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PERIODIC SOLUTIONS FOR P-LAPLACIAN NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DEVIATING ARGUMENTS

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ABSTRACT. By means of Mawhin's continuation theorem, we prove the existence of periodic solutions for a p-Laplacian neutral functional differential equation with multiple deviating arguments

$$\begin{aligned}&(\varphi_p(x'(t)-c(t)x'(t-r)))'\\&=f(x(t))x'(t)+g(t,x(t),x(t-\tau_1(t)),\dots,x(t-\tau_m(t)))+e(t).\end{aligned}$$

1. INTRODUCTION

In recent years, periodic solutions involving the scalar p-Laplacian have been studied extensively by many researchers. Lu and Ge [4] discussed sufficient conditions for the existence of periodic solutions to second order differential equation, with a deviating argument,

$$x''(t)=f(t,x(t),x(t-\tau(t)),x'(t))+e(t).$$

Recently, Pan [5] studied the equation

$$x^{(n)}(t)=\sum_{i=1}^{n-1}b_ix^{(i)}(t)+f(t,x(t),x(t-\tau_1(t)),\dots,x(t-\tau_m(t)))+p(t).$$

Feng, Lixiang and Shiping [2] investigated the existence of periodic solutions for a p-Laplacian neutral functional differential equation

$$(\varphi_p(x'(t)-c(t)x'(t-r)))'=f(x(t))x'(t)+\beta(t)g(x(t-\tau(t)))+e(t),$$

where $c(t)$ and $\beta(t)$ are allowed to change signs.

The purpose of this article is to study the existence of periodic solution for p-Laplacian neutral functional differential equation

$$\begin{aligned}&(\varphi_p(x'(t)-c(t)x'(t-r)))'\\&=f(x(t))x'(t)+g(t,x(t),x(t-\tau_1(t)),\dots,x(t-\tau_m(t)))+e(t).\end{aligned}\tag{1.1}$$

Where $p>1$ is a fixed real number. The conjugate exponent of p is denoted by q; i.e., $\frac{1}{p}+\frac{1}{q}=1$. Let $\varphi_p:\mathbb{R}\to\mathbb{R}$ be the mapping defined by $\varphi_p(s)=|s|^{p-2}s$

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for $s \neq 0$, and $\varphi_p(0) = 0$, $f, e, c \in C(\mathbb{R}, \mathbb{R})$ are continuous T-periodic functions defined on $\mathbb{R}$ and $T > 0$, $r \in \mathbb{R}$ is a constant with $r > 0$, $g \in C(\mathbb{R}^{m+2}, \mathbb{R})$ and $g(t+T, u_0, u_1, \dots, u_m) = g(t, u_0, u_1, \dots, u_m)$, for all $(t, u_0, u_1, \dots, u_m) \in \mathbb{R}^{m+2}$, $\tau_i \in C(\mathbb{R}, \mathbb{R})(i = 1, 2, \dots, m)$ with $\tau_i(t+T) = \tau_i(t)$.

2. Preliminaries

For convenience, define $\mathcal{C}_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\}$ with the norm $|x|_\infty = \max |x(t)|_{t\in[0,T]}$. Clearly $\mathcal{C}_T$ is a Banach space. We also define a linear operator

$$A : \mathcal{C}_T \to \mathcal{C}_T, \quad (Ax)(t) = x(t) - c(t)x(t-r), \tag{2.1}$$

and constant $C_p = \begin{cases} 1 & \text{if } 1 < p \leq 2, \\ 2^{p-2} & \text{if } p > 2. \end{cases}$

To simplify the studying of the existence of periodic solution for (1.1) we cite the following lemmas.

Lemma 2.1 ([2]). *Let $p \in]1, +\infty[$ be a constant, $s \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ such that $s(t+T) \equiv s(t)$, for all $t \in [0, T]$. Then for for all $u \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ with $u(t+T) \equiv u(t)$, we have*

$$\int_0^T |u(t) - u(t - s(t))|^p dt \leq 2(\max_{t\in[0,T]} |s(t)|)^p \int_0^T |u'(t)|^p dt.$$

Lemma 2.2 ([2]). *Let $B : \mathcal{C}_T \to \mathcal{C}_T$, $(Bx)(t) = c(t)x(t-r)$. Then B satisfies the following conditions*

(1) $\|B\| \leq |c|_\infty$.
(2) $(\int_0^T |[B^j x](t)|^p dt)^{1/p} \leq |c|_\infty^j (\int_0^T |x(t)|^p dt)^{1/p}, \quad \forall x \in \mathcal{C}_T, p > 1,\ j \geq 1.$

Lemma 2.3 ([2]). *If $|c|_\infty < 1$, then A, defined by (2.1), has continuous bounded inverse A^{-1} with the following properties:*

(1) $\|A^{-1}\| \leq 1/(1 - |c|_\infty)$,
(2) $(A^{-1}f)(t) = f(t) + \sum_{j=1}^{\infty} \prod_{i=1}^{j} c(t-(i-1)r) f(t-jr)$, *for all* $f \in \mathcal{C}_T$,
(3) $\int_0^T |(A^{-1}f)(t)|^p dt \leq (\frac{1}{1-|c|_\infty})^p \int_0^T |f(t)|^p dt$ *for all* $f \in \mathcal{C}_T$.

Now, we recall Mawhin's continuation theorem which will be used in our study. Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero. Here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \ker L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 and such that $X = \ker L \oplus X_1$ and $Y = \operatorname{Im} L \oplus Y_1$ and let $P : X \to \ker L$ and $Q : Y \to Y_1$ be natural projections. Clearly, $\ker L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_p := L|_{D(L)\cap X_1}$ is invertible. Denote the inverse of L_p by K.

Now, let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$, a map $N : \overline{\Omega} \to Y$ is said to be L-compact on $\overline{\Omega}$. If $QN(\overline{\Omega})$ is bounded and the operator $K(I-Q)N : \overline{\Omega} \to Y$ is compact.

Lemma 2.4 ([3]). *. Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set, and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If all of the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in]0, 1]$;
(2) $Nx \notin \operatorname{Im} L$ *for all* $x \in \partial\Omega \cap \ker L$; *and*

(3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \ker L$ *is an isomorphism. Then the equation* $Lx = Nx$ *has at least one solution on* $\overline{\Omega} \cap D(L)$.

To use Mawhin's continuation theorem to study the existence of T-periodic solution for (1.1), we rewrite (1.1) in the system

$$\begin{aligned} x_1'(t) &= [A^{-1}\varphi_q(x_2)](t), \\ x_2'(t) &= f(x_1(t))[A^{-1}\varphi_q(x_2)](t) \\ &\quad + g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_m(t))) + e(t). \end{aligned} \tag{2.2}$$

Where $q > 1$ is constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^T$ is a T-periodic solution to equation set (2.2), then $x_1(t)$ must be a T-periodic solution to equation (1.1). Thus, to prove that (1.1) has a T-periodic solution, it suffices to show that equation set (2.2) has a T-periodic solution.

Now, we set $X = Y = \{x = (x_1(t), x_2(t))^T \in C(\mathbb{R}, \mathbb{R}^2) : x_1 \in C_T, x_2 \in C_T\}$ with the norm $\|x\| = \max\{|x_1|_\infty, |x_2|_\infty\}$. Obviously, X and Y are two Banach spaces. Meanwhile, let

$$L : D(L) \subset X \to Y, \quad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}. \tag{2.3}$$

and $N : X \to Y$ be defined by

$$[Nx](t) = \begin{pmatrix} [A^{-1}\varphi_q(x_2)](t) \\ f(x_1(t))[A^{-1}\varphi_q(x_2)](t) + g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_m(t))) + e(t) \end{pmatrix}. \tag{2.4}$$

It is easy to see that (2.2) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of L, we see that $\ker L = \mathbb{R}^2$, $\operatorname{Im} L = \{y : y \in Y, \int_0^T y(s)ds = 0\}$. So L is a Fredholm operator with index zero.

Let projections $P : X \to \ker L$ and $Q : Y \to \operatorname{Im} Q$ be defined by

$$Px = \frac{1}{T}\int_0^T x(s)ds, \quad Qy = \frac{1}{T}\int_0^T y(s)ds,$$

and let K represent the inverse of $L|_{\ker P \cap D(L)}$. Clearly, $\ker L = \operatorname{Im} Q = \mathbb{R}^2$ and

$$[Ky](t) = \int_0^T G(t,s)y(s)ds, \tag{2.5}$$

where

$$G(t,s) = \begin{cases} \frac{s}{T}, & \text{if } 0 \le s < t \le T \\ \frac{s-T}{T}, & \text{if } 0 \le t \le s \le T. \end{cases}$$

From (2.4) and (2.5), it is not hard to find that N is L-compact on $\overline{\Omega}$, where Ω is an arbitrary open bounded subset of X.

Lemma 2.5 ([6]). *If* $\omega \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ *and* $\omega(0) = \omega(T) = 0$ *then*

$$\int_0^T |\omega(t)|^p dt \le \Big(\frac{T}{\pi_p}\Big)^p \int_0^T |\omega'(t)|^p dt,$$

where

$$\pi_p = 2\int_0^{(p-1)^{1/p}} \frac{ds}{(1 - \frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\frac{\pi}{p})}.$$

Lemma 2.6 ([7]). *$\Omega \subset \mathbb{R}^n$ is an open bounded set, and symmetric with respect to $0 \in \Omega$. If $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $f(x) \neq \mu f(-x)$ for all $x \in \partial\Omega$ and all $\mu \in [0,1]$, then $\deg(f, \Omega, 0)$ is an odd number.*

Lemma 2.7 ([2]). *If $c(t) \in C_T$ is not a constant function, $|c|_\infty < 1/2$,*

$$(Ax)(t) = x(t) - c(t)x(t-r) \equiv d_1, \tag{2.6}$$

where d_1 is a nonzero constant, and $x(t) \in C_T$, then

(1) *$x(t) = A^{-1}d_1$ is not a constant function,*
(2) *$\int_0^T (A^{-1}d_1)(t)dt \neq 0$.*

3. Main results

For the next theorem, we assume that the following conditions:

(H1) There is a constant $d > 0$ such that:
  (1) $g(t, u_0, u_1, \dots, u_m) > |e|_\infty$, for all $(t, u_0, u_1, \dots, u_m) \in [0,T] \times \mathbb{R}^{m+1}$ with $u_i > d$ $(i = 0, 1, \dots, m)$.
  (2) $g(t, u_0, u_1, \dots, u_m) < -|e|_\infty$, for all $(t, u_0, u_1, \dots, u_m) \in [0,T] \times \mathbb{R}^{m+1}$ with $u_i < -d$ $(i = 0, 1, \dots, m)$.

(H2) The function g has the decomposition
$$g(t, u_0, u_1, \dots, u_m) = h_1(t, u_0) + h_2(t, u_0, \dots, u_m),$$
such that $u_0 h_1(t, u_0) \geq l|u_0|^n$, $|h_2(t, u_0, \dots, u_m)| \leq \sum_{i=0}^m \alpha_i |u_i|^{p-1} + \beta$, where $n, l, \alpha_i (i = 0, \dots, m), \beta$ are non-negative constants with $n \geq p$.

Theorem 3.1. *Assume* (H1)–(H2). *Then,* (1.1) *has at least one T-periodic solution, if $|c|_\infty < 1/2$ and if*

$$\Big(\frac{1}{1 - |c|_\infty}\Big)^p \Big[|c|_\infty (1 + |c|_\infty)^{p-1} + 2^{p-1}\delta\Big(\frac{T}{\pi_p}\Big)^p + C_p 2^{1/q}\frac{T}{\pi_p}\sum_{i=1}^m \alpha_i |\tau_i|_\infty^{p-1}\Big] < 1,$$

where $\delta = \max(C_p \sum_{i=1}^m \alpha_i - \alpha_0 - l, 0)$.

Proof. Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in]0,1]\}$ if $x(\cdot) = (x_1(\cdot), x_2(\cdot))^T \in \Omega_1$, then from (2.3) and (2.4), we have

$$\begin{aligned} x_1'(t) &= \lambda[A^{-1}\varphi_q(x_2)](t), \\ x_2'(t) &= \lambda f(x_1(t))[A^{-1}\varphi_q(x_2)](t) \\ &\quad + \lambda g(t, x_1(t), x_1(t - \tau_1(t)), \dots, x_1(t - \tau_m(t))) + \lambda e(t). \end{aligned} \tag{3.1}$$

From the first equation in (3.1), we have $x_2(t) = \varphi_p(\frac{1}{\lambda}(Ax_1')(t)$, together with the second formula of (3.1), which yields

$$\begin{aligned} [\varphi_p((Ax_1')(t))]' &= \lambda^{p-1} f(x_1(t))x_1'(t) \\ &\quad + \lambda^p g(t, x_1(t), x_1(t - \tau_1(t)), \dots, x_1(t - \tau_m(t))) + \lambda^p e(t). \end{aligned} \tag{3.2}$$

Integrating both sides of (3.2) on the interval $[0,T]$ and applying integral mean value theorem, then there exists a constant $t_0 \in [0,T]$ such that

$$g(t, x_1(t_0), x_1(t_0 - \tau_1(t_0)), \dots, x_1(t_0 - \tau_m(t_0))) = -\frac{1}{T}\int_0^T e(t)dt. \tag{3.3}$$

We can prove that there is $t_1 \in [0,T]$ such that $|x_1(t_1)| \leq d$.

If $|x_1(t_0)| \leq d$, then taking $t_1 = t_0$ such that $|x_1(t_1)| \leq d$.

If $|x_1(t_0)| > d$. It follows from (H1) that there is some $i \in \{1, 2, \dots, m\}$ such that $|x_1(t_0 - \tau_i(t_0))| \le d$. Since $x_1(t)$ is continuous for $t \in \mathbb{R}$ and $x_1(t+T) = x_1(t)$, so there must be an integer k and a point $t_1 \in [0, T]$ such that $t_0 - \tau_i(t_0) = kT + t_1$. So $|x_1(t_1)| = |x_1(t_0 - \tau_i(t_0))| \le d$. Then, we have

$$|x_1(t)| = |x_1(t_1) + \int_{t_1}^{t} x_1'(s)ds| \le d + \int_{t_1}^{t} |x_1'(s)|ds, \quad t \in [t_1, t_1 + T],$$

and

$$|x_1(t)| = |x_1(t-T)| = |x(t_1) - \int_{t-T}^{t_1} x_1'(s)ds| \le d + \int_{t_1-T}^{t_1} |x_1'(s)|ds, \quad t \in [t_1, t_1 + T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x_1|_\infty &= \max_{t\in[0,T]} |x_1(t)| = \max_{t\in[t_1,t_1+T]} |x_1(t)| \\ &\le \max_{t\in[t_1,t_1+T]} \Big\{ d + \frac{1}{2}\Big(\int_{t_1}^{t} |x_1'(s)|ds + \int_{t-T}^{t_1} |x_1'(s)|ds \Big) \Big\} \\ &\le d + \frac{1}{2} \int_0^T |x_1'(s)|ds. \end{aligned} \tag{3.4}$$

On the hand, multiplying both sides of (3.2) by $x_1(t)$ and integrating it from 0 to T, we obtain

$$\begin{aligned} &\int_0^T [\varphi_p((Ax_1')(t))]' x_1(t)dt \\ &= \lambda^{p-1} \int_0^T f(x_1(t))x_1'(t)x_1(t)dt \\ &\quad + \lambda^p \int_0^T g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_m(t)))x_1(t)dt \\ &\quad + \lambda^p \int_0^T e(t)x_1(t)dt. \end{aligned} \tag{3.5}$$

On the other hand we have

$$\begin{aligned} &\int_0^T [\varphi_p((Ax_1')(t))]' x_1(t)dt \\ &= -\int_0^T \varphi_p((Ax_1')(t))x_1'(t)dt \\ &= -\int_0^T \varphi_p((Ax_1')(t))[x_1'(t) - c(t)x_1'(t-r) + c(t)x_1'(t-r)]dt \\ &= -\int_0^T |(Ax_1')(t)|^p dt - \int_0^T c(t)x_1'(t-r)\varphi_p((Ax_1')(t))dt. \end{aligned} \tag{3.6}$$

Meanwhile,

$$\int_0^T f(x_1(t))x_1'(t)x_1(t)dt = 0. \tag{3.7}$$

Substituting (3.6)-(3.7) into (3.5) we obtain

$$\begin{aligned}
&\int_0^T |(Ax_1')(t)|^p dt \\
&= -\int_0^T c(t)x_1'(t-r)\varphi_p((Ax_1')(t))dt \\
&\quad - \lambda^p \int_0^T g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_m(t)))x_1(t)dt \\
&\quad - \lambda^p \int_0^T e(t)x_1(t)dt.
\end{aligned} \tag{3.8}$$

In view of (H2), we obtain

$$\begin{aligned}
&\int_0^T |(Ax_1')(t)|^p dt \\
&= -\int_0^T c(t)x_1'(t-r)\varphi_p((Ax_1')(t))dt \\
&\quad - \lambda^p \int_0^T g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_m(t)))x_1(t)dt - \lambda^p \int_0^T e(t)x_1(t)dt \\
&= -\lambda^p \int_0^T h_1(t, x_1)x_1(t)dt \\
&\quad - \lambda^p \int_0^T h_2(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_m(t)))x_1(t)dt \\
&\quad - \lambda^p \int_0^T e(t)x_1(t)dt.
\end{aligned} \tag{3.9}$$

Define $\Delta_1 = \{t \in [0, T] : |x_1(t)| \le 1\}$, $\Delta_2 = \{t \in [0, T] : |x_1(t)| > 1\}$, in view of (H2) again we have

$$\begin{aligned}
-\lambda^p \int_0^T h_1(t, x_1)x_1(t)dt &\le -\lambda^p l \int_0^T |x_1(t)|^n dt \\
&= -\lambda^p l \Big(\int_{\Delta_1} + \int_{\Delta_2}\Big)|x_1(t)|^n dt \\
&\le -\lambda^p l \int_{\Delta_2} |x_1(t)|^n dt \\
&\le -\lambda^p l \int_{\Delta_2} |x_1(t)|^p dt \\
&= -\lambda^p l \int_0^T |x_1(t)|^p dt + \lambda^p l \int_{\Delta_1} |x_1(t)|^p dt \\
&\le -\lambda^p l \int_0^T |x_1(t)|^p dt + lT.
\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9),

$$
\begin{aligned}
&\int_0^T |(Ax_1')(t)|^p dt\\
&\le |c|_\infty \int_0^T |\varphi_p((Ax_1')(t))||x_1'(t-r)|dt - \lambda^p l \int_0^T |x_1(t)|^p dt\\
&\quad + \lambda^p \int_0^T |h_2(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_m(t)))||x_1(t)|dt\\
&\quad + \lambda^p |e|_\infty \int_0^T |x_1(t)|dt + lT.
\end{aligned}
\tag{3.11}
$$

Moreover, by using Hölder's inequality and Minkowski inequality, we obtain

$$
\begin{aligned}
&\int_0^T |\varphi_p((Ax_1')(t))||x_1'(t-r)|dt\\
&\le \Big(\int_0^T |\varphi_p((Ax_1')(t))|^q dt\Big)^{1/q} \Big(\int_0^T |x_1'(t-r)|^p dt\Big)^{1/p}\\
&= \Big(\int_0^T |(Ax_1')(t)|^p dt\Big)^{1/q} \Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/p}\\
&= \Big[\Big(\int_0^T |x_1'(t) - c(t)x_1'(t-r)|^p dt\Big)^{1/p}\Big]^{p/q} \Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/p}\\
&\le \Big[\Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/p} + \Big(\int_0^T |c(t)x_1'(t-r)|^p dt\Big)^{1/p}\Big]^{p/q} \Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/p}\\
&\le \Big[\Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/p} + |c|_\infty \Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/p}\Big]^{p/q} \Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/p}\\
&= (1+|c|_\infty)^{p-1} \int_0^T |x_1'(t)|^p dt.
\end{aligned}
\tag{3.12}
$$

By (3.11) and (3.12) and combining with (H2) and Lemma 2.1, we obtain

$$
\begin{aligned}
&\int_0^T |(Ax_1')(t)|^p dt\\
&\le |c|_\infty (1+|c|_\infty)^{p-1} \int_0^T |x_1'(t)|^p dt - \lambda^p l \int_0^T |x_1(t)|^p dt\\
&\quad + \lambda^p \int_0^T |h_2(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_m(t)))||x_1(t)|dt\\
&\quad + \lambda^p |e|_\infty \int_0^T |x_1(t)|dt + lT\\
&\le |c|_\infty (1+|c|_\infty)^{p-1} \int_0^T |x_1'(t)|^p dt + \lambda^p(\alpha_0 - l) \int_0^T |x_1(t)|^p dt\\
&\quad + \lambda^p \int_0^T \sum_{i=1}^m \alpha_i |x_1(t-\tau_i(t))|^{p-1}|x_1(t)|dt + \lambda^p(\beta + |e|_\infty) \int_0^T |x_1(t)|dt + lT\\
&\le |c|_\infty (1+|c|_\infty)^{p-1} \int_0^T |x_1'(t)|^p dt + \lambda^p(\alpha_0 - l) \int_0^T |x_1(t)|^p dt
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda^p C_p\sum_{i=1}^m\alpha_i\int_0^T|x_1(t-\tau_i(t))-x_1(t)|^{p-1}|x_1(t)|dt\\
&+\lambda^p C_p\sum_{i=1}^m\alpha_i\int_0^T|x_1(t)|^p dt+\lambda^p(\beta+|e|_\infty)\int_0^T|x_1(t)|dt+lT\\
&\le|c|_\infty(1+|c|_\infty)^{p-1}\int_0^T|x_1'(t)|^p dt+\lambda^p(C_p\sum_{i=1}^m\alpha_i+\alpha_0-l)\int_0^T|x_1(t)|^p dt\\
&+\lambda^p C_p\sum_{i=1}^m\alpha_i\Big(\int_0^T|x_1(t-\tau_i(t))-x_1(t)|^p dt\Big)^{1/q}\Big(\int_0^T|x_1(t)|^p dt\Big)^{1/p}\\
&+\lambda^p(\beta+|e|_\infty)T^{1/q}\Big(\int_0^T|x_1(t)|^p dt\Big)^{1/p}+lT\\
&\le|c|_\infty(1+|c|_\infty)^{p-1}\int_0^T|x_1'(t)|^p dt+\delta\int_0^T|x_1(t)|^p dt\\
&+C_p\sum_{i=1}^m\alpha_i 2^{1/q}|\tau_i|_\infty^{p-1}\Big(\int_0^T|x_1'(t)|^p dt\Big)^{1/q}\Big(\int_0^T|x_1(t)|^p dt\Big)^{1/p}\\
&+(\beta+|e|_\infty)T^{1/q}\Big(\int_0^T|x_1(t)|^p dt\Big)^{1/p}+lT.
\end{aligned}\tag{3.13}
$$

Let $\omega(t)=x_1(t+t_1)-x_1(t_1)$, then $\omega(T)=\omega(0)=0$ and from Lemma 2.1 we see that

$$
\int_0^T|\omega(t)|^p dt\le\Big(\frac{T}{\pi_p}\Big)^p\int_0^T|\omega'(t)|^p dt=\Big(\frac{T}{\pi_p}\Big)^p\int_0^T|x_1'(t)|^p dt.\tag{3.14}
$$

By (3.14) and the Minkowski inequality, we obtain

$$
\begin{aligned}
\Big(\int_0^T|x_1(t)|^p dt\Big)^{1/p}&=\Big(\int_0^T|\omega(t)+x_1(t_1)|^p dt\Big)^{1/p}\\
&\le\Big(\int_0^T|\omega(t)|^p dt\Big)^{1/p}+\Big(\int_0^T|x_1(t_1)|^p dt\Big)^{1/p}\\
&\le\frac{T}{\pi_p}\Big(\int_0^T|x_1'(t)|^p dt\Big)^{1/p}+dT^{1/p}.
\end{aligned}\tag{3.15}
$$

Substituting (3.15) into (3.13) yields

$$
\begin{aligned}
&\int_0^T|(Ax_1')(t)|^p dt\\
&\le|c|_\infty(1+|c|_\infty)^{p-1}\int_0^T|x_1'(t)|^p dt+\delta\Big[\frac{T}{\pi_p}\Big(\int_0^T|x_1'(t)|^p dt\Big)^{1/p}+dT^{1/p}\Big]^p\\
&+C_p 2^{1/q}\sum_{i=1}^m\alpha_i|\tau_i|_\infty^{p-1}\Big[\frac{T}{\pi_p}\Big(\int_0^T|x_1'(t)|^p dt\Big)^{1/p}+dT^{1/p}\Big]\Big(\int_0^T|x_1'(t)|^p dt\Big)^{1/q}\\
&+(\beta+|e|_\infty)T^{1/q}\Big[\frac{T}{\pi_p}\Big(\int_0^T|x_1'(t)|^p dt\Big)^{1/p}+dT^{1/p}\Big]+lT\\
&\le\Big[|c|_\infty(1+|c|_\infty)^{p-1}+2^{p-1}\delta\big(\frac{T}{\pi_p}\big)^p+C_p 2^{1/q}\frac{T}{\pi_p}\sum_{i=1}^m\alpha_i|\tau_i|_\infty^{p-1}\Big]\int_0^T|x_1'(t)|^p dt
\end{aligned}
$$

$$
\begin{aligned}
&+ C_p 2^{1/q}\sum_{i=1}^{m}\alpha_i|\tau_i|_\infty^{p-1} dT^{1/p}\Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/q}\\
&+ (\beta+|e|_\infty)T^{1/q}\frac{T}{\pi_p}\Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/p}\\
&+ 2^{p-1}\delta d^p T + (\beta+|e|_\infty)dT + lT.
\end{aligned} \tag{3.16}
$$

By applying the third part of Lemma 2.2, we obtain

$$
\int_0^T |x_1'(t)|^p dt = \int_0^T |(A^{-1}Ax_1')(t)|^p dt \le \Big(\frac{1}{1-|c|_\infty}\Big)^p \int_0^T |(Ax_1')(t)|^p dt. \tag{3.17}
$$

Then, substituting (3.17) into (3.16), we have

$$
\begin{aligned}
\int_0^T |x_1'(t)|^p dt \le \Big(\frac{1}{1-|c|_\infty}\Big)^p \Big[&|c|_\infty(1+|c|_\infty)^{p-1} + 2^{p-1}\delta\big(\frac{T}{\pi_p}\big)^p\\
&+ C_p 2^{1/q}\frac{T}{\pi_p}\sum_{i=1}^{m}\alpha_i|\tau_i|_\infty^{p-1}\Big]\int_0^T |x_1'(t)|^p dt\\
&+ \frac{C_p 2^{1/q}\sum_{i=1}^{m}\alpha_i|\tau_i|_\infty^{p-1} dT^{1/p}}{(1-|c|_\infty)^p}\Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/q}\\
&+ \frac{(\beta+|e|_\infty)T^{1/q}\frac{T}{\pi_p}}{(1-|c|_\infty)^p}\Big(\int_0^T |x_1'(t)|^p dt\Big)^{1/p} + \frac{2^{p-1}\delta d^p T}{(1-|c|_\infty)^p}\\
&+ \frac{(\beta+|e|_\infty)dT}{(1-|c|_\infty)^p} + \frac{lT}{(1-|c|_\infty)^p}.
\end{aligned} \tag{3.18}
$$

As

$$
\big(\frac{1}{1-|c|_\infty}\big)^p\Big[|c|_\infty(1+|c|_\infty)^{p-1} + 2^{p-1}\delta\big(\frac{T}{\pi_p}\big)^p + C_p 2^{1/q}\frac{T}{\pi_p}\sum_{i=1}^{m}\alpha_i|\tau_i|_\infty^{p-1}\Big] < 1,
$$

$1/p<1$, $1/q<1$, then from (3.18), there exists a constant $M>0$ such that

$$
\int_0^T |x_1'(t)|^p dt \le M. \tag{3.19}
$$

Which together with (3.4) gives

$$
|x_1|_\infty \le d + \frac{1}{2}T^{1/q}M^{1/p} =: M_1. \tag{3.20}
$$

Again, from the first equation in (3.1), we have

$$
\int_0^T (A^{-1}\varphi_q(x_2))(t)dt = 0.
$$

Then there is a constant $\eta\in[0,T]$, such that $(A^{-1}\varphi_q(x_2))(\eta)=0$, which together with the second part of lemma 2.3 gives

$$
(A^{-1}\varphi_q(x_2))(\eta) = \varphi_q(x_2(\eta)) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(\eta-(i-1)r)\varphi_q(x_2(\eta-jr)) = 0,
$$

$$
\begin{aligned}
|x_2(\eta)|^{q-1} &= |\varphi_q(x_2(\eta))| \\
&= \Big|\sum_{j=1}^{\infty}\prod_{i=1}^{j} c(\eta-(i-1)r)\varphi_q(x_2(\eta-jr))\Big| \\
&\le \sum_{j=1}^{\infty}|c|_\infty^j |x_2|_\infty^{q-1} = \frac{|c|_\infty}{1-|c|_\infty}|x_2|_\infty^{q-1}\,.
\end{aligned}
$$

It follows that

$$
|x_2(\eta)| \le \Big(\frac{|c|_\infty}{1-|c|_\infty}\Big)^{1/(q-1)}|x_2|_\infty. \tag{3.21}
$$

Let $M_f = \max_{|u|\le M_1}|f(u)|$, $M_g = \max_{t\in[0,T],|u_0|\le M_1,\dots,|u_m|\le M_1}|g(t,u_0,\dots,u_m)|$ and from (3.1), we have

$$
x_2'(t) = \lambda f(x_1(t))x_1'(t) + \lambda g(t,x_1(t),x_1(t-\tau_1(t)),\dots,x_1(t-\tau_m(t))) + \lambda e(t),
$$

and

$$
\begin{aligned}
&\int_0^T |x_2'(t)|dt \\
&\le \int_0^T |f(x_1(t))x_1'(t)|dt + \int_0^T |g(t,x_1(t),x_1(t-\tau_1(t)),\dots,x_1(t-\tau_m(t)))|dt \\
&\quad + \int_0^T |e(t)| \\
&\le M_f\int_0^T |x_1'(t)|dt + T(M_g+|e|_\infty) \\
&\le M_f T^{1/q}M^{1/p} + T(M_g+|e|_\infty) =: M_2.
\end{aligned} \tag{3.22}
$$

By (3.21) and (3.22)

$$
\begin{aligned}
|x_2(t)| = |x_2(\eta) + \int_\eta^t x_2'(s)ds| &\le \Big(\frac{|c|_\infty}{1-|c|_\infty}\Big)^{1/(q-1)}|x_2|_\infty + \int_0^T |x_2'(s)|ds \\
&\le \Big(\frac{|c|_\infty}{1-|c|_\infty}\Big)^{1/(q-1)}|x_2|_\infty + M_2, \quad t\in[0,T]\,.
\end{aligned} \tag{3.23}
$$

Since $|c|_\infty < \frac{1}{2}$, $(\frac{|c|_\infty}{1-|c|_\infty})^{1/(q-1)} < 1$, and (3.23), it follows that there exists a positive constant M_3 such that

$$
|x_2|_\infty \le M_3. \tag{3.24}
$$

Let $\Omega_2 = \{x\in\ker L, QNx = 0\}$. If $x\in\Omega_2$ then $x\in\mathbb{R}^2$ is a constant vector, and

$$
\begin{gathered}
\frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt = 0, \\
\begin{aligned}
&\frac{1}{T}\int_0^T [f(x_1(t))[A^{-1}\varphi_q(x_2)](t) \\
&+ g(t,x_1(t),x_1(t-\tau_1(t)),\dots,x_1(t-\tau_m(t))) + e(t)]dt = 0.
\end{aligned}
\end{gathered} \tag{3.25}
$$

By the first formula in (3.25) and the second part of Lemma 2.7, we have $x_2 = 0$. Which together with the second equation in (3.25) yields

$$\frac{1}{T}\int_0^T [g(t, x_1, x_1, \dots, x_1) + e(t)]dt = 0.$$

In view of (H1), we see that $|x_1| \le d$. Now, we let $\Omega = \{x | x = (x_1, x_2)^T \in X, |x_1| < M_1 + d, |x_2| < M_3 + d\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$. So from (3.20) and (3.24), it is easy to see that conditions (1) and (2) in Lemma 2.4 are satisfied.

Next, we verify the condition (3) in Lemma 2.4. To do this, we define the isomorphism $J : \operatorname{Im} Q \to \ker L$, $J(x_1, x_2)^T = (x_1, x_2)^T$. Then

$$JQN(x) = \begin{pmatrix} \frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt \\ \frac{1}{T}\int_0^T [f(x_1(t))[A^{-1}\varphi_q(x_2)](t) + g(t, x_1, x_1, \dots, x_1) + e(t)]dt \end{pmatrix},$$

$x \in \overline{\ker L \cap \Omega}$. By Lemma 2.6, we need to prove that

$$JQN(x) \ne \mu(JQN(-x)), \quad \forall x \in \partial(\Omega \cap \ker L), \quad \mu \in [0, 1]$$

Case1. If $x = (x_1, x_2)^T \in \partial(\Omega \cap \ker L) \setminus \{(M_1 + d, 0)^T, (-M_1 - d, 0)^T\}$, then $x_2 \ne 0$ which, together with the second part of Lemma 2.7, gives us $\int_0^T [A^{-1}\varphi_q(x_2)](t)dt \ne 0$,

$$\Big(\frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt\Big)\Big(\frac{1}{T}\int_0^T [A^{-1}\varphi_q(-x_2)](t)dt\Big) < 0,$$

obviously, for all $\mu \in [0, 1]$, $JQN(x) \ne \mu(JQN(-x))$.

Case2. If $x = (M_1 + d, 0)^T$ or $x = (-M_1 - d, 0)^T$ then

$$JQN(x) = \begin{pmatrix} 0 \\ \frac{1}{T}\int_0^T [g(t, x_1, x_1, \dots, x_1) + e(t)]dt \end{pmatrix}$$

which, together with (H1), yields $JQN(x) \ne \mu(JQN(-x))$ for all $\mu \in [0, 1]$. Thus, condition (3) of Lemma 2.4 is also satisfied. Therefore, by applying Lemma 2.4, we conclude that the equation $Lx = Nx$ has at least one T-periodic solution on $\overline{\Omega}$, so (1.1) has at least one T-periodic solution. This completes the proof. □

4. Example

In this section, we provide an example to illustrate Theorem 3.1. Let us consider the equation

$$\begin{aligned} &(\varphi_p(x'(t) - 0.1\sin(20\pi t)x'(t - r)))' \\ &= f(x(t))x'(t) + g(t, x(t), x(t - \frac{\cos 20\pi t}{90}), x(t - \frac{\sin 20\pi t}{100})) + \cos(20\pi t), \end{aligned} \tag{4.1}$$

where $p = 3$, $T = 1/10$, $c(t) = 0.1\sin(20\pi t)$, $\tau_1(t) = \cos(20\pi t)/90$, $\tau_2(t) = \sin(20\pi t)/100$,

$$g(t, u, v, w) = u^3\big(2 + \sin(20\pi t)\big) + \frac{3}{225}\big(\operatorname{sgn}(v)v^2 + \operatorname{sgn}(w)w^2\big)|\cos(20\pi t)|,$$

$e(t) = \cos(20\pi t)$. Therefore we can choose $l = 1$, $d = 1$, $\alpha_0 = 0$, $\alpha_1 = \alpha_2 = 0,014$. We can easily check condition (H1), (H2) in Theorem 3.1 hold. We can check that

$$\big(\frac{1}{1 - |c|_\infty}\big)^p \Big[|c|_\infty(1 + |c|_\infty)^{p-1} + 2^{p-1}\delta\big(\frac{T}{\pi_p}\big)^p + C_p 2^{1/q}\sum_{i=1}^m \alpha_i\big(\frac{T}{\pi_p}\big)|\tau_i|_\infty^{p-1}\Big] < 1.$$

By Theorem 3.1, equation (4.1) has at least one $\frac{1}{10}$-periodic solution.

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