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GENERATORS WITH INTERIOR DEGENERACY ON SPACES OF L^2 TYPE

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Abstract. We consider operators in divergence and in nondivergence form with degeneracy at the interior of the space domain. Characterizing the domain of the operators, we prove that they generate positive analytic semigroups on spaces of L^2 type. Finally, some applications to linear and semilinear parabolic evolution problems and to linear hyperbolic ones are presented.

1. Introduction

This article is concerned with the generation property of a second order ordinary differential degenerate operator in divergence or in nondivergence form under Dirichlet boundary conditions in the real setting. In particular, we consider both the operators $A_1u := (au')'$ and $A_2u := au''$ with a suitable domain, where the function a vanishes at an interior point of the domain. Degenerate parabolic operators naturally arise in many problems. For some examples involving degeneracy, let us recall some applications arising in aeronautics (the Crocco equation, see, e.g., [10]), in physics (boundary layer models, see, e.g., [6]), in genetics (Wright-Fisher and Fleming-Viot models, see, e.g., [24, 27]) and in mathematical finance (Black-Merton-Scholes models, see, e.g., [12, 20, 23]). Moreover, degenerate operators have been extensively studied since Feller's investigations in [15, 16], whose main motivation was the probabilistic interest of the associated parabolic equation for transition probabilities. After that, the degenerate operator A_1u or A_2u has been studied under different boundary conditions, see, for example, [7, 9, 11, 14, 30, 17, 19, 22, 28, 29, 31]. In particular, [30, 22, 28, 29] develop a functional analytic approach to the construction of Feller semigroups generated by degenerate elliptic operators with Wentzell boundary conditions. In [19], the authors consider degenerate operators with boundary conditions of Dirichlet, Neumann, periodic, or nonlinear Robin type. In [25], A. Stahel proves that the Dirichlet parabolic problem associated to a degenerate operator in divergence form with degeneracy at the boundary of the domain, has a unique solution under suitable assumptions on the degenerate function, using an approximation technique. In [1, 9, 17], the authors consider the degenerate operator in divergence and in non

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divergence form with Dirichlet or Neumann boundary conditions, giving more importance to controllability problems of the associated parabolic evolution equations. However, all the previous papers deal with a degenerate operator with degeneracy at the boundary of the domain. For example, as a, one can consider the double power function

$$a(x) = x^k(1-x)^\alpha, \quad x \in [0,1],$$

where k and α are positive constants.

To the best of our knowledge, Stahel's paper [26] is the first treating a problem with a degeneracy which may be interior. In particular, Stahel considers a parabolic problem in $\mathbb{R}^N$ with Dirichlet, Neumann or mixed boundary conditions, associated with a $N \times N$ matrix a, which is positive definite and symmetric, but whose smallest eigenvalue might converge to 0 as the space variable approaches a singular set contained in the closure of the space domain. In this case, he proves that the corresponding abstract Cauchy problem has a solution, provided that $\underline{a}^{-1} \in L^q(\Omega, \mathbb{R})$ for some $q > 1$, where

$$\underline{a}(x) := \min\{a(x)\xi \cdot \xi : \|\xi\| = 1\}.$$

In the present paper we generalize his assumption when $N = 1$ and the degeneracy is interior. More precisely, we shall admit two types of degeneracy for a, namely weak and strong degeneracy according to the following definitions:

Definition 1.1. The operators $A_1 u := (au')'$ and $A_2 u = au''$ are weakly degenerate if there exists $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1] \setminus \{x_0\}$, $a \in C[0,1]$ and $\frac{1}{a} \in L^1(0,1)$.

For example, as a, we can consider $a(x) = |x - x_0|^\alpha,\ 0 < \alpha < 1$.

Definition 1.2. The operators $A_1 u := (au')'$ and $A_2 u = au''$ are strongly degenerate if there exists $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1] \setminus \{x_0\}$, $a \in W^{1,\infty}(0,1)$ and $\frac{1}{a} \notin L^1(0,1)$.

For example, as a, one can consider $a(x) = |x - x_0|^\alpha,\ \alpha \geq 1$.

We remark that, while in [26] only the existence of a solution for the parabolic problem is considered, here we analyze in detail the underlying degenerate operator in the spaces $L^2(0,1)$ with or without weight, proving that under suitable assumptions it is nonpositive and selfadjoint, hence it generates an analytic semigroup which is positivity preserving (see Theorems 2.2-2.7, 3.3 - 3.11).

Moreover, in the strongly degenerate case, we are able to characterize the domain of the operator both in divergence and in non divergence cases (see Propositions 2.4, 3.8) under, possibly, additional assumptions on the function a. We point out that the generation property of the operator in nondivergence form cannot be deduced by the generation property of the operator in divergence form without additional assumptions on the function a. In fact, the operator

$$A_2 u = au''$$

can be recast using divergence form as follows

$$A_2 u = A_1 u - a'u' \tag{1.1}$$

at the price of adding the drift term $-a'u'$ to the divergence form operator A_1 defined in Definition 1.1 or Definition 1.2. Such an addition has major consequences.

For example, as described in [4], degenerate operators of the form (1.1), where a degenerates at the boundary of the domain, generate a strongly continuous semigroup in $L^2(0,1)$ under the structural assumption

$$|a'(x)| \le C\sqrt{a(x)} \tag{1.2}$$

for any $x \in [0,1]$, where C is a positive constant. Hence, it is natural to expect that a similar result holds also if a has an interior degeneracy. Similar considerations also hold interchanging the role of divergence and nondivergence operators. This is proved in Section 3, where, however, we establish the generation property of $A_2u = au''$ on a suitable weighted space without any additional technical assumption on a. The generator property of $A_1u = (au')'$ is proved without any further assumptions on a in Section 2.

The paper is organized in the following way. In Section 2 we consider the degenerate operator in divergence form and we prove that it is nonpositive and selfadjoint on $L^2(0,1)$. In Section 3 we prove the same result for the operator in nondivergence form on the space $L^2_{1/a}(0,1)$. Moreover, under hypothesis (1.2), we prove that both degenerate operators in nondivergence and in divergence form generate analytic semigroups on $L^2(0,1)$ and $L^2_{1/a}(0,1)$, respectively (see Theorems 3.10 and 3.11). As a consequence of the results proved in Sections 2 and 3, in Section 4 we obtain existence results for linear and semilinear parabolic evolution problems and linear hyperbolic evolution problems associated with the operators under consideration. Finally, for the reader convenience, in the Appendix we give some compactness theorems which are crucial for our proofs in the semilinear parabolic cases. In this paper we will always consider spaces of real valued functions.

2. Divergence form

In this section we consider the operator in divergence form, that is $A_1u = (au')'$, and we distinguish two cases: the weakly degenerate case and the strongly one.

2.1. **Weakly degenerate operator.**

Throughout this subsection we assume that the operator is weakly degenerate. To prove that A_1, with a suitable domain, generates a strongly continuous semigroup, we introduce, as in [1], the following weighted space:

$$\begin{aligned} H^1_a(0,1) := \big\{ & u \in L^2(0,1) : u \text{ absolutely continuous in } [0,1], \\ & \sqrt{a}u' \in L^2(0,1) \text{ and } u(0) = u(1) = 0 \big\} \end{aligned}$$

with the norm

$$\|u\|_{H^1_a(0,1)} := \Big(\|u\|^2_{L^2(0,1)} + \|\sqrt{a}u'\|^2_{L^2(0,1)}\Big)^{1/2}$$

and consider

$$H^2_a(0,1) := \{u \in H^1_a(0,1) : au' \in H^1(0,1)\}.$$

Then, define the operator A_1 by

$$D(A_1) = H^2_a(0,1),$$

and for any $u \in D(A_1)$,

$$A_1u := (au')'.$$

We prove the following Green's formula:

Lemma 2.1. *For all $(u,v) \in H^2_a(0,1) \times H^1_a(0,1)$ one has*

$$\int_0^1 (au')'v dx = -\int_0^1 au'v'dx. \tag{2.1}$$

Proof. Let $(u,v) \in H^2_a(0,1) \times H^1_a(0,1)$. For any sufficiently small $\delta > 0$ one has

$$\begin{aligned} \int_0^1 (au')'v dx &= \int_0^{x_0-\delta} (au')'v dx + \int_{x_0-\delta}^{x_0+\delta} (au')'v dx + \int_{x_0+\delta}^1 (au')'v dx \\ &= (au'v)(x_0-\delta) - (au'v)(0) \\ &\quad - \int_0^{x_0-\delta} au'v'dx + \int_{x_0-\delta}^{x_0+\delta} (au')'v dx \\ &\quad + (au'v)(1) - (au'v)(x_0+\delta) - \int_{x_0+\delta}^1 au'v'dx \\ &= (au'v)(x_0-\delta) - \int_0^{x_0-\delta} au'v'dx + \int_{x_0-\delta}^{x_0+\delta} (au')'v dx \\ &\quad - (au'v)(x_0+\delta) - \int_{x_0+\delta}^1 au'v'dx, \end{aligned} \tag{2.2}$$

since $au' \in H^1(0,1)$ and $v(0) = v(1) = 0$. Now, we prove that

$$\lim_{\delta\to 0} \int_0^{x_0-\delta} au'v'dx = \int_0^{x_0} au'v'dx, \quad \lim_{\delta\to 0} \int_{x_0+\delta}^1 au'v'dx = \int_{x_0}^1 au'v'dx$$

and

$$\lim_{\delta\to 0} \int_{x_0-\delta}^{x_0+\delta} (au')'v dx = 0. \tag{2.3}$$

Toward this end, observe that

$$\int_0^{x_0-\delta} au'v'dx = \int_0^{x_0} au'v'dx - \int_{x_0-\delta}^{x_0} au'v'dx \tag{2.4}$$

and

$$\int_{x_0+\delta}^1 au'v'dx = \int_{x_0}^1 au'v'dx - \int_{x_0}^{x_0+\delta} au'v'dx. \tag{2.5}$$

Moreover, $(au')'v$ and $au'v' \in L^1(0,1)$. Thus, for any $\epsilon > 0$, by the absolute continuity of the integral, there exists $\delta := \delta(\epsilon) > 0$ such that

$$\begin{gathered} \Big| \int_{x_0-\delta}^{x_0} au'v'dx \Big| \le \Big| \int_{x_0-\delta}^{x_0} |au'v'|dx \Big| < \epsilon, \\ \Big| \int_{x_0-\delta}^{x_0+\delta} (au')'v dx \Big| \le \Big| \int_{x_0-\delta}^{x_0+\delta} |(au')'v|dx \Big| < \epsilon, \\ \Big| \int_{x_0}^{x_0+\delta} au'v'dx \Big| \le \Big| \int_{x_0}^{x_0+\delta} |au'v'|dx \Big| < \epsilon. \end{gathered}$$

Now, take such a δ in (2.2). Thus, ϵ being arbitrary,

$$\lim_{\delta\to 0} \int_{x_0-\delta}^{x_0} au'v'dx = \lim_{\delta\to 0} \int_{x_0-\delta}^{x_0+\delta} (au')'v dx = \lim_{\delta\to 0} \int_{x_0}^{x_0+\delta} au'v'dx = 0.$$

The previous equalities and (2.4), (2.5) imply

$$\lim_{\delta\to 0}\int_0^{x_0-\delta} au'v'dx = \int_0^{x_0} au'v'dx \quad\text{and}\quad \lim_{\delta\to 0}\int_{x_0+\delta}^1 au'v'dx = \int_{x_0}^1 au'v'dx.$$

To obtain the desired result it is sufficient to prove that

$$\lim_{\delta\to 0}(au'v)(x_0-\delta) = \lim_{\delta\to 0}(au'v)(x_0+\delta). \tag{2.6}$$

Since $au' \in H^1(0,1)$ and $v \in H^1_a(0,1)$,

$$\lim_{\delta\to 0}(au'v)(x_0-\delta) = (au'v)(x_0) = \lim_{\delta\to 0}(au'v)(x_0+\delta). \tag{2.7}$$

Thus, by (2.2) – (2.3) and (2.6), it follows that

$$\int_0^1 (au')'vdx = -\int_0^1 au'v'dx. \qquad \square$$

As a consequence of the previous lemma one has the next result.

Theorem 2.2. *The operator $A_1 : D(A_1) \to L^2(0,1)$ is nonpositive and self-adjoint on $L^2(0,1)$. Moreover, the semigroup $\{T(t) = e^{tA_1} : t \ge 0\}$ generated by A_1 is positivity preserving.*

Proof. Observe that $D(A_1)$ is dense in $L^2(0,1)$. To show that A_1 is nonpositive and self-adjoint it suffices to prove that A_1 is symmetric, nonpositive and $(I - A_1)(D(A_1)) = L^2(0,1)$ (see e.g. [3, Theorem B.14] or [18]).
***A_1* is symmetric.** Indeed, for any $u, v \in D(A_1)$, one has

$$\langle v, A_1u\rangle_{L^2(0,1)} = \int_0^1 v(au')'dx = -\int_0^1 au'v'dx = \int_0^1 (av')'udx = \langle A_1v, u\rangle_{L^2(0,1)}.$$

***A_1* is nonpositive.** By (2.1), it follows that, for any $u \in D(A_1)$

$$\langle A_1u, u\rangle_{L^2(0,1)} = \int_0^1 (au')'udx = -\int_0^1 a(u')^2dx \le 0.$$

***$I - A_1$* is surjective.** First of all, observe that $H^1_a(0,1)$ equipped with the inner product

$$(u, v)_1 := \int_0^1 (uv + au'v')dx,$$

for any $u, v \in H^1_a(0,1)$, is a Hilbert space. Moreover,

$$H^1_a(0,1) \hookrightarrow L^2(0,1) \hookrightarrow (H^1_a(0,1))^*, \tag{2.8}$$

where $(H^1_a(0,1))^*$ is the dual space of $H^1_a(0,1)$ with respect to $L^2(0,1)$ (cf. (2.8)). Indeed, the continuous embedding of $H^1_a(0,1)$ in $L^2(0,1)$ is readily seen. In addition, for any $f, \varphi \in L^2(0,1)$

$$|\langle f, \varphi\rangle_{L^2(0,1)}| = \Big|\int_0^1 f\varphi dx\Big| \le \|f\|_{L^2(0,1)}\|\varphi\|_{L^2(0,1)} \le \|f\|_{H^1_a(0,1)}\|\varphi\|_{L^2(0,1)}.$$

Hence, $L^2(0,1) \hookrightarrow (H^1_a(0,1))^*$. Then, $(H^1_a(0,1))^*$ is the completion of $L^2(0,1)$ with respect to the norm of $(H^1_a(0,1))^*$. Now, for $f \in L^2(0,1)$, consider the functional $F : H^1_a(0,1) \to \mathbb{R}$ defined as $F(v) := \int_0^1 fvdx$. Since $H^1_a(0,1) \hookrightarrow L^2(0,1)$, we have that $F \in (H^1_a(0,1))^*$. As a consequence, by Riesz's Theorem, there exists a unique $u \in H^1_a(0,1)$ such that for all $v \in H^1_a(0,1)$

$$(u,v)_1 = \int_0^1 fv dx. \tag{2.9}$$

In particular, since $C_c^\infty(0,1) \subset H^1_a(0,1)$, (2.9) holds for all $v \in C_c^\infty(0,1)$; i.e.,

$$\int_0^1 au'v'dx = \int_0^1 (f-u)vdx, \quad \text{for all } v \in C_c^\infty(0,1).$$

Thus, the distributional derivative of au' is a function in $L^2(0,1)$, that is $au' \in H^1(0,1)$ (recall that $\sqrt{a}u' \in L^2(0,1)$) and $(au')' = u - f$ a.e. in $(0,1)$. Then $u \in H^2_a(0,1)$ and, by (2.9) and Lemma 2.1, we have

$$\int_0^1 (u-(au')'-f)vdx = 0.$$

Consequently,

$$u \in D(A_1) \quad \text{and} \quad u - A_1u = f. \tag{2.10}$$

As an immediate consequence of the Stone-von Neumann spectral theorem and functional calculus associated with the spectral theorem, one has that the operator $(A_1, D(A_1))$ generates a cosine family and an analytic semigroup of angle $\frac{\pi}{2}$ on $L^2(0,1)$. Positivity preserving follows as a consequence of the positive minimum principle. $\square$

2.2. **Strongly degenerate operator.** In this subsection we assume that the operator is strongly degenerate. Following [1], we introduce the weighted space

$$\begin{aligned} H^1_a(0,1) := \big\{&u \in L^2(0,1) : u \text{ locally absolutely continuous in } [0,x_0) \cup (x_0,1], \\ &\sqrt{a}u' \in L^2(0,1) \text{ and } u(0)=u(1)=0\big\} \end{aligned}$$

with the norm

$$\|u\|_{H^1_a(0,1)} := \Big(\|u\|^2_{L^2(0,1)} + \|\sqrt{a}u'\|^2_{L^2(0,1)}\Big)^{1/2}.$$

Define the operator A_1 by

$$D(A_1) = H^2_a(0,1),$$

and for any $u \in D(A_1)$,

$$A_1u := (au')',$$

where $H^2_a(0,1)$ is defined as before. Since in this case a function $u \in H^2_a(0,1)$ is locally absolutely continuous in $[0,1] \setminus \{x_0\}$ and not necessarily absolutely continuous in $[0,1]$ as for the weakly degenerate case, equality (2.7) is not true a priori. Thus, we have to prove again the Green formula. To do this, an idea is to characterize the domain of A_1. The next results hold:

Proposition 2.3. *Let*

$$\begin{aligned} X := \Big\{&u \in L^2(0,1) : u \textit{ locally absolutely continuous in } [0,1] \setminus \{x_0\}, \\ &\sqrt{a}u' \in L^2(0,1), au \textit{ is continuous at } x_0 \textit{ and} \\ &(au)(x_0) = 0 = u(0) = u(1)\Big\}. \end{aligned}$$

Then $H^1_a(0,1) = X$.

Proof. Obviously, $X \subseteq H^1_a$. Now we take $u \in H^1_a$, and we prove that $(au)(x_0) = 0$, that is $u \in X$. Toward this end, observe that $au \in H^1_0(0,1)$. Indeed, using the assumptions on u, one has that $au \in L^2(0,1)$ and $(au)(0) = (au)(1) = 0$. Moreover, since $a \in W^{1,\infty}(0,1)$, $(au)' = a'u + au' \in L^2(0,1)$. Thus $au \in H^1_0(0,1) \subset C[0,1]$. This implies that there exists $\lim_{x\to x_0}(au)(x) = (au)(x_0) = L \in \mathbb{R}$. If $L \neq 0$, then there exists $C > 0$ such that

$$|(au)(x)| \geq C$$

for all x in a neighborhood of x_0, $x \neq x_0$. Thus, setting $C_1 := \frac{C^2}{\max_{[0,1]} a(x)} > 0$, it follows that

$$|u^2(x)| \geq \frac{C^2}{a^2(x)} \geq \frac{C_1}{a(x)},$$

for all x in a neighborhood of x_0, $x \neq x_0$. But, since the operator is strongly degenerate, $\frac{1}{a} \notin L^1(0,1)$ thus $u \notin L^2(0,1)$. Hence $L = 0$, that is $(au)(x_0) = 0$. □

Using the previous result, one can prove the following characterization.

Proposition 2.4. *Let*

$$\begin{aligned} D := \big\{&u \in L^2(0,1) : u \text{ locally absolutely continuous in } [0,1] \setminus \{x_0\}, \\ &au \in H^1_0(0,1), au' \in H^1(0,1) \text{ and } (au)(x_0) = (au')(x_0) = 0\big\}. \end{aligned}$$

Then $D(A_1) = D$.

To prove Proposition 2.4, the following lemma is crucial:

Lemma 2.5. *For all* $u \in D$ *we have that*

$$|a(x)u(x)| \leq \|(au)'\|_{L^2(0,1)} \sqrt{|x - x_0|},$$

and

$$|a(x)u'(x)| \leq \|(au')'\|_{L^2(0,1)} \sqrt{|x - x_0|}, \tag{2.11}$$

for all $x \in [0,1]$.

Proof. Let $u \in D$. Since $(au)(x_0) = 0$, then

$$|(au)(x)| = \Big|\int_{x_0}^x (au)'(s)ds\Big| \leq \|(au)'\|_{L^2(0,1)} \sqrt{|x - x_0|},$$

for all $x \in [0,1]$. Analogously, using the fact that $(au')(x_0) = 0$,

$$|(au')(x)| = \Big|\int_{x_0}^x (au')'(s)ds\Big| \leq \|(au')'\|_{L^2(0,1)} \sqrt{|x - x_0|},$$

for all $x \in [0,1]$. □

Proof of Proposition 2.4. Let us prove that $D = D(A_1)$.

$\boldsymbol{D \subseteq D(A_1)}$ **:** Let $u \in D$. It is sufficient to prove that $\sqrt{a}u' \in L^2(0,1)$. Since $au' \in H^1(0,1)$ and $u(1) = 0$ (recall that $a > 0$ in $[0,1] \setminus \{x_0\}$), for $x \in (x_0, 1]$ we have

$$\int_x^1 [(au')'u](s)ds = [au'u]_x^1 - \int_x^1 (a(u')^2)(s)ds = -[au'u](x) - \int_x^1 (a(u')^2)(s)ds.$$

Thus

$$(au'u)(x) = -\int_x^1 [(au')'u](s)ds - \int_x^1 (a(u')^2)(s)ds.$$

Since $u \in D$, $(au')'u \in L^1(0,1)$. Hence, there exists

$$\lim_{x\to x_0^+}(au'u)(x) = L \in [-\infty,+\infty).$$

If $L \neq 0$, there exists $C > 0$ such that

$$|(au'u)(x)| \geq C$$

for all x in a right neighborhood of x_0, $x \neq x_0$. Thus, by (2.11), there exists $C_1 > 0$ such that

$$|u(x)| \geq \frac{C}{|(au')(x)|} \geq \frac{C_1}{\sqrt{(x-x_0)}},$$

for all x in a right neighborhood of x_0, $x \neq x_0$. This implies that $u \notin L^2(0,1)$. Hence $L = 0$ and

$$\int_{x_0}^1 [(au')'u](s)ds = -\int_{x_0}^1 (a(u')^2)(s)ds. \tag{2.12}$$

If $x \in [0,x_0)$, proceeding as before and using the condition $u(0) = 0$, it follows that:

$$(au'u)(x) = \int_0^x [(au')'u](s)ds + \int_0^x (a(u')^2)(s)ds$$

and there exists

$$\lim_{x\to x_0^-}(au'u)(x) = L \in (-\infty,+\infty].$$

As before, if $L \neq 0$, there exist two positive constants C' and C_1' such that

$$|u(x)| \geq \frac{C'}{|(au')(x)|} \geq \frac{C_1'}{\sqrt{(x_0-x)}},$$

for all x in a left neighborhood of x_0, $x \neq x_0$. This implies that $u \notin L^2(0,1)$. Hence $L = 0$ and

$$\int_0^{x_0} [(au')'u](s)ds = -\int_0^{x_0} (a(u')^2)(s)ds. \tag{2.13}$$

By (2.12) and (2.13), it follows that

$$\int_0^1 [(au')'u](s)ds = -\int_0^1 (a(u')^2)(s)ds.$$

Since $(au')'u \in L^1(0,1)$, then $\sqrt{a}u' \in L^2(0,1)$. Hence, $D \subseteq D(A_1)$.

$\boldsymbol{D(A_1) \subseteq D:}$ Let $u \in D(A_1)$. As in the proof of Proposition 2.3, we can prove that $au \in H_0^1(0,1)$. Moreover, by Proposition 2.3, $(au)(x_0) = 0$. Thus, it is sufficient to prove that $(au')(x_0) = 0$. Toward this end, observe that, since $au' \in H^1(0,1)$, there exists $L \in \mathbb{R}$ such that $\lim_{x\to x_0}(au')(x) = (au')(x_0) = L$. If $L \neq 0$, there exists $C > 0$ such that

$$|(au')(x)| \geq C,$$

for all x in a neighborhood of x_0, $x \neq x_0$. Thus

$$|(a(u')^2)(x)| \geq \frac{C^2}{a(x)},$$

for all x in a neighborhood of x_0, $x \neq x_0$. This implies that $\sqrt{a}u' \notin L^2(0,1)$. Hence $L = 0$, that is $(au')(x_0) = 0$. $\square$

We point out the fact that the condition $\frac{1}{a} \notin L^1(0,1)$ is crucial to prove the previous characterizations. Clearly this condition is not satisfied if the operator is weakly degenerate.

As for the weakly degenerate case and using the previous characterization, we can prove the following Green's formula.

Lemma 2.6. *For all $(u,v) \in H^2_a(0,1) \times H^1_a(0,1)$ one has*

$$\int_0^1 (au')'v dx = -\int_0^1 au'v' dx.$$

Proof. Let $(u,v) \in H^2_a(0,1) \times H^1_a(0,1)$. As for the weak case, one can prove that, for any $\delta > 0$:

$$\begin{aligned} \int_0^1 (au')'v dx = (au'v)(x_0-\delta) - \int_0^{x_0-\delta} au'v' dx \\ + \int_{x_0-\delta}^{x_0+\delta} (au')'v dx - (au'v)(x_0+\delta) - \int_{x_0+\delta}^1 au'v' dx. \end{aligned} \tag{2.14}$$

Moreover,

$$\lim_{\delta\to 0} \int_0^{x_0-\delta} au'v' dx = \int_0^{x_0} au'v' dx, \quad \lim_{\delta\to 0} \int_{x_0+\delta}^1 au'v' dx = \int_{x_0}^1 au'v' dx \tag{2.15}$$

and

$$\lim_{\delta\to 0} \int_{x_0-\delta}^{x_0+\delta} (au')'v dx = 0. \tag{2.16}$$

To obtain the desired result it is sufficient to prove that

$$\lim_{\delta\to 0} (au'v)(x_0-\delta) = \lim_{\delta\to 0} (au'v)(x_0+\delta). \tag{2.17}$$

First of all, observe that

$$(au'v)(x_0-\delta) = \int_0^{x_0-\delta} ((au')'v)(s) ds + \int_0^{x_0-\delta} (au'v')(s) ds$$

and

$$(au'v)(x_0+\delta) = -\int_{x_0+\delta}^1 ((au')'v)(s) ds - \int_{x_0+\delta}^1 (au'v')(s) ds.$$

Since $(au')'$, $v \in L^2(0,1)$ and $\sqrt{a}u'$, $\sqrt{a}v' \in L^2(0,1)$, by Hölder's inequality, $(au')'v \in L^1(0,1)$ and $au'v' \in L^1(0,1)$. Thus, there exist L_1, $L_2 \in \mathbb{R}$ such that

$$\begin{aligned} \lim_{\delta\to 0} (au'v)(x_0-\delta) &= \lim_{\delta\to 0} \int_0^{x_0-\delta} ((au')'v)(s) ds + \lim_{\delta\to 0} \int_0^{x_0-\delta} (au'v')(s) ds \\ &= \int_0^{x_0} ((au')'v)(s) ds + \int_0^{x_0} (au'v')(s) ds = L_1 \end{aligned}$$

and

$$\begin{aligned} \lim_{\delta\to 0} (au'v)(x_0+\delta) &= -\lim_{\delta\to 0} \int_{x_0+\delta}^1 ((au')'v)(s) ds - \lim_{\delta\to 0} \int_{x_0+\delta}^1 (au'v')(s) ds \\ &= -\int_{x_0}^1 ((au')'v)(s) ds - \int_{x_0}^1 (au'v')(s) ds = L_2. \end{aligned}$$

If $L_1 \neq 0$, then there exists $C > 0$ such that

$$|(au'v)(x)| \geq C$$

for all x in a left neighborhood of x_0, $x \neq x_0$. Thus, by (2.11),

$$|v(x)| \geq \frac{C}{|(au')(x)|} \geq \frac{C_1}{\sqrt{x_0 - x}}$$

for all x in a left neighborhood of x_0, $x \neq x_0$, and for a suitable positive constant C_1. This implies that $v \notin L^2(0,1)$. Hence $L_1 = 0$. Analogously, one can prove that $L_2 = 0$. Thus (2.17) holds. In particular,

$$\lim_{\delta \to 0}(au'v)(x_0 - \delta) = \lim_{\delta \to 0}(au'v)(x_0 + \delta) = 0. \qquad \square$$

As for the weakly degenerate case, one has the next result.

Theorem 2.7. *The operator $A_1 : D(A_1) \to L^2(0,1)$ is self-adjoint and nonpositive on $L^2(0,1)$. Moreover, the semigroup $\{T(t) = e^{tA_1} : t \geq 0\}$ generated by A_1 is positivity preserving.*

3. Non divergence form

Now, we consider the operator $A_2 u = au''$ and we distinguish again between the weakly and the strongly degenerate cases.

3.1. **Weakly degenerate operator.** Throughout this subsection we consider the weakly degenerate operator and, as in [9], we consider the following Hilbert spaces:

$$L^2_{1/a}(0,1) := \Big\{u \in L^2(0,1) : \int_0^1 \frac{u^2}{a} dx < \infty\Big\},$$
$$H^1_{1/a}(0,1) := L^2_{1/a}(0,1) \cap H^1_0(0,1)$$

with the norms

$$\|u\|^2_{L^2_{1/a}(0,1)} := \int_0^1 \frac{u^2}{a} dx,$$

and

$$\|u\|_{H^1_{1/a}(0,1)} := \Big(\|u\|^2_{L^2_{1/a}(0,1)} + \|u'\|^2_{L^2(0,1)}\Big)^{1/2},$$

respectively. Using the previous spaces, we define the operator A_2 by $D(A_2) = H^2_{1/a}(0,1)$, and for any $u \in D(A_2)$,

$$A_2 u := au'',$$

where

$$H^2_{1/a}(0,1) := \big\{u \in H^1_{1/a}(0,1) \cap W^{2,1}_{\rm loc}(0,1) : au'' \in L^2_{1/a}(0,1)\big\}.$$

The following characterization is immediate.

Corollary 3.1. *The spaces $H^1_{1/a}(0,1)$ and $H^1_0(0,1)$ coincide algebraically. Moreover the two norms are equivalent.*

Proof. Clearly $H^1_{1/a}(0,1) \subseteq H^1_0(0,1)$. Now, if $u \in H^1_0(0,1)$ then

$$\int_0^1 \frac{u^2}{a} dx \leq \max_{[0,1]} u^2 \int_0^1 \frac{1}{a} dx \in \mathbb{R},$$

using the fact that $\frac{1}{a} \in L^1(0,1)$, that is $u \in L^2_{1/a}(0,1)$. This implies $u \in H^1_{1/a}(0,1)$.

Moreover, using the embedding of $H^1_0(0,1)$ in $C[0,1]$, one has, for all $u \in H^1_0(0,1)$,

$$\int_0^1 \frac{u^2}{a}dx \le \|u\|^2_{C[0,1]}\|\frac{1}{a}\|_{L^1(0,1)} \le C\|u\|^2_{H^1_0(0,1)},$$

for a positive constant C. Thus, for all $u \in H^1_0(0,1)$,

$$\|u\|_{H^1_{1/a}(0,1)} \le (C+1)\|u\|_{H^1_0(0,1)} \le (C+1)\|u\|_{H^1_{1/a}(0,1)},$$

for a positive constant C. $\square$

As a consequence of the previous corollary one has that $C_c^\infty(0,1)$ is dense in $H^1_{1/a}(0,1)$.

Also for the weakly degenerate case, in order to prove that the operator generates a cosine family and hence an analytic semigroup on $L^2_{1/a}(0,1)$, we have to use a Green formula similar to the one stated in Lemma 2.1 or Lemma 2.6. However, we have to prove it again, since we cannot adapt the proof of these Lemma. In fact, to proceed as before, we would need $u', v' \in H^1(0,1)$, but this is not the case.

The following Green's formula holds.

Lemma 3.2. *For all $(u,v) \in H^2_{1/a}(0,1) \times H^1_{1/a}(0,1)$ one has*

$$\int_0^1 u''v dx = -\int_0^1 u'v'dx. \tag{3.1}$$

Proof. First, we prove that $H^1_c(0,1) := \{v \in H^1(0,1) : \text{supp}\{v\} \subset (0,1) \setminus \{x_0\}\}$ is dense in $H^1_{1/a}(0,1)$. Indeed, if we consider the sequence $(v_n)_{n \ge \max\{\frac{4}{x_0}, \frac{4}{1-x_0}\}}$, where $v_n := \xi_n v$ for a fixed function $v \in H^1_{1/a}(0,1)$ and

$$\xi_n(x) := \begin{cases} 0, & x \in [0,1/n] \cup [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}] \cup [1-1/n, 1], \\ 1, & x \in [2/n, x_0 - 2/n] \cup [x_0 + 2/n, 1-2/n], \\ nx - 1, & x \in (1/n, 2/n), \\ n(x_0 - x) - 1, & x \in (x_0 - 2/n, x_0 - 1/n), \\ n(x - x_0) - 1, & x \in (x_0 + 1/n, x_0 + 2/n), \\ n(1-x) - 1, & x \in (1-2/n, 1-1/n), \end{cases}$$

it is easy to see that $v_n \to v$ in $L^2_{1/a}(0,1)$. Moreover, one has that

$$\begin{aligned} &\int_0^1 ((v_n - v)')^2 dx \\ &\le 2\int_0^1 (1-\xi_n)^2(v')^2 dx + 2\int_0^1 (\xi_n')^2 v^2 dx \\ &= 2\int_0^1 (1-\xi_n)^2(v')^2 dx \\ &\quad + 2n^2\Big(\int_{1/n}^{2/n} v^2 dx + \int_{x_0-\frac{2}{n}}^{x_0-\frac{1}{n}} v^2 dx + \int_{x_0+\frac{1}{n}}^{x_0+\frac{2}{n}} v^2 dx + \int_{1-\frac{2}{n}}^{1-\frac{1}{n}} v^2 dx\Big). \end{aligned} \tag{3.2}$$

Obviously, the first term in the last member of (3.2) converges to zero. Furthermore, since $v \in H^1_0(0,1)$, by Hölder's inequality, one has that

$$v^2(x) \le x\int_0^x (v')^2(y)dy,$$

for all $x \in [0,1]$. Therefore,

$$n^2 \int_{1/n}^{2/n} v^2 dx \le n^2 \int_0^{2/n} (v')^2 dx \int_{1/n}^{2/n} x\,dx \to 0, \quad \text{as } n \to \infty.$$

Since the remaining terms in (3.2) can be similarly estimated, our claim is proved.

Now, set $\Phi(v) := \int_0^1 (u'v)'dx$, with $u \in H^2_{1/a}(0,1)$. It follows that

$$\Phi(v) = 0,$$

for all $v \in H^1_c(0,1)$. In fact, let $v \in H^1_c(0,1)$. Let $\delta > 0$ be such that $v(x_0+\delta) = v(x_0-\delta) = 0$. Then, there holds

$$\begin{aligned}
\int_0^1 u''v dx &= \int_0^{x_0-\delta} u''v dx + \int_{x_0-\delta}^{x_0+\delta} u''v dx + \int_{x_0+\delta}^1 u''v dx \\
&= (u'v)(x_0-\delta) - (u'v)(0) - \int_0^{x_0-\delta} u'v' dx + \int_{x_0-\delta}^{x_0+\delta} u''v dx \\
&\quad + (u'v)(1) - (u'v)(x_0+\delta) - \int_{x_0+\delta}^1 u'v' dx \\
&= -\int_0^{x_0-\delta} u'v' dx + \int_{x_0-\delta}^{x_0+\delta} u''v dx - \int_{x_0+\delta}^1 u'v' dx,
\end{aligned} \tag{3.3}$$

since $v \in H^1_c(0,1)$. Now we prove that

$$\lim_{\delta\to 0} \int_0^{x_0-\delta} u'v' dx = \int_0^{x_0} u'v' dx, \quad \lim_{\delta\to 0} \int_{x_0+\delta}^1 u'v' dx = \int_{x_0}^1 u'v' dx$$

and

$$\lim_{\delta\to 0} \int_{x_0-\delta}^{x_0+\delta} u''v dx = 0. \tag{3.4}$$

Toward this end, observe that

$$\int_0^{x_0-\delta} u'v' dx = \int_0^{x_0} u'v' dx - \int_{x_0-\delta}^{x_0} u'v' dx \tag{3.5}$$

and

$$\int_{x_0+\delta}^1 u'v' dx = \int_{x_0}^1 u'v' dx - \int_{x_0}^{x_0+\delta} u'v' dx. \tag{3.6}$$

Moreover, using the Hölder inequality, one can prove that $u''v$ and $u'v' \in L^1(0,1)$. Thus, for any $\epsilon > 0$, by the absolute continuity of the integral, there exists $\delta := \delta(\epsilon) > 0$ such that

$$\begin{gathered}
\Big| \int_{x_0-\delta}^{x_0} u'v' dx \Big| \le \Big| \int_{x_0-\delta}^{x_0} |u'v'| dx \Big| < \epsilon, \\
\Big| \int_{x_0-\delta}^{x_0+\delta} u''v dx \Big| \le \Big| \int_{x_0-\delta}^{x_0+\delta} |u''v| dx \Big| < \epsilon, \\
\Big| \int_{x_0}^{x_0+\delta} u'v' dx \Big| \le \Big| \int_{x_0}^{x_0+\delta} |u'v'| dx \Big| < \epsilon.
\end{gathered}$$

Now, take such a δ in (3.3). Thus, ϵ being arbitrary,

$$\lim_{\delta\to 0} \int_{x_0-\delta}^{x_0} u'v' dx = \lim_{\delta\to 0} \int_{x_0-\delta}^{x_0+\delta} u''v dx = \lim_{\delta\to 0} \int_{x_0}^{x_0+\delta} u'v' dx = 0.$$

The previous equalities and (3.5), (3.6) imply
$$\lim_{\delta\to 0}\int_0^{x_0-\delta} u'v'dx = \int_0^{x_0} u'v'dx \quad\text{and}\quad \lim_{\delta\to 0}\int_{x_0+\delta}^1 u'v'dx = \int_{x_0}^1 u'v'dx.$$
Thus, by the previous equalities and by (3.3)– (3.4), it follows that
$$\int_0^1 u''vdx = -\int_0^1 u'v'dx \quad\text{if and only if}\quad \varPhi(v) = \int_0^1 (u'v)'dx = 0,$$
for all $v \in H^1_c(0,1)$. Then, $\varPhi$ is a bounded linear functional on $H^1_{1/a}(0,1)$ such that $\varPhi = 0$ on $H^1_c(0,1)$. Thus, $\varPhi = 0$ on $H^1_{1/a}(0,1)$, that is, (3.1) holds. □

As a consequence of the previous lemma one has the next proposition, whose proof is similar to the proof of Theorem 2.2.

Theorem 3.3. *The operator $A_2 : D(A_2) \to L^2_{1/a}(0,1)$ is self-adjoint and nonpositive on $L^2_{1/a}(0,1)$. Moreover, the semigroup $\{T(t) = e^{tA_2} : t \ge 0\}$ generated by A_2 is positivity preserving.*

3.2. **Strongly degenerate operator.** Assume that the operator A_2 is strongly degenerate and consider, as in [9], the spaces introduced in Section 3.1. Observe that also in this case the space $C^\infty_c(0,1)$ is dense in $H^1_{1/a}(0,1)$, since it is clearly dense in $H^1_0(0,1)$ and dense in $L^2_{\frac{1}{a}}(0,1)$. Then, the conclusions of Lemma 3.2 and Theorem 3.3 hold in this case.

Theorem 3.4. *The operator $A_2 : D(A_2) \to L^2_{1/a}(0,1)$ is self-adjoint and nonpositive on $L^2_{1/a}(0,1)$. Moreover, the semigroup $\{T(t) = e^{tA_2} : t \ge 0\}$ generated by A_2 is positivity preserving.*

Moreover in this case, under an additional assumption on the function a, one can characterize the spaces $H^1_{1/a}(0,1)$ and $H^2_{1/a}(0,1)$. We point out the fact that in non divergence form, the characterization of the domain of the operator is not important to prove the Green formula as in divergence form.

From now on, we make the following assumption on a.

Hypothesis 3.5. Assume that there exists a positive constant K such that $\frac{1}{a(x)} \le \frac{K}{|x-x_0|^2}$, for all $x \in [0,1] \setminus \{x_0\}$ (e.g. $a(x) = |x-x_0|^K,\ 1 \le K \le 2$).

Proposition 3.6. *Let*
$$X := \{u \in H^1_{1/a}(0,1) : u(x_0) = 0\}.$$
If Hypothesis 3.5 is satisfied, then
$$H^1_{1/a}(0,1) = X$$
and the norm $\|u\|_{H^1_{\frac{1}{a}}(0,1)}$ and $\big(\int_0^1 (u')^2dx\big)^{1/2}$ are equivalent.

To prove Proposition 3.6 the following lemma is crucial.

Lemma 3.7. *Assume that Hypothesis 3.5 is satisfied. Then, there exists a positive constant C such that*
$$\int_0^1 v^2\frac{1}{a}dx \le C\int_0^1 (v')^2dx,$$
for all $v \in X$.

Proof. Let $v \in X$. By assumption, there exists $K > 0$ such that $\frac{1}{a(x)} \le \frac{K}{|x-x_0|^2}$, for all $x \in [0,1] \setminus \{x_0\}$. Then, for a suitable $\varepsilon > 0$ and using the assumption on a and the Hardy inequality, one has

$$
\begin{aligned}
\int_0^1 v^2 \frac{1}{a} dx &= \int_0^{x_0-\epsilon} v^2 \frac{1}{a} dx + \int_{x_0-\epsilon}^{x_0+\epsilon} v^2 \frac{1}{a} dx + \int_{x_0+\epsilon}^1 v^2 \frac{1}{a} dx \\
&\le \frac{1}{\min_{[0,x_0-\epsilon]} a(x)} \int_0^{x_0-\epsilon} v^2 dx + \int_{x_0-\epsilon}^{x_0+\epsilon} v^2 \frac{1}{a} dx \\
&\quad + \frac{1}{\min_{[x_0+\epsilon,1]} a(x)} \int_{x_0+\epsilon}^1 v^2 dx \\
&\le \frac{1}{\min_{[0,x_0-\epsilon]\cup[x_0+\epsilon,1]} a(x)} \int_0^1 v^2 dx + \int_{x_0-\epsilon}^{x_0} v^2 \frac{1}{a} dx + \int_{x_0}^{x_0+\epsilon} v^2 \frac{1}{a} dx \\
&\le \frac{1}{\min_{[0,x_0-\epsilon]\cup[x_0+\epsilon,1]} a(x)} \int_0^1 v^2 dx + K \int_{x_0-\epsilon}^{x_0} v^2 \frac{1}{|x-x_0|^2} dx \\
&\quad + K \int_{x_0}^{x_0+\epsilon} v^2 \frac{1}{|x-x_0|^2} dx \\
&\le \frac{1}{\min_{[0,x_0-\epsilon]\cup[x_0+\epsilon,1]} a(x)} \int_0^1 v^2 dx \\
&\quad + C_H \int_{x_0-\epsilon}^{x_0} (v')^2 dx + C_H \int_{x_0}^{x_0+\epsilon} (v')^2 dx \\
&\le C\Big(\int_0^1 v^2 dx + \int_0^1 (v')^2 dx \Big),
\end{aligned}
$$

for a positive constant C. Here C_H is the Hardy constant. By Poincaré's inequality, it follows that

$$
\int_0^1 v^2 \frac{1}{a} dx \le C \int_0^1 (v')^2 dx
$$

for a suitable constant C. □

Proof of Proposition 3.6. Obviously $X \subseteq H^1_{1/a}(0,1)$. Now, take $u \in H^1_{1/a}(0,1)$ and prove that $u(x_0) = 0$, that is $u \in X$. Since $u \in H^1_0(0,1)$, then there exists

$$
\lim_{x \to x_0} u(x) = u(x_0) = L \in \mathbb{R}.
$$

If $L \neq 0$, then there exists $C > 0$ such that

$$
|u(x)| \ge C,
$$

for all x in a neighborhood of x_0, $x \neq x_0$. Thus,

$$
\frac{u^2(x)}{a(x)} \ge \frac{C^2}{a(x)},
$$

for all x in a neighborhood of x_0, $x \neq x_0$. Since the operator is strongly degenerate, $\frac{1}{a} \notin L^1(0,1)$, then $u \notin L^2_{1/a}(0,1)$. Hence $L = 0$.

Now, we prove that the two norms are equivalent. Take $u \in X$. Then, by Lemma 3.7, there exists a positive constant C such that

$$
\|u'\|^2_{L^2(0,1)} \le \|u\|^2_{H^1_{1/a}(0,1)} \le C\|u'\|^2_{L^2(0,1)}.
$$

Hence the two norms are equivalent. □

An immediate consequence is the following result.

Proposition 3.8. *Let*

$$D := \big\{u \in H^1_{1/a}(0,1) : au'' \in L^2_{1/a}(0,1),\ au' \in H^1(0,1),\ u(x_0) = (au')(x_0) = 0\big\}.$$

If Hypothesis 3.5 is satisfied, then $H^2_{1/a}(0,1) = D$.

Proof. Obviously, $D \subseteq H^2_{1/a}(0,1)$. Now, we take $u \in H^2_{1/a}(0,1)$ and we prove that $u \in D$. By Proposition 3.6, $u(x_0) = 0$. Thus, it is sufficient to prove that $au' \in H^1(0,1)$ and $(au')(x_0) = 0$. Since $u \in H^1_0(0,1)$ and $a \in W^{1,\infty}(0,1)$, then $u' \in L^2(0,1)$, $\sqrt{a}u' \in L^2(0,1)$ and $au' \in L^2(0,1)$. Moreover $(au')' = a'u' + au'' \in L^2(0,1)$ (recall that $au'' \in L^2_{1/a}(0,1) \subset L^2(0,1)$). Thus $au' \in H^1(0,1)$. This implies that there exists $\lim_{x\to x_0}(au')(x) = (au')(x_0) = L \in \mathbb{R}$. If $L \neq 0$, then there exists $C > 0$ such that

$$|(au')(x)| \geq C$$

for all x in a neighborhood of x_0, $x \neq x_0$. Thus,

$$|a(u')^2(x)| \geq \frac{C^2}{a(x)},$$

for all x in a neighborhood of x_0, $x \neq x_0$. But $\frac{1}{a} \notin L^1(0,1)$, thus $\sqrt{a}u' \notin L^2(0,1)$. Hence $L = 0$, that is $(au')(x_0) = 0$. □

We point out the fact that also in non divergence form the condition $\frac{1}{a} \notin L^1(0,1)$ is crucial to characterize the domain of the strongly degenerate operator.

3.3. The operator in non divergence form in the space $\mathbf{L^2(0,1)}$.

In the previous subsections we have seen that without any additional assumptions on the function a, the operator in non divergence form generates a cosine family, and hence an analytic semigroup, on the space $L^2_{1/a}(0,1)$. In this subsection we will prove that this operator generates an analytic semigroup also on $L^2(0,1)$ under a suitable assumption on a. In particular we make the following hypothesis:

Hypothesis 3.9.

$$\begin{gathered}\text{In the } \textit{weakly degenerate case}\text{, assume further that } a \in W^{1,\infty}(0,1)\\ \text{and there exists } C \geq 0 \text{ such that } |a'(x)| \leq C\sqrt{a(x)} \text{ a.e. } x \in (0,1).\end{gathered} \tag{3.7}$$

$$\begin{gathered}\text{In the } \textit{strongly degenerate case}\text{, assume further that there exists}\\ C \geq 0 \text{ such that } |a'(x)| \leq C\sqrt{a(x)} \text{ a.e. } x \in (0,1).\end{gathered} \tag{3.8}$$

Now, define the operator A by $D(A) = H^2_a(0,1)$, and for any $u \in D(A)$,

$$Au := au'',$$

where $H^2_a(0,1)$ is the Sobolev space introduced in Section 2. Observe that if a satisfies Hypothesis 3.9, then

$$au'' \in L^2(0,1) \text{ if and only if } (au')' \in L^2(0,1),$$

for all $u \in D(A)$.

Theorem 3.10. *Assume that the operator A is weakly or strongly degenerate. If a satisfies Hypothesis 3.9, then A generates an analytic semigroup on $L^2(0,1)$.*

Proof. Let $Bu := a'u'$, for all $u \in D(A)$. Then $Au = A_1u - Bu$, where, we recall, $A_1u := (au')'$. Therefore, one has that B is A_1- bounded with A_1- bound $b_0 = 0$, where, we recall,

$$\begin{aligned} b_0 := \inf \big\{ b \geq 0 :\ & \text{there exists } c \in \mathbb{R}_+ \text{ such that} \\ & \|Bu\|_{L^2(0,1)} \leq b\|A_1u\|_{L^2(0,1)} + c\|u\|_{L^2(0,1)} \big\} \end{aligned} \tag{3.9}$$

(see, for example, [13, Definition III.2.1]). Indeed, using the assumption on a, one has

$$\begin{aligned} \|Bu\|^2_{L^2(0,1)} &= \int_0^1 (a')^2(u')^2 dx \leq C \int_0^1 au'u'dx \\ &= -C \int_0^1 (au')'udx \leq \epsilon \frac{K}{2} \int_0^1 ((au')')^2 dx + \frac{K}{\epsilon} \int_0^1 u^2 dx \\ &= \epsilon \frac{K}{2} \|A_1u\|^2_{L^2(0,1)} + \frac{K}{\epsilon} \|u\|^2_{L^2(0,1)} \end{aligned}$$

for all $u \in D(A_1)$, for all $\epsilon > 0$ and for a positive constant K. Hence B is a Kato perturbation of A_1. The conclusion follows by [13, Theorem III.2.10] and Theorems 2.2 and 2.7. □

3.4. **The operator in divergence form in the space $\mathbf{L^2_{1/a}(0,1)}$.** As in the previous subsection, interchanging the role of the divergence and nondivergence operators, we can prove that the operator in divergence form generates an analytic semigroup also on $L^2_{1/a}(0,1)$ under Hypothesis 3.9. Indeed, define the operator A by $D(A) = H^2_{1/a}(0,1)$, and for any $u \in D(A)$,

$$Au := (au')',$$

where $H^2_{1/a}(0,1)$ is the Sobolev space introduced in Section 3. Also in this case we have that if a satisfies Hypothesis 3.9, then

$$au'' \in L^2_{1/a}(0,1) \text{ if and only if } (au')' \in L^2_{1/a}(0,1),$$

for all $u \in D(A)$. The following theorem holds.

Theorem 3.11. *Assume that the operator A is weakly or strongly degenerate. If a satisfies Hypothesis 3.9, then A generates an analytic semigroup on $L^2_{1/a}(0,1)$.*

Proof. Let $Bu := a'u'$, for all $u \in D(A)$. Then $Au = A_2u + Bu$, where, we recall, $A_2u := au''$. As before, B is A_2- bounded with A_2- bound $b_0 = 0$, where b_0 is defined in (3.9). Indeed, using the assumption on a, one has

$$\begin{aligned} \|Bu\|^2_{L^2_{1/a}(0,1)} &= \int_0^1 \frac{(a')^2(u')^2}{a} dx \leq C \int_0^1 u'u'dx \\ &= -C \int_0^1 u''udx = -C \int_0^1 \sqrt{a}u'' \frac{u}{\sqrt{a}} dx \\ &\leq \epsilon \frac{K}{2} \int_0^1 a(u'')^2 dx + \frac{K}{\epsilon} \int_0^1 \frac{u^2}{a} dx \\ &= \epsilon \frac{K}{2} \|A_2u\|^2_{L^2_{1/a}(0,1)} + \frac{K}{\epsilon} \|u\|^2_{L^2_{1/a}(0,1)} \end{aligned}$$

for all $u \in D(A_2)$, for all $\epsilon > 0$ and for a positive constant K. Hence B is a Kato perturbation of A_2. The conclusion follows by [13, Theorem III.2.10] and Theorems 3.3 and 3.4. $\square$

4. Applications

4.1. **Linear problems.** As an application of the previous theorems, consider the linear operator $B(t)$ defined as

$$B(t)u := -b(t,\cdot)\frac{\partial u}{\partial x} - c(t,\cdot)u,$$

where $b, c \in L^\infty(\mathbb{R}_+ \times (0,1))$ and there exists a positive constant C such that $|b(t,x)| \le C\sqrt{a(x)}$, a.e. $x \in (0,1)$. Then $B(t)$ is an A_1- bounded operator on $L^2(0,1)$ or an A_2- bounded operator on $L^2_{1/a}(0,1)$. Observe that if $b \equiv 0$ and $c(t,\cdot) \le 0$, the operator $A_i - B(t)$ with domain $D(A_i)$, $i = 1, 2$, is still selfadjoint and nonpositive for each t. Moreover, $D(A_i - B(t))$ is independent of t. Thus using evolution operator theory (see e.g. [18], pp. 140-147), we can prove that the problem

$$\begin{gathered}\frac{\partial u}{\partial t} - Au + b(t,x)\frac{\partial u}{\partial x} + c(t,x)u = h(t,x), \quad (t,x) \in \mathbb{R}_+ \times (0,1),\\ u(t,0) = u(t,1) = 0, \quad t \ge 0,\\ u(0,x) = u_0(x), \quad x \in (0,1),\end{gathered} \tag{4.1}$$

is wellposed in the sense of evolution operator theory, provided that $A := A_1$ or $A := A_2$. In addition, setting $Q_T := (0,T) \times (0,1)$ for a fixed $T > 0$, the next results follow by Theorems 2.2, 2.7, 3.3, 3.4.

Theorem 4.1. *Assume that the operator $A = A_1$ is weakly or strongly degenerate. If $b(\cdot,x)$, $c(\cdot,x) \in C^1(\mathbb{R}_+)$ for all $x \in [0,1]$ and there exists a positive constant C such that $|b(t,x)| \le C\sqrt{a(x)}$, a.e. $x \in (0,1)$, then, for all $h \in L^2(Q_T)$ and $u_0 \in L^2(0,1)$, there exists a unique weak solution $u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_a(0,1))$ of (4.1) and*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(0,1)} + \int_0^T \|u(t)\|^2_{H^1_a(0,1)}dt \le C_T(\|u_0\|^2_{L^2(0,1)} + \|h\|^2_{L^2(Q_T)}), \tag{4.2}$$

for a positive constant C_T. Moreover, if $u_0 \in D(A_1)$, then

$$u \in H^1(0,T; L^2(0,1)) \cap L^2(0,T; H^2_a(0,1)) \cap C([0,T]; H^1_a(0,1)), \tag{4.3}$$

and there exists a positive constant C such that

$$\begin{aligned}&\sup_{t\in[0,T]} \Big(\|u(t)\|^2_{H^1_a(0,1)}\Big) + \int_0^T \Big(\|\frac{\partial u}{\partial t}\|^2_{L^2(0,1)} + \|\frac{\partial}{\partial x}(a\frac{\partial u}{\partial x})\|^2_{L^2(0,1)}\Big)dt\\ &\le C\Big(\|u_0\|^2_{H^1_a(0,1)} + \|h\|^2_{L^2(Q_T)}\Big).\end{aligned} \tag{4.4}$$

Remark 4.2. Actually, since $D(A_1)$ is dense in $H^1_a(0,1)$, one can prove that (4.3) and (4.4) also hold if $u_0 \in H^1_a(0,1)$.

Theorem 4.3. *Assume that the operator $A = A_2$ is weakly or strongly degenerate. If $b(\cdot,x), c(\cdot,x) \in C^1(\mathbb{R}_+)$ for all $x \in [0,1]$ and there exists a positive constant C such that $|b(t,x)| \le C\sqrt{a(x)}$, a.e. $x \in (0,1)$, then, for all $h \in L^2_{1/a}(Q_T)$ and*

$u_0 \in L^2_{1/a}(0,1)$, there exists a unique weak solution $u \in C\big([0,T];L^2_{1/a}(0,1)\big) \cap L^2\big(0,T;H^1_{1/a}(0,1)\big)$ of (4.1) and

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2_{1/a}(0,1)} + \int_0^T \|u(t)\|^2_{H^1_{1/a}(0,1)}dt \le C_T\Big(\|u_0\|^2_{L^2_{1/a}(0,1)} + \|h\|^2_{L^2_{1/a}(Q_T)}\Big), \tag{4.5}$$

for a positive constant C_T. Moreover, if $u_0 \in D(A_2)$, then

$$u \in H^1\Big(0,T;L^2_{1/a}(0,1)\Big) \cap L^2\Big(0,T;H^2_{1/a}(0,1)\Big) \cap C\Big([0,T];H^1_{1/a}(0,1)\Big), \tag{4.6}$$

and there exists a positive constant C such that

$$\begin{aligned} &\sup_{t\in[0,T]}\Big(\|u(t)\|^2_{H^1_{1/a}(0,1)}\Big) + \int_0^T \Big(\|\frac{\partial u}{\partial t}\|^2_{L^2_{1/a}(0,1)} + \|a\frac{\partial^2 u}{\partial x^2}\|^2_{L^2_{1/a}(0,1)}\Big)dt \\ &\le C\Big(\|u_0\|^2_{H^1_{1/a}(0,1)} + \|h\|^2_{L^2(Q_T)}\Big). \end{aligned} \tag{4.7}$$

Remark 4.4. Also in this case (4.6) and (4.7) hold if $u_0 \in H^1_{1/a}(0,1)$, since $D(A_2)$ is dense in $H^1_{1/a}(0,1)$.

By Theorem 3.10, one has the following result.

Theorem 4.5. *Assume that the operator $A = A_2$ is weakly or strongly degenerate. If a satisfies Hypothesis 3.9, $b(\cdot,x)$, $c(\cdot,x) \in C^1(\mathbb{R}_+)$ for all $x \in [0,1]$ and there exists a positive constant C such that $|b(t,x)| \le C\sqrt{a(x)}$, a.e. $x \in (0,1)$, then, for all $h \in L^2(Q_T)$ and $u_0 \in L^2(0,1)$, there exists a unique weak solution $u \in C([0,T];L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$ of (4.1). If $u_0 \in D(A_1)$, then*

$$u \in H^1\big(0,T;L^2(0,1)\big) \cap L^2\big(0,T;H^2_a(0,1)\big) \cap C\big([0,T];H^1_a(0,1)\big).$$

Moreover, (4.2) and (4.4) hold.

By Theorem 3.11, one has the following result.

Theorem 4.6. *Assume that the operator $A = A_1$ is weakly or strongly degenerate. If a satisfies Hypothesis 3.9, $b(\cdot,x)$, $c(\cdot,x) \in C^1(\mathbb{R}_+)$ for all $x \in [0,1]$ and there exists a positive constant C such that $|b(t,x)| \le C\sqrt{a(x)}$, a.e. $x \in (0,1)$, then, for all $h \in L^2_{1/a}(Q_T)$ and $u_0 \in L^2_{1/a}(0,1)$, there exists a unique weak solution $u \in C\left([0,T];L^2_{1/a}(0,1)\right) \cap L^2\left(0,T;H^1_{1/a}(0,1)\right)$ of (4.1). If $u_0 \in D(A_2)$, then*

$$u \in H^1\big(0,T;L^2_{1/a}(0,1)\big) \cap L^2\big(0,T;H^2_{1/a}(0,1)\big) \cap C\big([0,T];H^1_{1/a}(0,1)\big).$$

Moreover, (4.5) and (4.7) hold.

Let

$$B(t)u := -c(t,\cdot)u,$$

where $c \in L^\infty_{\rm loc}(\mathbb{R}_+ \times [0,1])$ and $c(\cdot,x) \in C^1(\mathbb{R}_+)$ for each $x \in [0,1]$. Consider the nonautonomous wave equation

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} - Au + c(t,x)\frac{\partial u}{\partial t} = h(t,x), \quad (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = 0, \quad t \ge 0, \\ u(0,x) = u_0(x), \quad x \in (0,1), \\ \frac{\partial u}{\partial t}(0,x) = u_1(x), \quad x \in (0,1). \end{gathered} \tag{4.8}$$

We take $A = A_j$ acting in H_j, for $j = 1, 2$, where

$$H_1 = L^2(0,1), \quad H_2 = L^2_{1/a}(0,1).$$

In either case $A_j = A_j^* \le 0$. Let D_j be the completion of $D(A_j)$ in the norm

$$\|f\|_{D_j} = \|(-A_j)^{1/2} f\|_{H_j}.$$

Then (4.8) can be rewritten as

$$\begin{gathered}\frac{d}{dt}U(t) = GU(t) + P(t)U(t) + H(t),\\ U(0) = U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},\end{gathered}$$

where

$$G = \begin{pmatrix} 0 & I \\ A_j & 0 \end{pmatrix}, \quad P(t) = \begin{pmatrix} 0 & 0 \\ 0 & -c(t,\cdot) \end{pmatrix}, \quad H(t) = \begin{pmatrix} 0 \\ h(t,\cdot) \end{pmatrix},$$

for $j = 1, 2$. By [18, Section II.7.1], on the completion K_j of $D_j \oplus H_j$ in the norm

$$\Big\| \begin{pmatrix} v \\ w \end{pmatrix} \Big\|_{K_j} = \Big(\|(-A_j)^{1/2} v\|^2_{H_j} + \|w\|^2_{H_j} \Big)^{1/2},$$

G is skewadjoint and generates a (C_0) unitary group. Moreover, each $P(t)$ is bounded and $P \in C^1(\mathbb{R}_+, K_j)$. Finally, we assume

$$h \in L^\infty_{\rm loc}(\mathbb{R}_+ \times [0,1])$$

and $h(\cdot, x) \in C^1(\mathbb{R}_+)$ for each $x \in [0,1]$. Then it follows that

$$H \in C^1(\mathbb{R}_+, K_j).$$

The wellposedness of (4.8) now follows immediately by a simple modification of [18, Theorem II.13.9].

4.2. **Semilinear problems.** In this subsection we extend the existence results obtained in the previous theorems to semilinear degenerate parabolic systems of the type:

$$\begin{gathered} u_t - Au + f(t,x,u,u_x) = h(t,x), \quad (t,x) \in (0,T) \times (0,1),\\ u(t,1) = u(t,0) = 0, \quad t \in (0,T),\\ u(0,x) = u_0(x), \quad x \in (0,1), \end{gathered} \tag{4.9}$$

where $Au := A_1 u$ or $Au := A_2 u$, $u_t(t,x) := \frac{\partial u(t,x)}{\partial t}$ and $u_x(t,x) := \frac{\partial u(t,x)}{\partial x}$. In particular, recalling that $Q_T = (0,T) \times (0,1)$, we give the following definitions:

Definition 4.7. Assume that $Au := A_1 u$, $u_0 \in L^2(0,1)$ and $h \in L^2(Q_T)$. A function u is said to be a solution of (4.9) if

$$u \in C(0,T; L^2(0,1)) \cap L^2(0,T; H^1_a(0,1))$$

and satisfies

$$\begin{aligned} &\int_0^1 u(T,x)\varphi(T,x)dx - \int_0^1 u_0(x)\varphi(0,x)dx - \int_0^T \int_0^1 \varphi_t(t,x)u(t,x)\,dx\,dt \\ &= -\int_0^T \int_0^1 a u_x \varphi_x \,dx\,dt + \int_0^T \int_0^1 (-f(t,x,u,u_x) + h(t,x))\varphi(t,x)\,dx\,dt, \end{aligned}$$

for all $\varphi \in H^1(0,T; L^2(0,1)) \cap L^2(0,T; H^1_a(0,1))$.

Definition 4.8. Assume that $Au := A_2u$, $u_0 \in L^2_{1/a}(0,1)$ and $h \in L^2_{1/a}(Q_T)$. A function u is said to be a solution of (4.9) if
$$u \in C(0,T; L^2_{1/a}(0,1)) \cap L^2(0,T; H^1_{1/a}(0,1))$$
and satisfies
$$\begin{aligned}&\int_0^1 \frac{u(T,x)\varphi(T,x)}{a(x)}dx - \int_0^1 \frac{u_0(x)\varphi(0,x)}{a(x)}dx - \int_0^T\int_0^1 \frac{\varphi_t(t,x)u(t,x)}{a(x)}\,dx\,dt\\ &= -\int_0^T\int_0^1 u_x\varphi_x\,dx\,dt + \int_0^T\int_0^1 (-f(t,x,u,u_x) + h(t,x))\frac{\varphi(t,x)}{a(x)}\,dx\,dt,\end{aligned}$$
for all $\varphi \in H^1(0,T; L^2_{1/a}(0,1)) \cap L^2(0,T; H^1_{1/a}(0,1))$.

On the functions a and f we make the following assumptions:

Hypothesis 4.9. There exists $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1]\setminus\{x_0\}$, $a \in C[0,1]$ and $\frac{1}{a} \in L^1(0,1)$.

Hypothesis 4.10. Let $f : [0,T] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ be such that
$$\begin{gathered}\forall (q,p) \in \mathbb{R}^2, \quad (t,x) \mapsto f(t,x,q,p) \text{ is measurable;}\\ \text{for a.e. } (t,x) \in (0,T) \times (0,1), \quad f(t,x,0,0) = 0;\end{gathered}$$
$f_p(t,x,q,p)$ exists, f_p is a Carathéodory function; i.e.,
$$\begin{gathered}\forall (q,p) \in \mathbb{R}^2, \quad (t,x) \mapsto f_p(t,x,,q,p) \text{ is measurable and}\\ \text{for a.e. } (t,x) \in (0,T) \times (0,1), \quad (q,p) \mapsto f_p(t,x,q,p) \text{ is continuous,}\end{gathered}$$
and there exists $L > 0$ such that for a.e.$(t,x) \in (0,T) \times (0,1)$ and for any $(q,p) \in \mathbb{R} \times \mathbb{R}$,
$$|f_p(t,x,q,p)| \le L\sqrt{a(x)}; \tag{4.10}$$
$f_q(t,x,q,p)$ exists, f_q is a Carathéodory function and there exists $C > 0$ such that for a.e.$(t,x) \in (0,T) \times (0,1)$ and for all $(q,p) \in \mathbb{R} \times \mathbb{R}$
$$|f_q(t,x,q,p)| \le C. \tag{4.11}$$

However, to prove that (4.9) has a solution in the non divergence case, i.e. $Au := A_2u$, it is sufficient to substitute (4.10) with the more general condition: there exists $L > 0$ such that for a.e. $(t,x) \in (0,T) \times (0,1)$ and $\forall\,(q,p) \in \mathbb{R} \times \mathbb{R}$
$$|f_p(t,x,q,p)| \le L. \tag{4.12}$$

Moreover, to obtain the desired result, we introduce on the spaces $H^2_a(0,1)$ and $H^2_{1/a}(0,1)$ the scalar products
$$\int_0^1 u(x)v(x)dx + \int_0^1 a(x)u'(x)v'(x)dx + \int_0^1 (a(x)u'(x))'(a(x)v'(x))'dx,$$
for all $u,v \in H^2_a(0,1)$, and
$$\int_0^1 \frac{u(x)v(x)}{a(x)}dx + \int_0^1 u'(x)v'(x)dx + \int_0^1 au''(x)v''(x)dx,$$
for all $u,v \in H^2_{1/a}(0,1)$, respectively. The previous scalar products induce on $H^2_a(0,1)$ and $H^2_{1/a}(0,1)$ the following two norms
$$\|u\|_{H^2_a(0,1)} := \Big(\|u\|^2_{H^1_a(0,1)} + \|(au')'\|^2_{L^2(0,1)}\Big)^{1/2},$$

$$\|u\|_{H^2_{1/a}(0,1)} := \Big(\|u\|^2_{H^1_{1/a}(0,1)} + \|au''\|^2_{L^2_{1/a}(0,1)}\Big)^{1/2}.$$

Clearly, $H^2_a(0,1)$ and $H^2_{1/a}(0,1)$ are Hilbert spaces.

As a first step, we study (4.9) with $u_0 \in H^1_a(0,1)$, if $Au := A_1u$, or $u_0 \in H^1_{1/a}(0,1)$, if $Au := A_2u$.

To prove the existence results we will use, as in [1] or in [8], a fixed point method. To this aim, we rewrite, first of all, the function f in the following way $f(t,x,u,u_x) = b(t,x,u)u_x + c(t,x,u)u$, where

$$b(t,x,u) := \int_0^1 f_p(t,x,\lambda u,\lambda u_x)d\lambda, \quad c(t,x,u) := \int_0^1 f_q(t,x,\lambda u,\lambda u_x)d\lambda.$$

In fact

$$\begin{aligned} f(t,x,u,u_x) &= \int_0^1 \frac{d}{d\lambda} f(t,x,\lambda u,\lambda u_x)d\lambda \\ &= \int_0^1 f_q(t,x,\lambda u,\lambda u_x)u d\lambda + \int_0^1 f_p(t,x,\lambda u,\lambda u_x)u_x d\lambda. \end{aligned}$$

Using the fact that f_p and f_q are Carathéodory functions and the Lebesgue Theorem, we can prove the following properties:

Proposition 4.11. *For the functions b and c one has the following properties:*

- *$b(t,x,u(t,x))$ and $c(t,x,u(t,x))$ belong to $L^\infty((0,T)\times(0,1))$;*
- *$|b(t,x,u)| \le L\sqrt{a(x)}$;*
- *if $\lim_{k\to+\infty} v_k = v$ in X, then*

$$\lim_{k\to+\infty} \frac{b(t,x;v_k)}{\sqrt{a(x)}} = \frac{b(t,x;v)}{\sqrt{a(x)}}, \quad a.e.,$$

and

$$\lim_{k\to+\infty} c(t,x;v_k) = c(t,x;v), \quad a.e.$$

Here

$$X := C(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$$

or

$$X := C(0,T;L^2_{1/a}(0,1)) \cap L^2(0,T;H^1_{1/a}(0,1)),$$

and L is the same constant of (4.10).

Theorem 4.12. *Assume that Hypotheses 4.9 and 4.10 are satisfied. Then, for all $h \in L^2(Q_T)$, the problem*

$$\begin{gathered} u_t - (au_x)_x + f(t,x,u,u_x) = h(t,x), \quad (t,x) \in (0,T)\times(0,1), \\ u(t,1) = u(t,0) = 0, \quad t \in (0,T), \\ u(0,x) = u_0(x) \in H^1_a(0,1), \quad x \in (0,1), \end{gathered} \tag{4.13}$$

has a solution $u \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2_a(0,1))$.

Proof. Let $X := C(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$ and, for any $(t,x) \in (0,T)\times(0,1)$, set $b^v(t,x) := b(t,x,v(t,x))$ and $c^v(t,x) := c(t,x,v(t,x))$ for any $v \in X$. Then, consider the function

$$\mathcal{T} : v \in X \mapsto u^v \in X,$$

where u^v is the unique solution of

$$\begin{gathered} u_t - (a(x)u_x)_x + b^v(t,x)u_x + c^v(t,x)u = h(t,x), \\ u(t,1) = u(t,0) = 0, \\ u(0,x) = u_0(x) \in H^1_a(0,1). \end{gathered} \tag{4.14}$$

By Theorem 4.1, problem (4.14) has a unique weak solution $u \in X$. Hence, if we prove that $\mathcal{T}$ has a fixed point u^v, i.e. $\mathcal{T}(u^v) = u^v$, then u^v is solution of (4.13).

To prove that $\mathcal{T}$ has a fixed point, by the Schauder's Theorem, it is sufficient to prove that

1. $\mathcal{T} : B_X \to B_X$,
2. $\mathcal{T}$ is a compact function,
3. $\mathcal{T}$ is a continuous function.

Here $B_X := \{v \in X : \|v\|_X \le R\}$, $R := C_T(\|u_0\|^2_{L^2(0,1)} + \|h\|^2_{L^2((0,T)\times(0,1))})$ (C_T is the same constant of Theorem 4.1) and $\|v\|_X := \sup_{t\in[0,T]}\Big(\|u(t)\|^2_{L^2(0,1)}\Big) + \int_0^T \|\sqrt{a}u_x\|^2_{L^2(0,1)}dt$.

The first item is a consequence of Theorem 4.1. Indeed, one has that $\mathcal{T} : X \to B_X$ and in particular $\mathcal{T} :\ B_X \to B_X$. Moreover, it is easy to see that item (2) is a simple consequence of the compactness Theorem 5.4 below. This theorem is also useful for the proof of item (3). Indeed, let $v_k \in X$ be such that $v_k \to v$ in X, as $k \to +\infty$. We want to prove that $u^k := u^{v_k} \to u^v$ in X, as $k \to +\infty$. Here u^k and u^v are the solutions of (4.13) associated to v_k and v, respectively. By Remark 4.2, $u^k \in B_Y$, where $Y := H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2_a(0,1))$. Proceeding as in Theorem 5.2 below, one has that, up to subsequence, u^k converges weakly to some $\bar u$ in Y and, thanks to Theorem 5.4 below, strongly in X. Now, we prove that $\bar u = u^v$. Multiplying the equation

$$u^k_t - \big(a(x)u^k_x\big)_x + b^{v^k}(t,x)u^k_x + c^{v^k}(t,x)u^k = h(t,x),$$

by a test function $\varphi \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$ and integrating over Q_T (we recall that $Q_T := (0,T)\times(0,1)$), we have:

$$\begin{aligned} &\int_0^1 u^k(T,x)\varphi(T,x)dx - \int_0^1 u_0(x)\varphi(0,x)dx - \int_0^T\int_0^1 \varphi_t(t,x)u^k(t,x)\,dx\,dt \\ &= -\int_0^T\int_0^1 a(x)u^k_x(t,x)\varphi_x(t,x)\,dx\,dt - \int_0^T\int_0^1 b^{v^k}(t,x)u^k_x(t,x)\varphi(t,x)\,dx\,dt \\ &\quad -\int_0^T\int_0^1 c^{v^k}(t,x)u^k(t,x)\varphi(t,x)\,dx\,dt + \int_0^T\int_0^1 h(t,x)\varphi(t,x)\,dx\,dt. \end{aligned}$$

Since u^k converges strongly to $\bar u$ in X, it is immediate to prove

$$\begin{gathered} \lim_{k\to+\infty}\int_0^1 u^k(T,x)\varphi(T,x)dx = \int_0^1 \bar u(T,x)\varphi(T,x)dx, \\ \lim_{k\to+\infty}\int_0^T\int_0^1 \varphi_t(t,x)u^k(t,x)\,dx\,dt = \int_0^T\int_0^1 \varphi_t(t,x)\bar u(t,x)\,dx\,dt \\ \lim_{k\to+\infty}\int_0^T\int_0^1 a u^k_x\varphi_x\,dx\,dt = \int_0^T\int_0^1 a\bar u_x\varphi_x\,dx\,dt \end{gathered}$$

for all $\varphi \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$. Moreover

$$\lim_{k\to+\infty}\int_0^T\int_0^1 c^{v^k}(t,x)u^k(t,x)\varphi(t,x)\,dx\,dt = \int_0^T\int_0^1 c^v(t,x)\bar{u}(t,x)\varphi(t,x)\,dx\,dt. \tag{4.15}$$

Indeed,

$$\begin{aligned}
&\Big|\int_0^T\int_0^1 (c^{v^k}(t,x)u^k(t,x) - c^v(t,x)\bar{u}(t,x))\varphi(t,x)\,dx\,dt\Big| \\
&\le \int_0^T\int_0^1 \big|(u^k(t,x)-\bar{u}(t,x))c^{v^k}(t,x)\varphi(t,x)\big|\,dx\,dt \\
&\quad + \int_0^T\int_0^1 \big|c^{v^k}(t,x)-c^v(t,x)\big||\bar{u}(t,x)\varphi(t,x)|\,dx\,dt \\
&\le C\|u^k-\bar{u}\|_X\|\varphi\|_{L^2(0,1)} + \int_0^T\int_0^1 \big|c^{v^k}(t,x)-c^v(t,x)\big||\bar{u}(t,x)\varphi(t,x)|\,dx\,dt,
\end{aligned}$$

where C is the constant of (4.11). By the Lebesgue Theorem and by Proposition 4.11, it follows

$$\lim_{k\to+\infty}\int_0^T\int_0^1 \big|c^{v^k}(t,x)-c^v(t,x)\big||\bar{u}(t,x)\varphi(t,x)|\,dx\,dt = 0.$$

Thus, since u^k converges strongly to $\bar{u}$ in X, (4.15) holds.

Finally,

$$\lim_{k\to+\infty}\int_0^T\int_0^1 b^{v^k}(t,x)u^k_x(t,x)\varphi(t,x)\,dx\,dt = \int_0^T\int_0^1 b^v(t,x)\bar{u}_x(t,x)\varphi(t,x)\,dx\,dt. \tag{4.16}$$

Indeed,

$$\begin{aligned}
&\Big|\int_0^T\int_0^1 (b^{v^k}(t,x)u^k_x(t,x) - b^v(t,x)\bar{u}_x(t,x))\varphi(t,x)\,dx\,dt\Big| \\
&\le \int_0^T\int_0^1 \big|(u^k_x(t,x)-\bar{u}_x(t,x))b^{v^k}(t,x)\varphi(t,x)\big|\,dx\,dt \\
&\quad + \int_0^T\int_0^1 \big|b^{v^k}(t,x)-b^v(t,x)\big||\bar{u}_x(t,x)\varphi(t,x)|\,dx\,dt \\
&\le L\|u^k-\bar{u}\|_X\|\varphi\|_{L^2(0,1)} \\
&\quad + \int_0^T\int_0^1 \frac{\big|b^{v^k}(t,x)-b^v(t,x)\big|}{\sqrt{a(x)}}|\sqrt{a(x)}\bar{u}_x(t,x)\varphi(t,x)|\,dx\,dt,
\end{aligned} \tag{4.17}$$

where L is the constant of (4.10). The strongly convergence of u^k to $\bar{u}$ in X, Proposition 4.11, the Lebesgue Theorem and (4.17) imply that (4.16) holds. Thus $u^v = \bar{u}$, that is $\bar{u} \in Y$ is the solution of (4.13) associated to v.

Hence, $\mathcal{T}$ has a fixed point $u^v \in Y$ and, in particular, u^v solves (4.13). $\square$

To prove that the existence result holds also if the initial data u_0 is in $L^2(0,1)$, we observe that (4.10) implies for a.e. $(t,x) \in (0,T)\times(0,1)$ and for all $(u,p,q) \in \mathbb{R}^3$,

$$|f(t,x,u,p) - f(t,x,u,q)| \le L\sqrt{a(x)}|p-q|. \tag{4.18}$$

Moreover, by (4.11) and (4.18), it follows that for a.e. $(t,x) \in (0,T)\times(0,1)$ and for all $(u,v,p,q)\in\mathbb{R}^4$,

$$|(f(t,x,u,p)-f(t,x,v,q))(u-v)| \le M[|u-v|^2+\sqrt{a(x)}|p-q||u-v|], \tag{4.19}$$

for some positive constant M.

Hence, using the previous estimates and Theorem 4.12, one can prove, as in [8], the following result.

Theorem 4.13. *Assume that Hypotheses 4.9 and 4.10 are satisfied. Then, for all $h\in L^2(Q_T)$, the problem*

$$\begin{gathered} u_t-(au_x)_x+f(t,x,u,u_x)=h(t,x), \quad (t,x)\in(0,T)\times(0,1),\\ u(t,0)=0, \quad t\in(0,T),\\ u(0,x)=u_0(x)\in L^2(0,1), \quad x\in(0,1), \end{gathered} \tag{4.20}$$

has a solution $u\in C([0,T];L^2(0,1))\cap L^2(0,T;H^1_a(0,1))$.

Proof. Let $(u_0^j)_j \in H^1_a(0,1)$ be such that $\lim_{j\to+\infty}\|u_0^j-u_0\|_{L^2(0,1)}=0$. Denote by u^j the solutions of (4.20) with respect to u_0^j. By Theorem 4.12, $u^j\in H^1(0,T;L^2(0,1))\cap L^2(0,T;H^2_a(0,1))$. Then $(u^j)_j$ is a Cauchy sequence in $X:=C([0,T];L^2(0,1))\cap L^2(0,T;H^1_a(0,1))$. In fact u^j-u^i solves the system

$$\begin{gathered} (u^j-u^i)_t-(a(u^j-u^i)_x)_x+f(t,x,u^j,u^j_x)-f(t,x,u^i,u^i_x)=0,\\ (u^j-u^i)(t,1)=(u^j-u^i)(t,0)=0,\\ (u^j-u^i)(0,x)=(u^j_0-u^i_0)(x), \end{gathered}$$

where $(t,x)\in(0,T)\times(0,1)$. Multiplying

$$(u^j-u^i)_t-(a(u^j-u^i)_x)_x+f(t,x,u^j,u^j_x)-f(t,x,u^i,u^i_x)=0$$

by u^j-u^i and integrating over $(0,1)$, one has, using (4.19),

$$\frac12\frac{d}{dt}\int_0^1|u^j-u^i|^2dx+\int_0^1 a|u^j_x-u^i_x|^2dx \le \int_0^1 M[|u^j-u^i|^2+\sqrt{a}|u^j_x-u^i_x||u^j-u^i|]dx.$$

Integrating over $(0,t)$:

$$\begin{aligned} &\frac12\|(u^j-u^i)(t)\|^2_{L^2(0,1)}+\int_0^t\int_0^1 a|(u^j-u^i)_x|^2\,dx\,ds\\ &\le \frac12\|u^j_0-u^i_0\|^2_{L^2(0,1)}+M\int_0^t\int_0^1|u^j-u^i|^2\,dx\,ds+\frac{\epsilon M}{2}\int_0^t\int_0^1 a|u^j_x-u^i_x|^2\,dx\,ds\\ &\quad+\frac{M}{2\epsilon}\int_0^t\int_0^1|u^j-u^i|^2\,dx\,ds. \end{aligned}$$

Thus

$$\begin{aligned} &\frac12\|(u^j-u^i)(t)\|^2_{L^2(0,1)}+\big(1-\frac{\epsilon M}{2}\big)\int_0^t\int_0^1 a|u^j_x-u^i_x|^2\,dx\,ds\\ &\le \frac12\|u^j_0-u^i_0\|^2_{L^2(0,1)}+M_\epsilon\int_0^t\int_0^1|u^j-u^i|^2\,dx\,ds, \end{aligned} \tag{4.21}$$

where $M_\epsilon:=\frac{M(1+2\epsilon)}{2\epsilon}$. By Gronwall's Lemma

$$\|(u^j-u^i)(t)\|^2_{L^2(0,1)}\le e^{M_\epsilon t}\|u^j_0-u^i_0\|^2_{L^2(0,1)},$$

and

$$\sup_{t\in[0,T]} \|(u^j - u^i)(t)\|^2_{L^2(0,1)} \le e^{M_\epsilon T}\|u_0^j - u_0^i\|^2_{L^2(0,1)}. \tag{4.22}$$

This implies that $(u^j)_j$ is a Cauchy sequence in $C(0,T;L^2(0,1))$. Moreover, by (4.21), one has

$$\big(1-\frac{\epsilon M}{2}\big)\int_0^t\int_0^1 a|u_x^j - u_x^i|^2\,dx\,ds \le \frac{1}{2}\|u_0^j-u_0^i\|^2_{L^2(0,1)} + M_\epsilon\int_0^t\int_0^1 |u^j-u^i|^2\,dx\,ds.$$

Using (4.22), it follows

$$\begin{aligned}\int_0^t \|\sqrt{a}(u_x^j - u_x^i)\|^2_{L^2(0,1)}ds &\le M_{\epsilon,T}(\|u_0^j-u_0^i\|^2_{L^2(0,1)} + \sup_{t\in[0,T]}\|u^j-u^i\|^2_{L^2(0,1)})\\ &\le M_{\epsilon,T}\|u_0^j-u_0^i\|^2_{L^2(0,1)}.\end{aligned}$$

Thus $(u^j)_j$ is a Cauchy sequence also in $L^2(0,T;H^1_a(0,1))$. Then there exists $\bar u \in X$ such that

$$\lim_{j\to+\infty}\|u^j-\bar u\|_X = 0.$$

Proceeding as in the proof of Theorem 4.12, one can prove that $\bar u$ is a solution of (4.20). □

Analogously, recalling Definition 4.8, one can prove the following results:

Theorem 4.14. *Assume that Hypotheses 4.9 and 4.10 are satisfied. Then, for all $h \in L^2_{1/a}(Q_T)$, the problem*

$$\begin{gathered} u_t - au_{xx} + f(t,x,u,u_x) = h(t,x), \quad (t,x)\in(0,T)\times(0,1),\\ u(t,0)=0, \quad t\in(0,T),\\ u(0,x)=u_0(x)\in H^1_{1/a}(0,1), \quad x\in(0,1),\end{gathered}$$

has a solution $u \in H^1(0,T;L^2_{1/a}(0,1))\cap L^2(0,T;H^2_{1/a}(0,1))$.

Theorem 4.15. *Assume that Hypotheses 4.9 and 4.10 are satisfied. Then, for all $h \in L^2_{1/a}(Q_T)$, the problem*

$$\begin{gathered} u_t - au_{xx} + f(t,x,u,u_x) = h(t,x), \quad (t,x)\in(0,T)\times(0,1),\\ u(t,0)=0, \quad t\in(0,T),\\ u(0,x)=u_0(x)\in L^2_{1/a}(0,1), \quad x\in(0,1),\end{gathered}$$

has a solution $u \in C([0,T];L^2_{1/a}(0,1))\cap L^2(0,T;H^1_{1/a}(0,1))$.

We recall that Theorems 4.14 and 4.15 still hold if we substitute (4.10) with (4.12). In this case (4.18) becomes for a.e. $(t,x)\in(0,T)\times(0,1)$ and for all $(u,p,q)\in\mathbb{R}^3$,

$$|f(t,x,u,p)-f(t,x,u,q)| \le L|p-q|.$$

5. Appendix

In this section we will give some compactness theorems that we have used in the previous section.

Theorem 5.1. *Assume that the function a satisfies Hypothesis 4.9. Then, the space $H^1_a(0,1)$ is compactly imbedded in $L^2(0,1)$.*

Proof. Clearly, $H^1_a(0,1)$ is continuously imbedded in $L^2(0,1)$. Now, let $\epsilon > 0$ and $M > 0$. We want to prove that there exists $\delta > 0$ such that for all $u \in H^1_a(0,1)$ with $\|u\|^2_{H^1_a(0,1)} \le M$ and for all $|h| < \delta$ it results

$$\int_\delta^{1-\delta} |u(x+h) - u(x)|^2 dx < \epsilon, \tag{5.1}$$

$$\int_{1-\delta}^1 |u(x)|^2 dx + \int_0^\delta |u(x)|^2 dx < \epsilon. \tag{5.2}$$

Hence, let $u \in H^1_a(0,1)$ such that $\|u\|^2_{H^1_a(0,1)} \le M$ and take $\delta > 0$ such that

$$\delta < g(\epsilon) := \min\Big\{\frac{x_0}{2}, \frac{1+x_0}{2}, \sqrt{\frac{\epsilon}{2M \max_{[0,\frac{x_0}{2}]} \frac{1}{a}}}, \sqrt{\frac{\epsilon}{2M \max_{[\frac{1+x_0}{2},1]} \frac{1}{a}}}\Big\}. \tag{5.3}$$

Then,

$$\begin{aligned}\int_0^\delta |u(x)|^2 dx &\le \int_0^\delta \Big| \int_0^x \frac{1}{\sqrt{a(y)}} \sqrt{a(y)} u'(y) dy \Big|^2 dx \le M \int_0^\delta \delta \max_{[0,\frac{x_0}{2}]} \frac{1}{a} dx \\ &< M\delta^2 \max_{[0,\frac{x_0}{2}]} \frac{1}{a} < \frac{\epsilon}{2}.\end{aligned}$$

Analogously,

$$\int_{1-\delta}^1 |u(x)|^2 dx < \frac{\epsilon}{2}.$$

Now, let h be such that $|h| < \delta$ and, for simplicity, assume $h > 0$ (the case $h < 0$ can be treated in the same way). Then

$$|u(x+h) - u(x)|^2 \le \|u\|^2_{H^1_a(0,1)} \int_x^{x+h} \frac{dy}{a(y)}.$$

Integrating over $(\delta, 1-\delta)$, it results

$$\begin{aligned}\int_\delta^{1-\delta} |u(x+h) - u(x)|^2 dx &\le \|u\|^2_{H^1_a(0,1)} \int_\delta^{1-\delta} dx \int_x^{x+\delta} \frac{dy}{a(y)} \\ &\le M \int_\delta^1 \frac{dy}{a(y)} \int_{y-\delta}^y dx \le M\delta \Big\|\frac{1}{a}\Big\|_{L^1(0,1)}.\end{aligned}$$

Moreover, since $\lim_{\delta\to 0} M\delta\|\frac{1}{a}\|_{L^1(0,1)} = 0$, there exists $\eta(\epsilon) > 0$ such that if $\delta < \eta(\epsilon)$, then $M\delta\,\|\frac{1}{a}\|_{L^1(0,1)} < \epsilon$. Thus, taking $\delta < \min\{g(\epsilon), \eta(\epsilon)\}$, (5.1) and (5.2) are verified and the thesis follows (see, e.g., [5, Chapter IV]). $\square$

Using Theorem 5.1 one can prove the next theorem.

Theorem 5.2. *Assume that the function a satisfies Hypothesis 4.9. Then, the space $H^2_a(0,1)$ is compactly imbedded in $H^1_a(0,1)$.*

Proof. Take $(u_n)_n \in \overline{B}_{H^2_a(0,1)}$. Here $B_{H^2_a(0,1)}$ denotes the unit ball of $H^2_a(0,1)$. Since $H^2_a(0,1)$ is reflexive, then, up to subsequence, there exists $u \in H^2_a(0,1)$ such that u_n converges weakly to u in $H^2_a(0,1)$. In particular, u_n converges weakly to u in $H^1_a(0,1)$ and in $L^2(0,1)$. But, since by the previous Theorem $H^1_a(0,1)$ is compactly imbedded in $L^2(0,1)$, then, up to subsequence, there exists $v \in L^2(0,1)$ such that u_n converges strongly to v in $L^2(0,1)$. Thus u_n converges weakly to v in $L^2(0,1)$. By uniqueness $v \equiv u$. Then we can conclude that the sequence u_n converges strongly to u in $L^2(0,1)$.

Moreover, one has

$$\|\sqrt{a}u_n' - \sqrt{a}u'\|_{L^2(0,1)} \to 0, \quad \text{as } n \to +\infty.$$

Indeed, using the Hölder inequality, one has

$$\begin{aligned}
\|\sqrt{a}(u_n - u)'\|^2_{L^2(0,1)} &= \int_0^1 a(u_n - u)'(u_n - u)'dx \\
&= -\int_0^1 (a(u_n - u)')'(u_n - u)dx \\
&\le \|(a(u_n - u)')'\|_{L^2(0,1)}\|u_n - u\|_{L^2(0,1)} \to 0,
\end{aligned}$$

as $n \to +\infty$. Hence the sequence u_n converges strongly to u in $H^1_a(0,1)$. □

For the proof of Theorem 5.4 below we will use Aubin's Theorem, that we give here for the reader's convenience.

Theorem 5.3 ([2, Chapter 5] or [21, Theorem 5.1])**.** *Let X_0, X_1 and X_2 be three Banach spaces such that $X_0 \subset X_1 \subset X_2$, X_0, X_2 are reflexive spaces and the injection of X_0 into X_1 is compact. Let r_0, $r_1 \in (1, +\infty)$ and $a, b \in \mathbb{R}$, $a < b$. Then the space*

$$L^{r_0}(a, b; X_0) \cap W^{1,r_1}(a, b; X_2)$$

is compactly imbedded in $L^{r_0}(a, b; X_1)$.

Now we are ready to prove the next compactness Theorem.

Theorem 5.4. *Assume that function a satisfies Hypothesis 4.9. Then, the space $H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2_a(0,1))$ is compactly imbedded in $L^2(0,T;H^1_a(0,1)) \cap C(0,T;L^2(0,1))$.*

Proof. Using Aubin's Theorem 5.3 with $r_0 = r_1 = 2$, $X_0 = H^2_a(0,1)$, $X_1 = H^1_a(0,1)$, $X_2 = L^2(0,1)$, $a = 0$ and $b = T$, one has

$$H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2_a(0,1)) \hookrightarrow\hookrightarrow L^2(0,T;H^1_a(0,1)).$$

Moreover, since $H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2_a(0,1))) \hookrightarrow H^1(0,T;L^2(0,1))$ and $H^1(0,T;L^2(0,1)) \hookrightarrow\hookrightarrow C(0,T;L^2(0,1))$, then

$$H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2_a(0,1)) \hookrightarrow\hookrightarrow C(0,T;L^2(0,1)).$$

Thus

$$H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2_a(0,1)) \hookrightarrow\hookrightarrow L^2(0,T;H^1_a(0,1)) \cap C(0,T;L^2(0,1)).$$

□

Analogously to Theorem 5.4, one can obtain the following compactness result.

Theorem 5.5. *Assume that the function a satisfies Hypothesis 4.9. Then, the space $H^1(0,T;L^2_{1/a}(0,1)) \cap L^2(0,T;H^2_{1/a}(0,1))$ is compactly imbedded in the space $L^2(0,T;H^1_{1/a}(0,1)) \cap C(0,T;L^2_{1/a}(0,1))$.*

To prove Theorem 5.5, it is sufficient to prove the analogous results of Theorems 5.1 and 5.2. In particular, we have:

Theorem 5.6. *Assume that the function a satisfies Hypothesis 4.9. Then, the space $H^1_{1/a}(0,1)$ is compactly imbedded in $L^2_{1/a}(0,1)$.*

Proof. Clearly, $H^1_{1/a}(0,1)$ is continuously imbedded in $L^2_{1/a}(0,1)$. Now, let $\epsilon > 0$ and $M > 0$. We want to prove that there exists $\delta > 0$ such that for all $u \in H^1_{1/a}(0,1)$ with $\|u\|^2_{H^1_{1/a}(0,1)} \le M$ and for all $|h| < \delta$ it holds that

$$\int_\delta^{1-\delta} \frac{|u(x+h)-u(x)|^2}{a(x)} dx < \epsilon, \tag{5.4}$$

$$\int_{1-\delta}^1 \frac{|u(x)|^2}{a(x)} dx + \int_0^\delta \frac{|u(x)|^2}{a(x)} dx < \epsilon. \tag{5.5}$$

Hence, let $u \in H^1_{1/a}(0,1)$ such that $\|u\|^2_{H^1_{1/a}(0,1)} \le M$ and take $\delta > 0$ such that $\delta < g(\epsilon)$, where $g(\epsilon)$ is defined in (5.3). Then,

$$\int_0^\delta \frac{|u(x)|^2}{a(x)} dx \le \max_{[0,\frac{x_0}{2}]} \frac{1}{a} \int_0^\delta \Big| \int_0^x u'(y)dy\Big|^2 dx < M\delta^2 \max_{[0,\frac{x_0}{2}]} \frac{1}{a} < \frac{\epsilon}{2}.$$

Analogously,

$$\int_{1-\delta}^1 \frac{|u(x)|^2}{a(x)} dx < \frac{\epsilon}{2}.$$

Now, let h be such that $|h| < \delta$ and, for simplicity, assume $h > 0$ (the case $h < 0$ can be treated in the same way). Then

$$\int_\delta^{1-\delta} \frac{|u(x+h)-u(x)|^2}{a(x)} dx = \int_\delta^{1-\delta} \frac{1}{a(x)} Big| \int_x^{x+h} u'(y)dy\Big|^2 dx \le M\delta \|\frac{1}{a}\|_{L^1(0,1)}.$$

Moreover, since $\lim_{\delta\to 0} M\delta\|\frac{1}{a}\|_{L^1(0,1)} = 0$, there exists $\eta(\epsilon) > 0$ such that if $\delta < \eta(\epsilon)$, then $M\delta \left\|\frac{1}{a}\right\|_{L^1(0,1)} < \epsilon$. Thus, taking $\delta < \min\{g(\epsilon), \eta(\epsilon)\}$, (5.4) and (5.5) are verified and the thesis follows (see, e.g., [5, Chapter IV]). $\square$

Using Theorem 5.6 one can prove the next theorem.

Theorem 5.7. *Assume that the function a satisfies Hypothesis 4.9. Then, the space $H^2_{1/a}(0,1)$ is compactly imbedded in $H^1_{1/a}(0,1)$.*

Proof. Take $(u_n)_n \in \overline{B}_{H^2_{1/a}(0,1)}$. Here $B_{H^2_{1/a}(0,1)}$ denotes the unit ball of $H^2_{1/a}(0,1)$. As before, we can prove that, up to subsequence, there exists $u \in H^2_{1/a}(0,1)$ such that u_n converges strongly to u in $L^2_{1/a}(0,1)$.

Moreover, one has

$$\|u_n' - u'\|_{L^2(0,1)} \to 0, \quad \text{as } n \to +\infty.$$

Indeed, using the Hölder inequality, one has

$$\|(u_n-u)'\|^2_{L^2(0,1)} = \int_0^1 (u_n-u)'(u_n-u)'dx$$

$$= -\int_0^1 (u_n - u)''(u_n - u)dx = -\int_0^1 \sqrt{a}(u_n - u)'' \frac{u_n - u}{\sqrt{a}} dx$$
$$\le \|a(u_n - u)''\|_{L^2(0,1)} \|u_n - u\|_{L^2_{1/a}(0,1)} \to 0,$$

as $n \to +\infty$. Hence the sequence u_n converges strongly to u in $H^1_{1/a}(0,1)$. $\square$

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