts. Then, $\mathcal{N}$ satisfies (3.9), while from ii) of Lemma 3.1 $\mathcal{S}$ satisfies

$$\Delta\mathcal{S} \le -|\sigma_\alpha| < -\eta,$$

since, thanks to the interaction estimates (3.16), one has

$$\eta < |\sigma_p| \le O(1)\|h\|_{L^\infty}|\sigma_\alpha\sigma_\beta| \le |\sigma_\alpha|,$$

again by the smallness assumption on $\|h\|_{L^\infty}$ and the upper bound on $|\sigma_\beta|$, that is on $\|p\|_{L^\infty}$ and $\|h\|_{L^\infty}$, proved in [2]. This time, if b is chosen so that

$$b \le \frac{\eta}{2N^{2\widetilde{k}}(N-1)}, \tag{3.12}$$

finally we get

$$\Delta\mathcal{M} < -\eta + b\,N^{2\widetilde{k}}(N-1) \le -\frac{\eta}{2}. \tag{3.13}$$

In a similar way we prove also that $\mathcal{M}$ decreases when the interaction involves two waves of the second family both not crossing $p = 1$. If the interaction, instead, involves two waves of different families and the incoming one of the second family does not cross $p = 1$, we have already noticed that there can not be any partitioning and $\mathcal{M}$ decreases trivially.

Now suppose that the interaction involves (only) one of the fronts crossing $p = 1$, of size σ_α. Let σ_β be the size of the other incoming wave and σ_h and σ_p those of the outgoing waves, as before. If σ_β is a rarefaction of the first family, then the outgoing wave of the second family will maintain the sign of σ_α, while the one of the first family will be forced to be a shock. In this case,

$$k_\alpha = k_h = k_p \quad \text{and} \quad k_\beta = k_\alpha + 1. \tag{3.14}$$

Then, being that

$$\Delta\mathcal{N} = N^{2k_h} + N^{2k_p} - N^{2k_\alpha} - N^{2k_\beta} = N^{2k_\alpha}(1 - N^2) \le -(N^2-1) < 0$$

and $\Delta\mathcal{S} < 0$, one has $\Delta\mathcal{M} \le -b(N^2-1) < 0$. On the other hand, if σ_β is a shock of the first family, then the outgoing wave of the first family is a rarefaction that could be of strength larger than η. In that case,

$$\Delta\mathcal{N} \le NN^{2k_h} + N^{2k_p} - N^{2k_\alpha} - N^{2k_\beta} = N^{2k_\alpha+1} - N^{2(k_\alpha+1)}$$

$$= N^{2k_\alpha}(N - N^2) \le -(N^2 - N),$$

thus $\Delta\mathcal{M} \le -b(N^2 - N) < 0$.

If σ_β is of the second family (not crossing $p = 1$) and hits from the right σ_α, again we have (3.14). Moreover, if the outgoing wave of the first family is a shock or a rarefaction of strength smaller than η, it is easy to see that $\mathcal{M}$ decreases. If the outgoing wave of the first family, instead, is a rarefaction of strength $|\sigma_h| > \eta$, as before, one obtains $\Delta\mathcal{M} \le -b(N^2 - N)$. Otherwise, suppose σ_β is of the second family and hits σ_α from the left: now we have that (3.8) holds and if the outgoing wave of the first family is a rarefaction of strength $|\sigma_h| > \eta$, then

$$\Delta\mathcal{N} \le N^{2\widetilde{k}}(N - 1)$$

and

$$\Delta\mathcal{M} \le -\frac{c}{4}\eta + b\,N^{2\widetilde{k}}(N - 1) \le -\frac{c}{8}\eta,$$

choosing b as in (3.10). The last case to consider (i.e. when the interaction involves two fronts of the second family both crossing $p = 1$) can be treated in a similar way and leads to analogous results.

In conclusion, if b satisfies (3.10) and (3.12) then $\mathcal{M}$ is non-increasing, and choosing the constant μ as

$$\mu = \min\Big\{\frac{c}{8}\eta\,,\ \frac{\eta}{2}\,,\ b(N^2 - N)\Big\},$$

the last statement is proven. $\square$

The above lemma ensures that interactions generating more than two outgoing fronts can occur only finitely many times. Therefore, we can apply Lemma 2.1 to obtain that the total number of fronts and interactions is finite.

3.2. **The strength of a rarefaction.** To conclude the analysis of the algorithm we have to prove its convergence (up to subsequences) to a weak entropic solution of (3.3)-(3.2). It is sufficient to show that the approximate solutions are indeed ε-approximate front-tracking solutions in the sense of [8, Definition 7.1], and it only remains to prove that the maximum size of the rarefactions can be bounded by a small quantity. To do this, we will need to adapt the analysis done in [8, pp. 138–139].

The strength of a rarefaction front can increase only when it interacts with waves of a different characteristic family, while when interacting with a shock of the same family its strength decreases. This is expected for 1-waves since $\|\overline{h}\|_{L^\infty}$ is chosen small, but it is not straightforward for 2-waves when dealing with data with large $\|\overline{p}\|_{L^\infty}$, and it has to be proven. Indeed this comes from the following interaction estimates derived in [2, Lemma 3].

Lemma 3.3. *Consider two interacting wave-fronts, with left, middle, and right states (h_l, p_l), (h_m, p_m), (h_r, p_r) before interaction, respectively, and*

$$h_{\max} = \max\{h_l, h_m, h_r\}.$$

(i) *If two p-waves of size σ_p and $\tilde\sigma_p$ interact producing the outgoing waves of size σ_h^+ and σ_p^+ then*

$$|\sigma_h^+| + |\sigma_p^+ - (\sigma_p + \tilde\sigma_p)| = \mathcal{O}(1)\cdot h_l \cdot |\sigma_p\tilde\sigma_p|. \tag{3.15}$$

(ii) *If two h-waves of size σ_h and $\tilde\sigma_h$ interact producing the outgoing waves of size σ_h^+ and σ_p^+ then*

$$|\sigma_h^+ - (\sigma_h + \tilde\sigma_h)| + |\sigma_p^+| = \mathcal{O}(1) \cdot \min\{|p_l - 1|, |p_m - 1|\} \cdot (|\sigma_h| + |\tilde\sigma_h|)|\sigma_h \tilde\sigma_h|. \quad (3.16)$$

(iii) *If two waves of different family and size σ_p and σ_h interact producing the outgoing waves of size σ_h^+ and σ_p^+ then*

$$|\sigma_h^+ - \sigma_h| + |\sigma_p^+ - \sigma_p| = \mathcal{O}(1) \cdot h_{\max} \cdot |\sigma_h \sigma_p|. \quad (3.17)$$

If σ_h is a h-rarefaction interacting at time τ with a h-shock of size $\tilde\sigma_h$, either the rarefaction is canceled or by (3.16) its size σ_h^+ satisfies

$$\begin{aligned} |\sigma_h^+| - |\sigma_h| \le |\sigma_h^+ - \sigma_h - \tilde\sigma_h| - |\tilde\sigma_h| &\le \mathcal{O}(1)\|h\|_{L^\infty}|\sigma_h\tilde\sigma_h| - |\tilde\sigma_h| \\ &\le -|\tilde\sigma_h|\big(1 - \mathcal{O}(1)\|h\|_{L^\infty}V(\tau-)\big), \end{aligned}$$

where it was also used that $|\sigma_h| > |\tilde\sigma_h|$, which is valid thanks to the definition of the σ's. Using (3.15) the same argument for p-rarefactions gives

$$|\sigma_p^+| - |\sigma_p| \le -|\tilde\sigma_p|\big(1 - \mathcal{O}(1)\|h\|_{L^\infty}V(\tau-)\big).$$

As long as the approximate solution is defined the functional $V + c\mathcal{Q}$ is decreasing, hence $V(t)$ is uniformly bounded. Since $\|h\|_{L^\infty}$ is of order $\mathcal{O}(1)\|\overline{h}\|_{L^\infty}$ (see [2]), by choosing the constant δ in Lemma 3.1 sufficiently small we obtain that the strength of both h- and p-rarefactions decreases after interacting with waves of the same family. Hence the strength of a rarefaction can increase only after interacting with waves of the other family. Now we proceed almost like in [8, pp. 138–139].

Let $\sigma_\alpha(t)$ be the size at time t of a rarefaction front generated at $\tau_0 \in]t_{k-1}, t_k[$. The aim is to find a limitation to $|\sigma_\alpha(t)|$ by means of $\mathcal{Q}$ and a functional V_α to be introduced. In the case σ_α is of the first characteristic family we define the quantity

$$V_\alpha(t) := \sum_{\substack{i_\beta = 2 \\ x_\beta < x_\alpha}} |\sigma_\beta| + \sum_{\substack{i_\gamma = 1 \\ x_\gamma \neq x_\alpha}} |\sigma_\gamma|, \quad (3.18)$$

which is the *total wave strength* restricted to the wave-fronts which could interact with σ_α possibly causing an increase in its strength. Similarly, if the rarefaction is of the second family one considers

$$V_\alpha(t) := \sum_{\substack{i_\beta = 1 \\ x_\beta > x_\alpha}} |\sigma_\beta| + \sum_{\substack{i_\gamma = 2 \\ \sigma_\gamma \text{ shock}}} |\sigma_\gamma|.$$

The only difference with [8, p. 139] comes from the fact that, because of the structure of the system, a h-rarefaction may become a h-shock after colliding with a p-wave; this means that, after an interaction, a wave not approaching σ_α could turn into one which is approaching. Hence, two distinct h-waves are always potentially approaching, and this justifies the choice of the indexes of summation in the second sum in (3.18).

Now, suppose that at a time $\tau > \tau_0$ an interaction takes place. There are two possibilities: either it involves two wave-fronts different from the rarefaction under consideration, or it involves σ_α itself. In any case, thanks to Lemma 3.1 and Lemma 3.3 one can prove that either

$$\Delta|\sigma_\alpha| \le 0, \quad \Delta V_\alpha + c\,\Delta\mathcal{Q} \le 0,$$

or

$$\Delta|\sigma_\alpha| \le C_2|\sigma_\alpha\sigma_\beta|, \quad \Delta V_\alpha = -|\sigma_\beta|, \quad \Delta\mathcal{Q} \le 0,$$

for a suitable constant C_2. Hence we are in the same situation of [8, p. 139], and it is easy to prove that the map defined by

$$\Theta : t \mapsto |\sigma_\alpha(t)| \exp\{C_2[V_\alpha(t) + c\,\mathcal{Q}(t)]\}$$

is non-increasing for $t \in [\tau_0, \tau_1] \subseteq [t_{k-1}, t_k]$, where τ_0 is the time of generation and τ_1 the time of (possible) cancellation of the rarefaction.

Consequently, for $t \ge \tau_0$

$$|\sigma_\alpha(t)| \le \Theta(t) \le \Theta(\tau_0) \le |\sigma_\alpha(\tau_0)| \exp\{C_2[V_\alpha(t_{k-1}) + c\,\mathcal{Q}(t_{k-1})]\}$$

and it is clear that the strength of each rarefaction between t_{k-1} and t_k remains small. Indeed, since $V_\alpha(t_{k-1}) + c\,\mathcal{Q}(t_{k-1})$ is bounded and $|\sigma_\alpha(\tau_0)| \le \eta$ (where η is the small parameter controlling the size of the newly generated rarefactions), it follows that

$$|\sigma_\alpha(t)| \le \mathcal{O}(1)\eta.$$

Finally, combining the results of subsection 3.1 and 3.2 we obtain the following result.

Theorem 3.4. *Under the same assumptions of Lemma 3.1, for every $\eta > 0$ the algorithm $\mathcal{ALG}$ applied to system* (3.3) *is well-defined and provides a piecewise constant approximate solution of* (3.3) *defined for all $t \in [t_{k-1}, t_k[$ (indeed for all $t \ge t_{k-1}$ and all t_{k-1}). Choosing a sequence $\eta_\nu \to 0^+$, the corresponding sequence of approximate solutions converges in L^1_{loc}, up to subsequences, to a weak entropic solution of* (3.3)-(3.2).

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