*Electronic Journal of Differential Equations*, Vol. 2012 (2012), No. 190, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# WEAK COMPACTNESS OF BIHARMONIC MAPS

SHENZHOU ZHENG

Abstract. This article shows that if a sequence of weak solutions of a perturbed biharmonic map satisfies $\Phi_k \to 0$ in $(W^{2,2})^*$ and $u_k \rightharpoonup u$ weakly in $W^{2,2}$, then $u$ is a biharmonic map. In particular, we show that the space of biharmonic maps is sequentially compact under the weak-$W^{2,2}$ topology.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^4$ be a bounded smooth domain, and $(\mathcal{N}, h)$ be a $C^3$-smooth compact $k$-dimensional Riemannian manifold without boundary, embedded into the Euclidean space $\mathbb{R}^d$. For $1 \le l < +\infty$ and $1 \le p < +\infty$, define the Sobolev space $W^{l,p}(\Omega, \mathcal{N})$ by

$$W^{l,p}(\Omega, \mathcal{N}) = \{v \in W^{l,p}(\Omega, \mathbb{R}^d) : v(x) \in \mathcal{N} \text{ a.e. } x \in \Omega\}.$$

In this article, we discuss the limiting behavior of weakly convergent sequences of approximate (extrinsic) biharmonic maps $\{u_m\} \subset W^{2,2}(\Omega, \mathcal{N})$ in dimension four, especially weak compactness of sequence of biharmonic maps. Before stating them, let us recall some related notions.

Recall that an (extrinsic) biharmonic map is the critical point of the Hessian energy functional

$$E_2(v) := \int_\Omega |\Delta v|^2, \quad \forall v \in W^{2,2}(\Omega, \mathcal{N}). \tag{1.1}$$

More precisely;

**Definition 1.1.** A map $u \in W^{2,2}(\Omega, \mathcal{N})$ is called weakly biharmonic if $u$ is a critical point of the Hessian energy functional (1.1) with respect to compactly supported variations on $\mathcal{N}$. That is, if for all $\xi \in C_0^\infty(\Omega, \mathbb{R}^d)$, we have

$$\frac{d}{dt}\Big|_{t=0} E_2(\Pi(u + t\xi)) = 0 \tag{1.2}$$

where $\Pi$ denotes the nearest point projection from $\mathbb{R}^d$ to $\mathcal{N}$.

Note that for the tubular neighborhood $V_\delta$ of $\mathcal{N}$ in $\mathbb{R}^d$ for $\delta > 0$ sufficiently small, we write the smooth nearest point projection $\Pi_{\mathcal{N}} : V_\delta \to \mathcal{N}$. For $u(x) \in \mathcal{N}$, let $\mathbb{P}(u) : \mathbb{R}^d \to T_u\mathcal{N}$ be the orthogonal projection from $\mathbb{R}^d$ onto the tangent space $T_u\mathcal{N}$ of $\mathcal{N}$ at $u \in \mathcal{N}$, and $\mathbb{P}^\perp(u)$ be the orthogonal projection onto the normal space

2000 *Mathematics Subject Classification.* 35J48, 35G50, 58J05.
*Key words and phrases.* Biharmonic maps, conservation law, weak compactness.

Submitted July 15,2012. Published October 31,2012.

with $\mathbb{P}^{\perp}(u) = Id - \mathbb{P}(u)$. Then, from the point of geometric view we readily have $\Delta^2 u \perp T_u\mathcal{N}$ a. e. $\Omega$, which implies the following Euler-Lagrange equation thanks to Wang's work [15, 16].

$$\Delta^2 u = \Delta(\mathbb{B}(u)(\nabla u, \nabla u)) + 2\nabla\cdot\langle \Delta u, \nabla(\mathbb{P}(u))\rangle - \langle \Delta(\mathbb{P}(u)), \Delta u\rangle, \tag{1.3}$$

where $\mathbb{B}(y)(X,Y) = -\nabla_X \mathbb{P}(y)(Y)$, for $X, Y \in T_y\mathcal{N}$, is the second fundamental form on $\mathcal{N} \subset \mathbb{R}^d$.

It is well known that biharmonic maps are higher order extensions of harmonic maps. The study of biharmonic maps has generated considerable interests after the initial work by Chang-Wang-Yang [2], for further development the readers can refer to Wang [15, 16], Strzelecki [12], Lamm-Rivière [6], Struwe [11], Scheven [9, 10] etc. Generally speaking, a singular phenomenon, which appears as various critical problems related nonlinear systems, especially in the investigation of harmonic, biharmonic and $p$-harmonic maps, leads to defects of strong convergence. Typically, some part energy of variation functional is lost in the limit passage so that the sequence does not have to converge strongly. So it is an important and interesting problem to consider the compactness of the sequence of biharmonic maps in the weak topology.

**Definition 1.2.** A sequence of biharmonic maps $\{u_m\} \subset W^{2,2}(\Omega, \mathcal{N})$ is called a Palais-Smale sequence for the energy functional $E_2(u_m)$ on the admissible set $W^{2,2}(\Omega, \mathcal{N})$, if the following two conditions hold:
(a) $u_m \rightharpoonup u$ weakly in $W^{2,2}(\Omega, \mathcal{N})$, and
(b) $E_2'(u_m) \to 0$ in $(W^{2,2}(\Omega, \mathcal{N}))^*$,
where $(W^{2,2}(\Omega, \mathcal{N}))^*$ is the dual of $W^{2,2}(\Omega, \mathcal{N})$.

In this note, we are mainly interested in the compactness problem in the weak topology for the sequence of biharmonic maps in dimension four. We show stability results of biharmonic maps in $\mathbb{R}^4$ under the weak convergence, which is done by a divergence structure of Euler-Lagrange's equations concerning biharmonic maps to general manifolds due to Lamm-Rivière (cf.[6, 11, 7]). In this way we can avoid Lions' concentration compactness argument [3, 14]. In order to further analyze the behaviour of weakly convergent Palais-Smale sequences for the variation functional, we are supposed that $u_m$ is a Palais-Smale sequence for Hessian energy functional with disturbance, which satisfies

$$-\Delta^2 u_m + \Phi_m \perp T_{u_m}\mathcal{N} \tag{1.4}$$

and

$$u_m \text{ is bounded in } W^{2,2}(\Omega, \mathbb{R}^d), \quad \Phi_m \to 0 \text{ in } (W^{2,2}(\Omega, \mathbb{R}^d))^*. \tag{1.5}$$

**Theorem 1.3.** *Assume that $\{u_m\} \subset W^{2,2}(\Omega, \mathcal{N})$ is a Palais-Smale sequence of Hessian energy functionals* (1.1)*; i.e., they satisfy the relation* (1.4) *with $\Phi_m \to 0$ in $(W^{2,2}(\Omega, \mathcal{N}))^*$, and $u_m \rightharpoonup u$ weakly in $W^{2,2}(\Omega, \mathcal{N})$. Then $u \in W^{2,2}(\Omega, \mathcal{N})$ is a biharmonic map.*

Here, we would like to point out that if $\Phi_m$ is bounded in $L^p$ for $p > \frac{4}{3}$. Theorem 1.3 is a consequence of the main theorem by Wang-Zheng[17, 18].

**Remark 1.4.** While $u_k \in W^{1,2}(\Omega, \mathcal{N})$ is a Palais-Smale sequence of harmonic maps, Bethuel [1] was the first to obtain the conclusion of its weak compactness. Later, Freire-Müller-Struwe [3] simplified Bethuel's proof by way of Lions' concentration argument. With similar approach to the Freire-Müller-Struwe's approach, Strzelecki-Zatorska [13] and Wang [14] derived the conclusions of compactness to higher dimensional H-system and n-harmonic maps into manifold in the critical dimension, respectively. Recently, Strzelecki [12, Proposition 1.2] and Goldstein-Strzelecki-Zatorska [4, Theorem 1.2] also achieved the compactness for biharmonic maps and polyharmonic maps into sphere in the critical dimension respectively, who made use of transferring into the equation with divergence form on the basis of the geometric nature of sphere, which avoids to employ Lions' concentration argument.

As an immediate corollary, we obtain a compactness conclusion of any weakly convergent sequence of weak solutions of biharmonic maps as follows. More precisely, we have

**Corollary 1.5.** *Assume that $\{u_m\} \subset W^{2,2}(\Omega, \mathcal{N})$ are a sequences of biharmonic maps converging weakly to u in $W^{2,2}(\Omega, \mathcal{N})$. then u is a biharmonic map.*

## 2. Conservation law for biharmonic maps

In what follows, it is convenient to rewrite the geometric expression (1.4) into the divergence structure, due to the initial work of Lamm-Rivière [6] on biharmonic maps. To this end, we introduce some notation. Let $M(d)$ be the space of $d \times d$ matrices and $so(d) := \{A \in M(d) : A^t + A = 0\}$, $\wedge^l \mathbb{R}^4$ be $l$-tuple exterior differential form on $\mathbb{R}^4$, and $L^2 \cdot W^{1,2}(B_1, M(d))$ be the space of linear combination of products of an $L^2$ map with an $W^{1,2}$ map form $B_1$ into the space $M(d)$. We observe the following fourth order elliptic systems in 4-dimension raised by Lamm-Rivière [6] on biharmonic maps to Riemannian manifold $\mathcal{N}$.

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \operatorname{div} w \nabla u) + \nabla \omega \nabla u + F \nabla u. \tag{2.1}$$

The study of this equation can be traced back to Rivière's work [8] on the conservation law of conformally invariant variational problems including harmonic maps and H-surfaces, which makes the second order elliptic systems $-\Delta u = \Omega \cdot \nabla u$ to be in divergence form, where $\Omega = \Omega_k \partial_{x_k}$ is a vectorfield tensored $d \times d$ antisymmetric matrices. Later, Lamm-Rivière [6] and Struwe [11] discovered another conservation law obtained from (2.1) by extending their original idea concerning harmonic maps to the fourth-order setting concerning biharmonic maps. Now, let us proceed to a Hodge decomposition of $dPP^\perp - P^\perp dP = d\alpha + d^*\beta$ with $\alpha \in W^{2,2}(B_1, so(d)), \beta \in W_0^{2,2}(B_1, \wedge^2(\mathbb{R}^4) \otimes M(d))$ (see [5]) so that

$$\begin{gathered}
\Delta\alpha = \Delta P P^\perp - P^\perp \Delta P \in L^2(B_1), \\
\Delta d^*\beta = \Delta P \wedge dP^\perp - \Delta P^\perp \wedge dP \in L^2 \cdot W^{1,2}(B_1, \wedge^2(\mathbb{R}^4) \otimes M(d)).
\end{gathered}$$

Thanks to [7, Proposition 1.1], we can rewrite the geometric expression (1.4) as

$$\Delta^2 u_m = \Delta(V_m \cdot \nabla u_m) + \operatorname{div} w_m \nabla u_m) + \nabla \omega_m \nabla u_m + F_m \nabla u_m + \Phi_m, \tag{2.2}$$

in the distributional sense, where

$$\begin{gathered}
V_m = -2\nabla P_m^\perp - (\nabla P_m P_m^\perp - P_m^\perp \nabla P_m) \\
w_m = \Delta P_m^\perp - 2\nabla(\nabla P_m P_m^\perp - P_m^\perp \nabla P_m) \\
\omega_m = \Delta\alpha_m \\
F_m = \Delta d^*\beta_m + 2\nabla_y P_m^\perp \nabla(\nabla P_m^\perp \nabla u_m) + 2\nabla_y \nabla P_m^\perp \Delta u_m,
\end{gathered} \tag{2.3}$$

and

$$
\begin{aligned}
|V_m| &\le C|\nabla u_m|,\\
|w_m| + |\nabla V_m| + |\omega_m| &\le C(|\nabla^2 u_m| + |\nabla u_m|^2),\\
|F_m| &\le C(|\nabla^2 u_m||\nabla u_m| + |\nabla u_m|^3),
\end{aligned} \tag{2.4}
$$

furthermore, $V_m \in W^{1,2}(B_1, M(d)\otimes \wedge^1\mathbb{R}^4)$, $w_m \in L^2(B_1, M(d))$, $\Phi_m \in L^p(B_1)$, $W_m = \nabla\omega_m + F_m$ such that $\omega_m \in L^2(B_1, so(d))$, $F_m \in L^2 \cdot W^{1,2}(B_1, M(d)\otimes\wedge^1\mathbb{R}^4)$ and $\Phi_m \to 0$ in $(W^{2,2}(\Omega,\mathcal{N}))^*$.

First, recall a decomposition result to the fourth order systems from Lamm-Rivière's work (see [6, Theorem 1.5]), which is related to biharmonic maps.

**Proposition 2.1.** *Let $V \in W^{1,2}(B_1, M(d)\otimes\wedge^1\mathbb{R}^4)$, $w \in L^2(B_1, M(d))$, and $W = \nabla\omega + F$ with $\omega \in L^2(B_1, so(d))$, $F \in L^2\cdot W^{1,2}(B_1, M(d)\otimes\wedge^1\mathbb{R}^4)$. If the smallness condition*

$$
\|V\|_{W^{1,2}} + \|w\|_{L^2} + \|\omega\|_{L^2} + \|F\|_{L^2\cdot W^{1,2}} < \varepsilon, \tag{2.5}
$$

*is satisfied for some $\varepsilon > 0$, then there exist $A \in L^\infty \cap W^{2,2}(B_1, GL(d))$ and $B \in W^{1,\frac{4}{3}}(B_1, M(d)\otimes\wedge^2\mathbb{R}^4)$ such that*

$$
\nabla\Delta A + \Delta A V - \nabla A w + AW = \operatorname{curl} B, \quad \text{in } B_1, \tag{2.6}
$$

*with the estimate*

$$
\begin{aligned}
&\|A\|_{W^{2,2}} + \|\operatorname{dist}(A, SO(d))\|_{L^\infty} + \|B\|_{W^{1,\frac{4}{3}}}\\
&\le C\big(\|V\|_{W^{1,2}} + \|w\|_{L^2} + \|\omega\|_{L^2} + \|F\|_{L^2\cdot W^{1,2}}\big),
\end{aligned}
$$

*see* [7, Theorem 2.1].

Note that for the Palais-Smale sequence $u_m \in W^{2,2}(B_1)$ of biharmonic maps in $\mathbb{R}^4$, by a simple computation from the estimates (2.4) we obtain that

$$
\|V_m\|^2_{W^{1,2}} + \|w_m\|^2_{L^2} + \|\omega_m\|^2_{L^2} + \|F_m\|^2_{L^2\cdot W^{1,2}} \le C\int_{B_1} |\nabla^2 u_m|^2 + |\nabla u_m|^4, \tag{2.7}
$$

$$
\|A_m\|^2_{W^{2,2}} + \|\operatorname{dist}(A_m, SO(d))\|^2_{L^\infty} + \|B_m\|^{\frac{4}{3}}_{W^{1,\frac{4}{3}}} \le C\int_{B_1} |\nabla^2 u_m|^2 + |\nabla u_m|^4. \tag{2.8}
$$

Therefore, under the smallness assumption $\int_{B_1} |\nabla^2 u_m|^2 + |\nabla u_m|^4 \le \varepsilon$, we obtain from Proposition 2.1 and (2.2) that

$$
\Delta(A_m \Delta u_m) = \operatorname{div} K_m + A_m\Phi_m, \quad x \in B_1, \tag{2.9}
$$

where

$$
\begin{aligned}
K_m = \Big(&2\nabla A_m \Delta u_m - \Delta A_m \nabla u_m + A_m w_m \nabla u_m\\
&- \nabla A_m (V_m\cdot\nabla u_m) + A_m\nabla(V_m\cdot\nabla u_m) + B_m\cdot\nabla u_m\Big).
\end{aligned} \tag{2.10}
$$

**Lemma 2.2** ($\varepsilon$-weak compactness)**.** *In dimension four, there exists an $\varepsilon_0 > 0$ such that if $\{u_m\} \subset W^{2,2}(B_1,\mathcal{N})$ is a Palais-Smale sequence of (extrinsic) biharmonic maps (1.4) and (1.5) with $\Omega$ replaced by $B_1$, $u_m \rightharpoonup u$ weakly in $W^{2,2}(B_1,\mathcal{N})$, and satisfy*

$$
\int_{B_1} |\nabla^2 u_m|^2 + |\nabla u_m|^4 \le \varepsilon^2; \tag{2.11}
$$

*then $u \in W^{2,2}(B_1,\mathcal{N})$ is also a biharmonic map.*

*Proof.* Under the smallness condition $\int_{B_1} |\nabla^2 u_m|^2 + |\nabla u_m|^4 \le \varepsilon^2$, it follows from Proposition 2.1 that the divergent structure (2.9) is valid. Now we analyze the convergence of sequence to any weak solutions of equation (2.9) under the disturbance conditions (1.5). Thanks to

$$u_m \rightharpoonup u \text{ in } W^{2,2}(B_1, \mathbb{R}^d), \quad \Phi_m \to 0 \text{ in } (W^{2,2}(B_1, \mathbb{R}^d))^*,$$

up to taking subsequences, we derive from the estimates (2.7) and (2.8) that $V_m \rightharpoonup V$ weakly in $W^{1,2}(B_1)$, $w_m \rightharpoonup w$ weakly in $L^2(B_1)$,$F_m \rightharpoonup F$ weakly in $L^2 \cdot W^{1,2}(B_1)$, $A_m \rightharpoonup A$ weakly in $W^{2,2}(B_1)$ and $B_m \rightharpoonup B$ weakly in $W^{1,\frac{4}{3}}(B_1)$. By Sobolev's compact imbedding theorem, it follows that $\nabla u_m \to \nabla u$ strongly in $L^2(B_1)$, $\nabla A_m \to \nabla A$ strongly in $L^2(B_1)$ and $A_m \to A$ strongly in $L^2(B_1)$. Hence, it gives

$$\Delta(A_m \Delta u_m) \to \Delta(A\Delta u), \quad \mathcal{D}'(B_1), \tag{2.12}$$

$$\big|A_m \Phi_m\big|_{\{W^{2,2},(W^{2,2})^*\}} \le \|A_m\|_{W^{2,2}(B)} \|\Phi_m\|_{(W^{2,2}(B))^*} \to 0; \tag{2.13}$$

where $\mathcal{D}'$ is Schwartz distributional spaces of $\mathcal{D}$.

For $K_m$, we readily see that $K_m \in L^{\frac{4}{3}}(B_1)$. Then we have

$$\begin{aligned} &2\nabla A_m \Delta u_m - \Delta A_m \nabla u_m + A_m w_m \nabla u_m - \nabla A_m (V_m \cdot \nabla u_m) + A_m \nabla (V_m \cdot \nabla u_m) \\ &\rightharpoonup 2\nabla A \Delta u - \Delta A \nabla u + A w \nabla u - \nabla A (V \cdot \nabla u) + A \nabla (V \cdot \nabla u) \end{aligned} \tag{2.14}$$

weakly $L^1(B_1)$. In addition, since $B_m \in W^{1,\frac{4}{3}}(B_1)$, it implies $B_m \rightharpoonup B$ weakly in $L^2(B_1)$. Then, for $B_m \cdot \nabla u_m$ we have

$$B_m \cdot \nabla u_m \rightharpoonup B \cdot \nabla u$$

weakly in $L^1(B_1)$, and

$$\operatorname{div} K_m \to \operatorname{div} K, \quad \mathcal{D}'(B_1). \tag{2.15}$$

Combining (2.12),(2.14) and (2.15), we obtain

$$\Delta(A\Delta u) = \operatorname{div} K, \quad x \in B_1, \tag{2.16}$$

in $\mathcal{D}'(B_1)$, where

$$K = \Big(2\nabla A \Delta u - \Delta A \nabla u + A w \nabla u - \nabla A (V \cdot \nabla u) + A \nabla (V \cdot \nabla u) + B \cdot \nabla u\Big).$$

That is, $u$ is a biharmonic map of $B_1$. $\square$

## 3. Proof of main result

*Proof of Theorem 1.3.* Assume that $\{u_m\}_{m=1}^{\infty} \in W^{2,2}(\Omega, \mathcal{N})$ is a Palais-Smale sequence of biharmonic maps with disturbance (1.5), and $u_m \rightharpoonup u$ weakly in the space $W^{2,2}(\Omega, \mathcal{N})$. Since $u_m$ is a bounded sequence in $W^{2,2}(\Omega, \mathcal{N})$, we have that $\mu_m := \int_\Omega |\nabla^2 u_m|^2 + |\nabla u_m|^4$ is a family of nonnegative Radon measures with $M = \sup_m \mu_m(\Omega) < \infty$. Therefore, we can assume, after passing up to subsequences, that there is a nonnegative Radon measure $\mu$ on $\Omega$ such that

$$\mu_m := \int_\Omega |\nabla^2 u_m|^2 + |\nabla u_m|^4 \to \mu,$$

as a convergence of Radon measures. Let $\varepsilon_0 > 0$ be the same constant as in Lemma 2.2 and define singular set $\Sigma$ by

$$\Sigma := \{x \in \Omega : \mu(\{x\}) \ge \varepsilon_0^2\}. \tag{3.1}$$

Then, by a simple covering argument we have that $\Sigma$ is a finite set, and

$$H^0(\Sigma) \le \frac{M}{\epsilon_0^2}, \quad M = \sup_m \int_\Omega |\nabla^2 u_m|^2 + |\nabla u_m|^4 < +\infty.$$

Here $H^0$ is 0-dimensional Hausdorff measure. In fact, let $\Sigma' = \{x_1, \dots, x_m\} \subset \Sigma$ be any finite subset of $\Sigma$, then there is a small $\delta_0 > 0$ such that $\{B_{\delta_0}(x_i)\}_{i=1}^s$ are mutually disjoint balls such that

$$\liminf_{m\to\infty} \int_{B_{\delta_0}(x_i)} |\nabla^2 u_m|^2 + |\nabla u_m|^4 \ge \varepsilon_0^2, \quad i = 1, 2, \dots, s;$$

which implies that there is a natural number $K_s \in \mathbf{N}$ such that for any $m \ge K_s$ satisfying

$$\int_{B_{\delta_0}(x_i)} |\nabla^2 u_m|^2 + |\nabla u_m|^4 \ge \varepsilon_0^2, \quad i = 1, 2, \dots, s.$$

Therefore, for any $m \ge K_s$ we have

$$\begin{aligned} s\varepsilon_0^2 &\le \sum_{i=1}^s \int_{B_{\delta_0}(x_i)} |\nabla^2 u_m|^2 + |\nabla u_m|^4 \\ &= \int_{\bigcup_{i=1}^s B_{\delta_0}(x_i)} |\nabla^2 u_m|^2 + |\nabla u_m|^4 \\ &\le \int_\Omega |\nabla^2 u_m|^2 + |\nabla u_m|^4 \le M < \infty. \end{aligned}$$

This implies $s \le M\varepsilon_0^{-2}$.

Therefore, for any $x_0 \in \Omega\backslash\Sigma$ there exists an $r_0 > 0$ such that $\mu(B_{2r_0}(x_0)) < \varepsilon_0^2$. On account of

$$\limsup_{m\to\infty} \int_{B_{r_0}(x_0)} |\nabla^2 u_m|^2 + |\nabla u_m|^4 \le \mu(B_{2r_0}(x_0)),$$

which states clearly that we may assume that there exists $K_0 \ge 1$ such that while $m > K_0$, it follows that

$$\int_{B_{r_0}(x_0)} |\nabla^2 u_m|^2 + |\nabla u_m|^4 \le \varepsilon_0^2. \tag{3.2}$$

Therefore, from Lemma 2.2 it implies that $u$ is a biharmonic map in $B_{r_0}(x_0)$. Since $x_0 \in \Omega\backslash\Sigma$ is arbitrary, we conclude that $u$ is a biharmonic map in $\Omega\backslash\Sigma$. It is standard argument to show that $u$ is biharmonic map in $\Omega$ (see [15, 16]). The proof of Theorem 1.3 is complete. □

**Acknowledgements.** The work is partially supported by grant 11071012 from the NSFC . The author would like to thank Professor Changyou Wang for his constructive and interesting discussions, and to the Department of Mathematics at the University of Kentucky for their kind hospitality while visiting them.

## References

[1] Bethuel, F.: On the singular set of stationary harmonic maps. *Manuscripta Math.,* **78**, 417-443 (1993).

[2] Chang, A., Wang, L., Yang, P.: A regularity theory of biharmonic maps. *Comm. Pure Appl. Math.,* **52**, 1113-1137 (1999).

[3] Freire, A., Müller, S., Struwe, M.: Weak compactness of wave maps and harmonic maps, *Ann. Inst. H. Poincar Anal. Non Linaire,* **15**(6), 725-754(1998).

[4] Goldstein, P., Strzelecki, P., Zatorska-Goldstein,A.: On polyharmonic maps into spheres in the critical dimension, *Ann. Inst. H. Poincaré Anal. NonLinéaire*, **26**(4), 1387-1405(2011).
[5] Iwaniec, T., Martin, G.: Quasiregular mappings in even dimensions. *Acta Math.*, **170**, 29-81 (1993).
[6] Lamm, T., Rivière, T.: Conservation laws for fourth order systems in four dimensions. *Comm. PDE.*, **33**, 245-262 (2008).
[7] Laurain, P., Rivière, T.: Angular Energy Quantization for Linear Elliptic Systems with Antisymmetric Potentials and Applications. arXiv: 1109.3599, preprint.
[8] Rivière, T.: Conservation laws for conformally invariant variational problems. *Invent. Math.*, **168**, 1-22(2007).
[9] Scheven, C.: Dimension reduction for the singular set of biharmonic maps. *Adv. Calc. Var.*, **1**(1), 53-91 (2008).
[10] Scheven, C.: An optimal partial regularity result for minimizers of an intrinsically defined second-order functional. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **26**(5), 1585-1605 (2009).
[11] Struwe, M.: Partial regularity for biharmonic maps, revisited. *Calc. Var. Partial Differential Equations,* **33** (2), 249-262 (2008).
[12] Strzelecki, P.: On biharmonic maps and their generalizations. *Calc. Var. Partial Differential Equations,* **18** (4), 401-432 (2003).
[13] Strzelecki, P., Zatorska-Goldstein, A.: A compactness theorem for weak solutions of higher-dimensional H-systems, *Duke Math. J.*,**121**(2), 269-284(2004).
[14] Wang, C. Y.: A compactness theorem of $n$-harmonic maps, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **22** (4), 509-519 (2005).
[15] Wang, C. Y.: Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold. *Math. Z.*, **247**, 65-87 (2004).
[16] Wang, C. Y.: Stationray biharmonic maps from $\mathbb{R}^m$ into a Riemannian manifold. *Comm. Pure Appl. Math.*, **57**, 0419-0444 (2004).
[17] Wang C. Y. and Zheng S. Z.: Energy identity for a class of approximate biharmonic maps into sphere in dimension four. *Discrete Contin. Dyn. Syst. Ser. A*, **33** (2), 861-878(2013).
[18] Wang C. Y. and Zheng S. Z.: Energy identity of approximate biharmonic maps to Riemannian manifolds and its application. *J. Funct. Anal.*, **263**, 960-987 (2012).

Shenzhou Zheng
Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China
*E-mail address*: shzhzheng@bjtu.edu.cn, Tel: +1-859-257-4802