Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 196, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STABILITY OF SOLUTIONS FOR SOME INVERSE PROBLEMS

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Abstract. In this article we establish three stability results for some inverse problems. More precisely we consider the following boundary value problem: $\Delta u + \lambda u + \mu = 0$ in Ω, $u = 0$ on $\partial\Omega$, where λ and μ are real constants and $\Omega \subset \mathbb{R}^2$ is a smooth bounded simply-connected open set. The inverse problem consists in the identification of λ and μ from knowledge of the normal flux $\partial u/\partial\nu$ on $\partial\Omega$ corresponding to some nontrivial solution.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded simply-connected open set. Consider the elliptic boundary-value problem

$$\Delta u + \lambda u + \mu = 0 \quad \text{in } \Omega\,, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega\,, \tag{1.2}$$

where λ and μ are real constants. An interesting problem is to examine whether one can identify the constants λ and μ from knowledge of the normal flux $\partial u/\partial\nu$ on $\partial\Omega$ corresponding to some nontrivial solution of (1.1)-(1.2). For more general sources this inverse problem arises for instance in plasma physics in connection with the modelling of Tokamaks [2]. Actually when $\lambda u + \mu$ is replaced by $f(u)$ where $f \geq 0$ we have a simplified version of the Grad-Shafranov equation (see [3]). The identifiability of f from $\partial u/\partial\nu$ depends on the shape of $\partial\Omega$. But even in the very particular case of an affine term the problem is difficult. It is well known that if Ω is a disk then such identification of (λ, μ) is completely impossible, even in the case where a sign condition is imposed on the affine term: It is shown in [12] that there is a continuum of coefficient pairs $(\lambda, \mu_\lambda) \in \mathbb{R}^2$, and therefore a continuum of affine functions, which give rise to the same normal derivative on the boundary. We refer the reader to paper [12] for a more detailed description of the problem in general and the difficulties encountered.

A partial answer to this problem was first obtained by Vogelius in [12], and more recently we have also given a contribution [4]-[6]: Under some conditions on the domain and on the normal derivative, there exist at most finitely many pairs of coefficients.

Our purpose is to show that in some cases uniqueness of the above inverse problem is stable under analytic perturbation of the data.

2000 *Mathematics Subject Classification.* 35J05, 35R30.
Key words and phrases. Inverse problem; stability.

Submitted September 10, 2012. Published November 8, 2012.

Let χ denote the torsion function relative to Ω, that is,

$$\Delta\chi + 1 = 0 \quad \text{in } \Omega\,, \quad \chi = 0 \quad \text{on } \partial\Omega\,.$$

In [7] we proved the following result.

Theorem 1.1. *Assume that Ω is a true ellipse. If there exist λ, $\mu \in \mathbb{R}$ and u satisfying* (1.1)-(1.2) *and*

$$\frac{\partial u}{\partial \nu} = \frac{\partial \chi}{\partial \nu} \quad \text{on } \partial\Omega\,, \tag{1.3}$$

then $\lambda = 0$, $\mu = 1$ and $u = \chi$.

Remark 1.2. It is shown in [7] that, when Ω is the unit disk, there is a continuum of coefficient pairs (λ, μ_λ) and u_λ which solve the problem raised in Theorem 1.1.

Remark 1.3. We conjectured in [7] that disks are the only smooth bounded simply-connected open sets for which problem (1.1)-(1.2) and (1.3) has more than one solution.

In our first two results we study the case where problem (1.1)-(1.3) has a unique solution. We first consider an analytic perturbation of the normal derivative of the torsion function and we establish the uniqueness of the inverse problem. Then we treat the case of an analytic perturbation of the domain. In our last result we assume that, for some given φ on $\partial\Omega$, (λ, μ) is uniquely determined. We show that uniqueness holds under an analytic perturbation of φ.

We begin with the following theorem.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a $C^{4,\alpha}$ bounded simply-connected open set ($\alpha \in (0,1]$) such that the identification problem* (1.1)-(1.3) *has a unique solution ($\lambda = 0$, $\mu = 1$ and $u = \chi$). Let $T > 0$ and $\psi : [0,T) \to C^{4,\alpha}(\partial\Omega)$ depending analytically on $t \in [0,T)$ and such that $\psi(0) = \partial\chi/\partial\nu$ on $\partial\Omega$. Suppose that for any $t \in [0,T)$, there exist $\lambda(t)$, $\mu(t) \in \mathbb{R}$ and $u(t) \in C^{4,\alpha}(\overline{\Omega})$ such that*

$$\Delta u(t) + \lambda(t)u(t) + \mu(t) = 0 \quad \text{in } \Omega\,,$$
$$u(t) = 0\,, \quad \frac{\partial u(t)}{\partial \nu} = \psi(t) \quad \text{on } \partial\Omega\,.$$

Suppose moreover that λ and μ depend analytically on $t \in [0,T)$. Then $\lambda(t)$ and $\mu(t)$ are unique.

Now let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected open set with real analytic boundary. Let $\Omega_t \subset \mathbb{R}^2$, $t \in [0,T)$ be a family of bounded simply-connected open sets with real analytic boundary depending analytically on the parameter t. We also suppose that $\Omega_0 = \Omega$. We denote by $\nu(t)$ (resp. $\nu = \nu(0)$) the outward unit normal to Ω_t (resp. Ω) and by $\chi(t)$ (resp. $\chi = \chi(0)$) the torsion function relative to Ω_t (resp. Ω). We have the following theorem.

Theorem 1.5. *In the above setting suppose that the identification problem* (1.1)-(1.3) *has a unique solution ($\lambda = 0$, $\mu = 1$ and $u = \chi$) and that for any $t \in [0,T)$, there exist $\lambda(t)$, $\mu(t) \in \mathbb{R}$ and $u(t)$ analytic on $\overline{\Omega}_t$ such that*

$$\Delta u(t) + \lambda(t)u(t) + \mu(t) = 0 \quad \text{in } \Omega_t\,,$$
$$u(t) = 0\,, \quad \frac{\partial u(t)}{\partial \nu(t)} = \frac{\partial \chi(t)}{\partial \nu(t)} \quad \text{on } \partial\Omega_t\,.$$

Suppose moreover that λ and μ depend analytically on $t \in [0,T)$. Then $\lambda(t) = 0$ and $\mu(t) = 1$ for $t \in [0,T)$.

Remark 1.6. In the setting of Theorem 1.5, for any $t \in [0,T)$, Ω_t is not a disk by Remark 1.2.

A smooth bounded simply-connected open set $\Omega \subset \mathbb{R}^2$ is said to have the *Schiffer property* if (for any λ) the following overdetermined boundary-value problem

$$\Delta v + \lambda v + 1 = 0 \quad \text{in } \Omega\,, \tag{1.4}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,, \tag{1.5}$$

has no solution. It is well known that disks do not have the Schiffer property. Indeed let J_z denote the z-th Bessel function. For any $\lambda > 0$ such that $J_1(\sqrt{\lambda}) = 0$, the function

$$v_\lambda(x) = \frac{1}{\lambda}\Big(\frac{J_0(\sqrt{\lambda}|x|)}{J_0(\sqrt{\lambda})} - 1\Big)\,, \quad |x| < 1\,,$$

satisfies (1.4)-(1.5) when Ω is the unit disk.

The *Schiffer conjecture* asserts that disks are the only smooth bounded simply-connected open sets for which (1.4)-(1.5) has a solution for even a single value of λ. Wide classes of smooth bounded simply-connected open sets in $\mathbb{R}^2$ having the Schiffer property were studied in [11] and the references therein. In [8] and [9] we gave some elementary results allowing to exhibit very simple examples of planar domains having the Schiffer property. Now we can state our last result.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^2$ be a $C^{3,\alpha}$ bounded simply-connected open set ($\alpha \in (0,1]$) and let $\varphi \in C^{3,\alpha}(\partial\Omega)$. We assume that*

- (i) *Ω has the Schiffer property;*
- (ii) *$\varphi \not\equiv 0$;*
- (iii) *$\int_{\partial\Omega} \varphi(x)\, d\sigma(x) = 0$, or*
  $\int_{\partial\Omega} \varphi(x)(x_1\nu_2(x) - x_2\nu_1(x))^2\, d\sigma(x) = 0$ and $\partial\Omega$ is real analytic;
- (iv) *There exists a unique pair of coefficients $(\lambda,\ \mu) \in \mathbb{R}^2$ such that problem (1.1)-(1.2) has a solution $u \in C^{3,\alpha}(\overline{\Omega})$ satisfying*

$$\frac{\partial u}{\partial \nu} = \varphi \quad \text{on } \partial\Omega\,.$$

Let $T > 0$ and $\psi : [0,T) \to C^{3,\alpha}(\partial\Omega)$ depending analytically on $t \in [0,T)$ and such that $\psi(0) = \varphi$ on $\partial\Omega$. Suppose that for any $t \in [0,T)$, there exist $\lambda(t)$, $\mu(t) \in \mathbb{R}$ and $u(t) \in C^{3,\alpha}(\overline{\Omega})$ such that

$$\Delta u(t) + \lambda(t)u(t) + \mu(t) = 0 \quad \text{in } \Omega\,,$$

$$u(t) = 0\,, \quad \frac{\partial u(t)}{\partial \nu} = \psi(t) \quad \text{on } \partial\Omega\,.$$

Suppose moreover that λ and μ depend analytically on $t \in [0,T)$. Then $\lambda(t)$ and $\mu(t)$ are unique.

We shall use the following theorem obtained by Bennett [1].

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a $C^{4,\alpha}$ domain ($\alpha \in (0,1]$). Suppose that there exists $u \in C^4(\overline{\Omega})$ such that*

$$\Delta^2 u = 1 \quad \text{in } \Omega\,,$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad on\ \partial\Omega\,,$$
$$\Delta u = c \quad on\ \partial\Omega$$

where c is a constant. Then Ω is an open ball of radius $R = (cN(N+2))^{1/2}$ and

$$u(x) = \frac{1}{8N(N+2)}(R^2 - |x-x_0|^2)^2\,, \quad x \in \Omega\,,$$

where x_0 denotes the center of Ω.

Theorems 1.4–1.7 are proved in Sections 2–4 respectively.

2. Proof of Theorem 1.4

As λ, μ and ψ depend analytically on the parameter t, the function u does also. Let us consider the Taylor decompositions

$$u(t)(x) = \sum_{n=0}^{\infty} u_n(x)t^n\,, \quad \psi(t)(y) = \sum_{n=0}^{\infty} \psi_n(y)t^n\,,$$
$$\lambda(t) = \sum_{n=0}^{\infty} \lambda_n t^n\,, \quad \mu(t) = \sum_{n=0}^{\infty} \mu_n t^n\,,$$

for $x \in \overline{\Omega}$, $y \in \partial\Omega$ and $t \in [0,T)$, where λ_n, $\mu_n \in \mathbb{R}$, $u_n \in C^{4,\alpha}(\overline{\Omega})$ and $\psi_n \in C^{4,\alpha}(\partial\Omega)$. For $n \in \mathbb{N}$ we have

$$\Delta u_n + v_n + \mu_n = 0 \quad \text{in } \Omega\,, \tag{2.1}$$
$$u_n = 0\,, \quad \frac{\partial u_n}{\partial \nu} = \psi_n \quad \text{on } \partial\Omega\,, \tag{2.2}$$

where

$$v_n = \sum_{k=0}^{n} \lambda_{n-k} u_k\,.$$

Now (2.1) and (2.2) with $n = 0$ give

$$\Delta u_0 + \lambda_0 u_0 + \mu_0 = 0 \quad \text{in } \Omega\,,$$
$$u_0 = 0\,, \quad \frac{\partial u_0}{\partial \nu} = \psi_0 = \frac{\partial \chi}{\partial \nu} \quad \text{on } \partial\Omega\,,$$

and our assumption implies that $\lambda_0 = 0$, $\mu_0 = 1$ and $u_0 = \chi$. Then

$$v_n = \sum_{k=0}^{n-1} \lambda_{n-k} u_k\,, \quad \text{if } n \ge 1\,.$$

Integrating (2.1) and using (2.2) with $n = 1$ we obtain

$$\lambda_1 \int_\Omega \chi(x)\,dx + \mu_1 |\Omega| = -\int_{\partial\Omega} \psi_1(x)\,d\sigma(x)\,. \tag{2.3}$$

We have $u_1 = \lambda_1 v + \mu_1 \chi$ where

$$\Delta v + \chi = 0 \text{ in } \Omega \quad \text{and} \quad v = 0 \text{ on } \partial\Omega\,. \tag{2.4}$$

Then (2.2) with $n = 1$ gives

$$\lambda_1 \frac{\partial v}{\partial \nu} + \mu_1 \frac{\partial \chi}{\partial \nu} = \psi_1\,. \tag{2.5}$$

We claim that

$$\exists\, y \in \partial\Omega \text{ such that } \frac{\partial v}{\partial \nu}(y) - \frac{1}{|\Omega|}\frac{\partial \chi}{\partial \nu}(y)\int_\Omega \chi(x)\,dx \neq 0\,. \tag{2.6}$$

Suppose the contrary and let

$$z = v - \frac{1}{|\Omega|}\chi \int_\Omega \chi(x)\,dx\,.$$

Then we have

$$\begin{gathered} \Delta^2 z = 1 \quad \text{in } \Omega\,,\\ z = \frac{\partial z}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,,\\ \Delta z = \frac{1}{|\Omega|}\int_\Omega \chi(x)\,dx = \text{const.} \quad \text{on } \partial\Omega\,, \end{gathered}$$

and using Theorem 1.8 we conclude that Ω is a disk. Our assumption and Remark 1.2 give a contradiction. Thus our claim is proved. Now (2.3), (2.5) and (2.6) imply that λ_1 and μ_1 are uniquely determined. Then u_1 is also uniquely determined. Assume that λ_j, μ_j and u_j , $j = 1, \dots, n-1$ ($n \geq 2$), are uniquely determined. Integrating (2.1) and using (2.2) we obtain

$$\lambda_n \int_\Omega \chi(x)\,dx + \mu_n|\Omega| = -\int_{\partial\Omega} \psi_n(x)\,d\sigma(x) - \sum_{k=1}^{n-1}\lambda_{n-k}\int_\Omega u_k(x)\,dx\,.$$

We have $u_n = \lambda_n v + \mu_n \chi + w_n$ where

$$w_n(x) = \sum_{k=1}^{n-1}\lambda_{n-k}\int_\Omega G(x,y)u_k(y)\,dy\,, \quad x \in \Omega\,,$$

where G denotes the Green's function of the operator $-\Delta$ on Ω with Dirichlet boundary conditions. Then (2.2) gives

$$\lambda_n \frac{\partial v}{\partial \nu} + \mu_n \frac{\partial \chi}{\partial \nu} = \psi_n - \frac{\partial w_n}{\partial \nu}\,.$$

and we conclude as before that (λ_n, μ_n) and u_n are uniquely determined. The proof of the theorem is complete.

Remark 2.1. Note that Theorem 1.4 also holds when $\Omega \subset \mathbb{R}^N$, $N \geq 3$.

3. Proof of Theorem 1.5

Let $\omega(t) : \Omega = \Omega_0 \to \Omega_t$ be a conformal mapping analytically depending on $t \in [0, T)$ and such that $\omega(0)$ is the identity mapping. It is well-known that $\omega(t)$ extends analytically on a neighborhood of $\overline{\Omega}$ to a neighborhood of $\overline{\Omega}_t$: See [10]. Define $v(t)(x) = u(t) \circ \omega(t)(x)$ and $\tilde\chi(t)(x) = \chi(t) \circ \omega(t)(x)$ for $x = (x_1, x_2) \in \overline{\Omega}$ and

$$\rho(t)(x) = \big(\frac{\partial \omega_j(t)(x)}{\partial x_1}\big)^2 + \big(\frac{\partial \omega_j(t)(x)}{\partial x_2}\big)^2, \quad x \in \overline{\Omega}\,,\ j = 1 \text{ or } 2\,,$$

where $\omega(t)(x) = (\omega_1(t)(x), \omega_2(t)(x))$. The behavior of the Laplacian under conformal mappings is well-known. Using elementary calculations and the fact that $\omega(t)$ maps $\partial\Omega$ on $\partial\Omega_t$ we obtain

$$\Delta\tilde\chi(t) + \rho(t) = 0 \quad \text{in } \Omega\,, \tag{3.1}$$

$$\tilde{\chi}(t) = 0\,, \quad \frac{\partial \tilde{\chi}(t)}{\partial \nu} = \rho(t)^{1/2} \frac{\partial \chi(t)}{\partial \nu(t)} \circ \omega(t) \quad \text{on } \partial\Omega\,, \tag{3.2}$$

and

$$\Delta v(t) + \lambda(t)\rho(t)v(t) + \mu(t)\rho(t) = 0 \quad \text{in } \Omega\,, \tag{3.3}$$

$$v(t) = 0\,, \quad \frac{\partial v(t)}{\partial \nu} = \rho(t)^{1/2} \frac{\partial \chi(t)}{\partial \nu(t)} \circ \omega(t) \quad \text{on } \partial\Omega\,. \tag{3.4}$$

Let us consider the Taylor decompositions

$$\tilde{\chi}(t)(x) = \sum_{n=0}^{\infty} \tilde{\chi}_n(x) t^n\,, \quad v(t)(x) = \sum_{n=0}^{\infty} v_n(x) t^n\,,$$

$$\frac{\partial \chi(t)}{\partial \nu(t)} \circ \omega(t)(y) = \sum_{n=0}^{\infty} \gamma_n(y) t^n\,, \quad \rho(t)^{1/2}(x) = \sum_{n=0}^{\infty} \alpha_n(x) t^n\,,$$

$$\lambda(t) = \sum_{n=0}^{\infty} \lambda_n t^n\,, \quad \mu(t) = \sum_{n=0}^{\infty} \mu_n t^n\,,$$

for $x \in \overline{\Omega}$, $y \in \partial\Omega$ and $t \in [0, T)$, where λ_n, $\mu_n \in \mathbb{R}$, v_n, α_n are analytic on $\overline{\Omega}$ and γ_n is analytic on $\partial\Omega$. We have $\alpha_0 = 1$, $\tilde{\chi}_0 = \chi$ and $\gamma_0 = \partial\chi/\partial\nu$. Now

$$\rho(t) = \sum_{n=0}^{\infty} \rho_n t^n\,, \quad \mu(t)\rho(t) = \sum_{n=0}^{\infty} \beta_n t^n\,, \quad \lambda(t)\rho(t)v(t) = \sum_{n=0}^{\infty} \delta_n t^n\,,$$

where

$$\rho_n = \sum_{k=0}^{n} \alpha_{n-k}\alpha_k\,, \quad \beta_n = \sum_{k=0}^{n} \mu_{n-k}\rho_k\,, \quad \delta_n = \sum_{k=0}^{n} \lambda_{n-k}\Big(\sum_{j=0}^{k} \rho_{k-j} v_j\Big)\,.$$

From (3.1)-(3.4) we obtain

$$\Delta \tilde{\chi}_n + \rho_n = 0 \quad \text{in } \Omega\,, \tag{3.5}$$

$$\tilde{\chi}_n = 0\,, \quad \frac{\partial \tilde{\chi}_n}{\partial \nu} = \sum_{k=0}^{n} \gamma_{n-k}\alpha_k \quad \text{on } \partial\Omega\,, \tag{3.6}$$

and

$$\Delta v_n + \delta_n + \beta_n = 0 \quad \text{in } \Omega\,, \tag{3.7}$$

$$v_n = 0\,, \quad \frac{\partial v_n}{\partial \nu} = \sum_{k=0}^{n} \gamma_{n-k}\alpha_k \quad \text{on } \partial\Omega\,, \tag{3.8}$$

for $n \in \mathbb{N}$. (3.7) and (3.8) with $n = 0$ give

$$\Delta v_0 + \lambda_0 v_0 + \mu_0 = 0 \quad \text{in } \Omega\,,$$

$$v_0 = 0\,, \quad \frac{\partial v_0}{\partial \nu} = \frac{\partial \chi}{\partial \nu} \quad \text{on } \partial\Omega\,,$$

and our assumption implies that $\lambda_0 = 0$, $\mu_0 = 1$ and $v_0 = \chi$. Now (3.7) and (3.8) with $n = 1$ give

$$\Delta v_1 + \lambda_1 \chi + \rho_1 + \mu_1 = 0 \quad \text{in } \Omega\,,$$

$$v_1 = 0\,, \quad \frac{\partial v_1}{\partial \nu} = \alpha_1 \frac{\partial \chi}{\partial \nu} + \gamma_1 \quad \text{on } \partial\Omega\,.$$

Then, using (3.5) and (3.6) with $n=1$ we obtain

$$\Delta(v_1-\tilde\chi_1)+\lambda_1\chi+\mu_1=0 \quad \text{in} \quad \Omega\,, \tag{3.9}$$

$$v_1-\tilde\chi_1=\frac{\partial(v_1-\tilde\chi_1)}{\partial\nu}=0 \quad \text{on } \partial\Omega\,. \tag{3.10}$$

Integrating (3.9) and using (3.10) we obtain

$$\lambda_1\int_\Omega \chi(x)\,dx+\mu_1|\Omega|=0\,.$$

Since $v_1-\tilde\chi_1=\lambda_1 v+\mu_1\chi$ where v is given by (2.4) we obtain

$$\lambda_1\frac{\partial v}{\partial\nu}+\mu_1\frac{\partial\chi}{\partial\nu}=0\,.$$

We conclude that $\lambda_1=\mu_1=0$ by using the same arguments as in the proof of Theorem 1.4. Then we also have $v_1=\tilde\chi_1$. Assume that $\lambda_j=\mu_j=0$ and $v_j=\tilde\chi_j$ for $j=1,\dots,n-1$ ($n\ge 2$). Then (3.7) becomes

$$\Delta v_n+\lambda_n\chi+\rho_n+\mu_n=0 \quad \text{in } \Omega\,. \tag{3.11}$$

With the help of (3.5), (3.6), (3.8) and (3.11) we obtain

$$\Delta(v_n-\tilde\chi_n)+\lambda_n\chi+\mu_n=0 \quad \text{in } \Omega\,, \tag{3.12}$$

$$v_n-\tilde\chi_n=\frac{\partial(v_n-\tilde\chi_n)}{\partial\nu}=0 \quad \text{on } \partial\Omega\,. \tag{3.13}$$

Integrating (3.12) and using (3.13) we obtain

$$\lambda_n\int_\Omega \chi(x)\,dx+\mu_n|\Omega|=0\,.$$

Since $v_n-\tilde\chi_n=\lambda_n v+\mu_n\chi$ where v is given by (2.4) we obtain

$$\lambda_n\frac{\partial v}{\partial\nu}+\mu_n\frac{\partial\chi}{\partial\nu}=0\,.$$

We conclude that $\lambda_n=\mu_n=0$ by using the same arguments as before. Then we also have $v_n=\tilde\chi_n$. The proof of the theorem is complete.

4. Proof of Theorem 1.7

We shall need the following lemma.

Lemma 4.1. *Let a, b and c be real constants. Let $z\in C^2(\overline{\Omega})$ and $u\in C(\overline{\Omega})$ satisfy*

$$\Delta z+az+bu+c=0 \quad \textit{in } \Omega\,,$$

$$z=\frac{\partial z}{\partial\nu}=u=0 \quad \textit{on } \partial\Omega\,.$$

Then

$$\frac{\partial^2 z}{\partial x_j\partial x_k}=-c\nu_j\nu_k \quad \textit{on } \partial\Omega\,.$$

Proof. Let $x=x(s)=(x_1(s),x_2(s))$, $s\in[0,L]$, be a parametrization of $\partial\Omega$ by arc length. We denote by $\tilde\nu(s)=(\tilde\nu_1(s),\tilde\nu_2(s))$ the exterior normal to $\partial\Omega$ at $x(s)$. Then $\tilde\nu_1(s)=x_2'(s)$ and $\tilde\nu_2(s)=-x_1'(s)$, $s\in[0,L]$. We have

$$\frac{\partial z}{\partial x_j}(x(s))=0\,, \quad s\in[0,L]\,,\ j=1,2\,. \tag{4.1}$$

Differentiating (4.1) with respect to s we obtain

$$-\frac{\partial^2 z}{\partial x_j \partial x_1}(x(s))\tilde{\nu}_2(s) + \frac{\partial^2 z}{\partial x_j \partial x_2}(x(s))\tilde{\nu}_1(s) = 0\,, \quad s \in [0, L]\,,$$

for $j = 1, 2$. Since

$$\frac{\partial^2 z}{\partial x_2^2}(x(s)) = -\frac{\partial^2 z}{\partial x_1^2}(x(s)) - c\,, \quad s \in [0, L]\,,$$

the lemma follows. □

As λ, μ and ψ depend analytically on the parameter t, the function u does also. Let us consider the Taylor decompositions

$$u(t)(x) = \sum_{n=0}^{\infty} u_n(x) t^n\,, \quad \psi(t)(y) = \sum_{n=0}^{\infty} \psi_n(y) t^n\,,$$

$$\lambda(t) = \sum_{n=0}^{\infty} \lambda_n t^n\,, \quad \mu(t) = \sum_{n=0}^{\infty} \mu_n t^n\,,$$

for $x \in \overline{\Omega}$, $y \in \partial\Omega$ and $t \in [0, T)$, where λ_n, $\mu_n \in \mathbb{R}$, $u_n \in C^{3,\alpha}(\overline{\Omega})$ and $\psi_n \in C^{3,\alpha}(\partial\Omega)$. We have $\lambda_0 = \lambda$, $\mu_0 = \mu$, $u_0 = u$ and $\psi_0 = \varphi$.

For $n \in \mathbb{N}$ we have

$$\Delta u_n + v_n + \mu_n = 0 \quad \text{in } \Omega\,, \tag{4.2}$$

$$u_n = 0\,, \quad \frac{\partial u_n}{\partial \nu} = \psi_n \quad \text{on } \partial\Omega\,, \tag{4.3}$$

where

$$v_0 = \lambda_0 u_0 = \lambda u \quad \text{and} \quad v_n = \sum_{k=0}^{n} \lambda_{n-k} u_k\,, \quad \text{if } n \geq 1\,.$$

When $n = 1$, (4.2) and (4.3) give

$$\Delta u_1 + \lambda u_1 + \lambda_1 u + \mu_1 = 0 \quad \text{in } \Omega\,,$$

$$u_1 = 0\,, \quad \frac{\partial u_1}{\partial \nu} = \psi_1 \quad \text{on } \partial\Omega\,.$$

Assume that there exist $\tilde{\lambda}_1$, $\tilde{\mu}_1 \in \mathbb{R}$ and $\tilde{u}_1 \in C^{3,\alpha}(\overline{\Omega})$ such that

$$\Delta \tilde{u}_1 + \lambda \tilde{u}_1 + \tilde{\lambda}_1 u + \tilde{\mu}_1 = 0 \quad \text{in } \Omega\,,$$

$$\tilde{u}_1 = 0\,, \quad \frac{\partial \tilde{u}_1}{\partial \nu} = \psi_1 \quad \text{on} \quad \partial\Omega\,.$$

Then, if $z = u_1 - \tilde{u}_1$, we have

$$\Delta z + \lambda z + (\lambda_1 - \tilde{\lambda}_1) u + \mu_1 - \tilde{\mu}_1 = 0 \quad \text{in } \Omega\,, \tag{4.4}$$

$$z = \frac{\partial z}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,. \tag{4.5}$$

Now we have two cases to consider.

Case 1. Assume that $\int_{\partial\Omega} \varphi(x)\, d\sigma(x) = 0$. Let $j = 1$ or 2. From (4.4), (4.5) and Lemma 4.1 we obtain

$$\Delta \frac{\partial z}{\partial x_j} + \lambda \frac{\partial z}{\partial x_j} + (\lambda_1 - \tilde{\lambda}_1) \frac{\partial u}{\partial x_j} = 0 \quad \text{in } \Omega\,,$$

$$\frac{\partial z}{\partial x_j} = 0\,, \ \frac{\partial}{\partial \nu}(\frac{\partial z}{\partial x_j}) = (\tilde{\mu}_1 - \mu_1)\nu_j \quad \text{on } \partial\Omega\,.$$

We also have

$$\Delta \frac{\partial u}{\partial x_j} + \lambda \frac{\partial u}{\partial x_j} = 0 \quad \text{in} \quad \Omega\,,$$

and, since $u = 0$ on $\partial\Omega$,

$$\frac{\partial u}{\partial x_j} = \varphi \nu_j \quad \text{on} \quad \partial\Omega\,.$$

Then we can write

$$\begin{aligned}
-\int_\Omega \frac{\partial u}{\partial x_j} \Delta \frac{\partial z}{\partial x_j}\,dx &= \lambda \int_\Omega \frac{\partial u}{\partial x_j}\frac{\partial z}{\partial x_j}\,dx + (\lambda_1 - \tilde{\lambda}_1)\int_\Omega \big(\frac{\partial u}{\partial x_j}\big)^2\,dx \\
&= -\int_\Omega \frac{\partial z}{\partial x_j}\Delta \frac{\partial u}{\partial x_j}\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial x_j}\frac{\partial}{\partial \nu}(\frac{\partial z}{\partial x_j})\,d\sigma(x) \\
&= \lambda \int_\Omega \frac{\partial z}{\partial x_j}\frac{\partial u}{\partial x_j}\,dx + (\mu_1 - \tilde{\mu}_1)\int_{\partial\Omega} \varphi \nu_j^2\,d\sigma(x)
\end{aligned}$$

which implies that

$$(\lambda_1 - \tilde{\lambda}_1)\int_\Omega \big(\frac{\partial u}{\partial x_j}\big)^2\,dx = (\mu_1 - \tilde{\mu}_1)\int_{\partial\Omega} \varphi \nu_j^2\,d\sigma(x)\,. \tag{4.6}$$

From (4.6) we deduce that

$$(\lambda_1 - \tilde{\lambda}_1)\int_\Omega |\nabla u|^2\,dx = (\mu_1 - \tilde{\mu}_1)\int_{\partial\Omega} \varphi\,d\sigma(x) = 0\,. \tag{4.7}$$

By (ii), $u \not\equiv 0$. Then (4.7) implies that $\lambda_1 = \tilde{\lambda}_1$. Suppose that $\mu_1 \neq \tilde{\mu}_1$. Define $v = z/(\mu_1 - \tilde{\mu}_1)$. Then (4.4) and (4.5) give

$$\begin{gathered}
\Delta v + \lambda v + 1 = 0 \quad \text{in } \Omega\,, \\
v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,,
\end{gathered}$$

a contradiction to (i). Therefore $\mu_1 = \tilde{\mu}_1$ and necessarily $u_1 = \tilde{u}_1$. Now assume that λ_j, μ_j and u_j, $j = 1, \dots, n-1$ ($n \geq 2$), are uniquely determined. We have

$$\begin{gathered}
\Delta u_n + \lambda u_n + \lambda_n u + \sum_{j=1}^{n-1} \lambda_{n-j} u_j + \mu_n = 0 \quad \text{in } \Omega\,, \\
u_n = 0\,, \quad \frac{\partial u_n}{\partial \nu} = \psi_n \quad \text{on } \partial\Omega\,.
\end{gathered}$$

Then we obtain the uniqueness of λ_n, μ_n and u_n in the same way.

Case 2. Assume that $\int_{\partial\Omega} \varphi(x)(x_1\nu_2(x) - x_2\nu_1(x))^2\,d\sigma(x) = 0$ and that $\partial\Omega$ is real analytic. If $f \in C^1(\overline{\Omega})$ we define

$$Pf(x) = x_1 \frac{\partial f}{\partial x_2}(x) - x_2 \frac{\partial f}{\partial x_1}(x)$$

for $x \in \overline{\Omega}$. From (4.4), (4.5) and Lemma 4.1 we obtain

$$\begin{gathered}
\Delta Pz + \lambda Pz + (\lambda_1 - \tilde{\lambda}_1) Pu = 0 \quad \text{in } \Omega\,, \\
Pz = 0\,, \quad \frac{\partial Pz}{\partial \nu} = (\tilde{\mu}_1 - \mu_1)(x_1\nu_2 - x_2\nu_1) \quad \text{on } \partial\Omega\,.
\end{gathered}$$

We also have

$$\begin{gathered}\Delta Pu + \lambda Pu = 0 \quad \text{in } \Omega\,, \\ Pu = \varphi(x_1\nu_2 - x_2\nu_1) \quad \text{on } \partial\Omega\,.\end{gathered}$$

Then we can write

$$\begin{aligned}-\int_\Omega Pu\Delta Pz\,dx &= \lambda\int_\Omega PuPz\,dx + (\lambda_1 - \tilde\lambda_1)\int_\Omega (Pu)^2\,dx \\ &= -\int_\Omega Pz\Delta Pu\,dx - \int_{\partial\Omega} Pu\frac{\partial Pz}{\partial\nu}\,d\sigma(x) \\ &= \lambda\int_\Omega PzPu\,dx + (\mu_1 - \tilde\mu_1)\int_{\partial\Omega}\varphi(x_1\nu_2 - x_2\nu_1)^2\,d\sigma(x)\end{aligned}$$

which implies

$$(\lambda_1 - \tilde\lambda_1)\int_\Omega (Pu)^2\,dx = (\mu_1 - \tilde\mu_1)\int_{\partial\Omega}\varphi(x_1\nu_2 - x_2\nu_1)^2\,d\sigma(x) = 0\,. \tag{4.8}$$

Suppose that $Pu = 0$ in Ω. Then $\varphi(x_1\nu_2 - x_2\nu_1) = 0$ on $\partial\Omega$. By (ii) $A = \{x \in \partial\Omega; \varphi(x) \neq 0\} \neq \emptyset$. Then $x_1\nu_2 - x_2\nu_1 = 0$ on A. Since $\partial\Omega$ is connected and real analytic we deduce that $x_1\nu_2 - x_2\nu_1 = 0$ on $\partial\Omega$, hence $\partial\Omega$ is a circle, a contradiction to i). Therefore $Pu \not\equiv 0$ and (4.8) implies that $\lambda_1 = \tilde\lambda_1$. Now we show that $\mu_1 = \tilde\mu_1$ and $u_1 = \tilde u_1$ as in Case 1. Then we use an induction argument as in Case 1 to obtain the uniqueness of λ_n, μ_n and u_n for all $n \geq 1$. The proof of the theorem is complete.

Remark 4.2. Assume that $\Omega \subset \mathbb{R}^N$, $N \geq 3$. If we replace the second condition in (iii) by:

(iv) for all j, $k \in \{1, \dots, N\}$ such that $j \neq k$

$$\int_{\partial\Omega}\varphi(x)(x_j\nu_k(x) - x_k\nu_j(x))^2\,dx = 0\,,$$

and $\partial\Omega$ is connected and real analytic,

then Theorem 1.7 also holds since the Schiffer property and Lemma 4.1 can be stated in any dimension.

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