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MINIMIZERS OF A VARIATIONAL PROBLEM INHERIT THE SYMMETRY OF THE DOMAIN

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Abstract. We give a general framework under which the minimizers of a variational problem inherit the symmetry of the ambient space. The main technique used is the moving plane (or sliding) method.

1. Introduction

It is a hot topic in PDE to determine wether or not a solution possess some kind of symmetry. Besides the celebrated result of [2], much effort has been put in addressing this problem in several situations, and many fundamental questions are still open (see, among the others, for instance [4, 5] and references therein).

Indeed, the so-called moving plane (or sliding) method has been widely used to prove radial symmetry for positive solutions of elliptic equations. The classical references are the papers [2, 3], where the authors proved that positive solutions of the semilinear equation $-\Delta u(x) + f(u(x)) = 0$ on a ball are radially symmetric provided that $u = 0$ on the boundary of the ball. The same conclusion holds in $x \in \mathbb{R}^N$ if one assumes that u decays to zero at infinity. They were also able to treat the equation with radially dependent nonlinearity $-\Delta u(x) + f(|x|, u(x)) = 0$ provided $\partial f(r, u)/\partial r < 0$ for every $r > 0$.

Later, in [7] and [8] radial symmetry or partial symmetry for global minimizers of functionals was considered, and the advantage of the reflection method considered there relies in its simplicity and generality. Indeed, there are at least three cases not covered by [2, 3] which have been taken into account in [7] and [8], which also includes more general boundary conditions, no requirement is taken on the sign of the minimizer, and domains like the annulus may be treated as well.

The technique of reflecting the minimizers has been also used in [9], where the use of the unique continuation principle of [7, 8] was replaced by suitable regularity assumptions.

Another approach to prove symmetry lies in the technique of symmetrization, which, for example, may be used to show the symmetry of minimizers assuming that the minimizer is positive and that the nonlinearity is monotone with respect to r (see [10]) The foliated Schwartz symmetrization can be used to prove the axial symmetry of the minimizers without any assumption on the sign of u and

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of $\partial f(r,u)/\partial r$ (see [10, 11]). We also refer to [1] for further insight on symmetry problems.

In this note, which is very elementary in spirit, we show that minimizers of variational problems inherit the natural symmetries induced by the domain and by the equation. For this, the use of the maximum principle or of the moving plane method is not necessary and so things are much easier, and much more general, than in the case of nonminimal solutions. Indeed, we will revisit the approach of [7, 8] to obtain symmetry in a more general setting. Our motivation also comes from the paper [6] where the authors use a generalization of [2] to obtain symmetry for positive solutions of $-\Delta u + f(u) = 0$ with Dirichlet boundary condition on a domain that could be, for instance, a David star, a square, a stellated cube or a Kepler's stella octangula. The general feature in common to these domains is the so-called Steiner-symmetry, which is one of the essential ingredients of [6]. Here we are able to treat non-Steiner-symmetric domains like the five star pentagon and the Kepler-Poinsot polyhedron (see, e.g., Example 2.8). The reader is referred to [6] for many pictures of such domains.

The method we use is somewhat classical, and the basic idea is already sketched on [4, p. 19], of which we repeat here the very clear exposition:

> Suppose u minimizes a strictly convex functional $\mathcal{I}(v)$ on a convex set of admissible functions v. Moreover v is defined on a symmetric set Ω; i.e., Ω is invariant under some group action. If g is an element of the group, $g(\Omega) = \Omega$ and consequently $u(x) = u(g(x))$; i.e., u is invariant under the group action; otherwise the convex combination $w(x) = [u(x)+u(g(x))]/2$ would have smaller "energy" $\mathcal{I}(w) < \mathcal{I}(u)$, a contradiction.

2. Statements of results

Now, we introduce formally our framework. Given n, m, $\ell \in \mathbb{N}$, we define

$$\mathbb{R}_\ell := \mathbb{R}^{nm\ell} \times \cdots \times \mathbb{R}^{nm} \times \mathbb{R}^m = \big[\prod_{j=1}^{\ell} \mathbb{R}^{nmj}\big] \times \mathbb{R}^m.$$

For an open set $\Omega \subseteq \mathbb{R}^n$, we consider a measurable function $\psi : \mathbb{R}_\ell \times \Omega \to \mathbb{R}$.

Let $\mathcal{W}(\Omega)$ be the set of functions from $\Omega \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ that are ℓ-times differentiable a.e. in Ω. For any $u \in \mathcal{W}(\Omega)$ and any $x \in \Omega$ we write

$$\Psi[u](x) := \psi\big(D^\ell u(x), \dots, Du(x), u(x), x\big).$$

We consider the set of admissible functions

$$\mathcal{A}(\Omega) := \big\{u \in \mathcal{W}(\Omega) \text{ s.t. } \Psi[u] \in L^1(\Omega)\big\}$$

and we define

$$\mathcal{I}_\Omega[u] := \begin{cases} \int_\Omega \Psi[u](x)\,dx & \text{if } u \in \mathcal{A}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Given u, $v \in \mathcal{W}(\Omega)$, $t \in (0,1)$ and $x \in \Omega$, we consider the convex combination

$$[u,v]_t(x) := tu(x) + (1-t)v(x).$$

We take $\mathcal{S}(\Omega) \subseteq \mathcal{A}(\Omega)$ such that

$$\text{if } u \text{ and } v \in \mathcal{S}(\Omega) \text{ then there exists } t \in (0,1) \text{ such that } [u,v]_t \in \mathcal{S}(\Omega). \tag{2.1}$$

We suppose that Ψ is strictly convex in $\mathcal{S}(\Omega)$; i.e., for every u, $v \in \mathcal{S}(\Omega)$, if $t \in (0,1)$ is as in (2.1), we have that

$$\begin{gathered} \Psi[[u,v]_t](x) \le t\Psi[u](x) + (1-t)\Psi[v](x) \text{ for every } x \in \Omega, \\ \text{and if equality holds for a.e. } x \in \Omega \text{ then } u = v \text{ a.e. in } \Omega. \end{gathered} \tag{2.2}$$

Given a Lipschitz bijection $S : \overline{\Omega} \to \overline{\Omega}$, we say that S is a symmetry for $\mathcal{I}_\Omega$ in $\mathcal{S}(\Omega)$ if the following conditions hold:

$$\text{if } u \in \mathcal{S}(\Omega) \text{ and } u_S(x) := u(S(x)) \text{ for any } x \in \Omega, \text{ we have that } u_S \in \mathcal{S}(\Omega) \tag{2.3}$$

and

$$\begin{aligned} &\psi\big(D^\ell u_S(x), \dots, Du_S(x), u_S(x), x\big) \\ &= \psi\big(D^\ell u(S(x)), \dots, Du(S(x)), u(S(x)), S(x)\big)\,|\det DS(x)| \quad \text{for a.e. } x \in \Omega. \end{aligned} \tag{2.4}$$

In this setting, minimizers inherit the symmetry of S:

Theorem 2.1. *Let $S : \overline{\Omega} \to \overline{\Omega}$ be a symmetry for $\mathcal{I}_\Omega$ in $\mathcal{S}(\Omega)$. Assume that there exists $u^\star \in \mathcal{S}(\Omega)$ such that*

$$\mathcal{I}_\Omega[u^\star] = \inf_{u \in \mathcal{S}(\Omega)} \mathcal{I}_\Omega[u] < +\infty.$$

Then $u^\star\big(S(x)\big) = u(x)$ for a.e. $x \in \Omega$.

Proof. The proof is a simple combination of two well-known principles. The first principle is the fact that strictly convex functionals attain at most one minimum. The second one is that uniqueness implies symmetry with respect to every transformation which leaves the functional values unchanged. Here is the argument in detail. By (2.3),

$$u^\star_S \in \mathcal{S}(\Omega). \tag{2.5}$$

Also, by (2.4),

$$\begin{aligned} \mathcal{I}_\Omega[u^\star_S] &= \int_\Omega \psi\big(D^\ell u^\star_S(x), \dots, u^\star_S(x), x\big)\,dx \\ &= \int_\Omega \psi\big(D^\ell u^\star(S(x)), \dots, u^\star(S(x)), S(x)\big)\,|\det DS(x)|\,dx \\ &= \int_\Omega \psi\big(D^\ell u^\star(y), \dots, u^\star(y), y\big)\,dy \\ &= \mathcal{I}_\Omega[u^\star]. \end{aligned}$$

Therefore, by (2.1), (2.2) and (2.5), there exists $t \in (0,1)$ such that the following computation holds:

$$\begin{aligned} \mathcal{I}_\Omega[u^\star] &= \inf_{u \in \mathcal{S}(\Omega)} \mathcal{I}_\Omega[u] \\ &\le \mathcal{I}_\Omega[[u^\star, u^\star_S]_t] \\ &= \int_\Omega \Psi[[u^\star, u^\star_S]_t](x)\,dx \\ &\le \int_\Omega t\Psi[u^\star](x) + (1-t)\Psi[u^\star_S](x)\,dx \\ &= t\mathcal{I}_\Omega[u^\star] + (1-t)\mathcal{I}_\Omega[u^\star_S] \\ &= \mathcal{I}_\Omega[u^\star]. \end{aligned}$$

Hence
$$\Psi[[u^\star, u^\star_S]_t] = t\Psi[u^\star](x) + (1-t)\Psi[u^\star_S](x)$$
a.e. $x \in \Omega$, so (2.2) implies that $u^\star = u^\star_S$ a.e. in Ω, as desired. □

Remark 2.2. Of course, given the simplicity of Theorem 2.1 and of its proof, we cannot really claim any priority or originality in it, but we think it could be useful to have the result stated and understood in such a general form.

Remark 2.3. In most of the applications, the symmetry S is a rigid motion (in particular, a reflection or a rotation), so $|\det DS| = 1$. However, we thought it was somewhat useful to speak about more general type of symmetries (see also the forthcoming Example 2.6 where $|\det DS| \neq 1$).

Remark 2.4. The space $\mathcal{S}(\Omega)$ is designed to include the boundary data (see the examples below).

Remark 2.5. In the particular case $\psi := |\nabla u|^2 + G(u)$, the convexity condition (2.2) boils down to the monotonicity of the nonlinearity $G'(u)$, i.e. on a sign condition on the linear term $G''(u)$ driving the linearized equation. In this case, this assumption reduces to the classical one which makes the maximum principle hold.

Though the statement of Theorem 2.1 is quite general, it may be useful to give some very simple, not exhaustive, but concrete, applications.

Example 2.6. Let $n = m = \ell = 1$. We take the rectangle $\Omega := [-1/2, 1] \times [-1, 1]$ and we define, for any $x = (x_1, x_2) \in \mathbb{R}^2$,
$$S(x) := \begin{cases} (-2x_1, x_2) & \text{if } x_1 < 0, \\ (-x_1/2, x_2) & \text{if } x_1 \geq 0. \end{cases}$$
Let also
$$a(x) := \begin{cases} 1 & \text{if } x_1 < 0, \\ 2 & \text{if } x_1 \geq 0. \end{cases}$$
We observe that, if $x_1 \neq 0$, then
$$a(S(x)) = \frac{2}{a(x)}. \tag{2.6}$$
Given $(r, x) \in \mathbb{R}^2 \times \Omega$, we define
$$\psi(r, x) := a(x) r_1^2 + \frac{r_2^2}{a(x)}.$$
We also take $\bar{u} \in C^\infty(\mathbb{R})$ and $u_o(x_1, x_2) = \bar{u}(x_2)$. We notice that
$$u_o(S(x)) = u_o(x). \tag{2.7}$$
Thus, if we define
$$\mathcal{S}(\Omega) := W^{1,2}_{u_o}(\Omega) = \{u \in W^{1,2}(\Omega) \text{ s.t. } u - u_o \in W^{1,2}_0(\Omega)\}, \tag{2.8}$$
we have that (2.3) holds, due to (2.7). Moreover, a careful computation shows that (2.4) is satisfied, due to (2.6).

Also, (2.2) follows from the strict convexity of the maps $r_1 \mapsto r_1^2$ and $r_2 \mapsto r_2^2$; notice that if equality holds in (2.2) then $\nabla u = \nabla v$, hence $u = v + c$, for some

$c \in \mathbb{R}$. From the boundary data in (2.8), we obtain that $c = 0$, and this shows that (2.2) is satisfied. Then, Theorem 2.1 applies to this case.

We remark that, in this case, S is not a rigid motion. In fact, more general types of symmetries and domains (such as the ones with the shape of a scamorza-cheese) may be treated with the same idea; i.e., decomposing S into a reflection and two dialations on the opposite halfplane. Of course, the more complicated Ω and S are, the more complicated needs to be the function ψ in order to satisfy the invariance in (2.2).

Example 2.7. Let $\ell = 1$, $n = 2$ and $m = 1$. We take $\Omega = [-1, 1] \times [-2, 2]$ and we consider the odd reflection $S(x) := -x$. Let $G \in C^\infty(\mathbb{R})$, $p \in (1, +\infty)$ and, for every $r \in \mathbb{R}^2$ and $\tau \in \mathbb{R}$, we set

$$\psi(r, \tau, x) := \frac{|r|^p}{p} - \frac{ar_1^2}{2} + G(|x|^2)\tau$$

and $\mathcal{S}(\Omega) := W_0^{1,p}(\Omega)$. The corresponding PDE (in the weak sense) is

$$\Delta_p u - a\partial_{11} u = G(|x|^2),$$

where, as usual, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator. Then, Theorem 2.1 applies and it gives that the minimal solution is odd.

The case of an even function $G(x_1, x_2)$ may be treated as well. Notice that, in general, some conditions are needed to ensure that the functional $\mathcal{I}_\Omega$ is bounded from below (but it is not the aim of this note to discuss such conditions, since we just assume in Theorem 2.1 the existence of a minimizer).

Example 2.8. Let $\ell = 1$, $n = 2$ and $m = 1$. Given G, $H \in C^\infty(\mathbb{R})$, $r \in \mathbb{R}^2$ and $\tau \in \mathbb{R}$, we define

$$\psi(r, \tau, x) := \frac{|r|^2}{2} + G(|x|^2)H(\tau).$$

We observe that the associated PDE has the form

$$\Delta u = G(|x|^2)H'(u). \tag{2.9}$$

Let $R \in \operatorname{Mat}(n \times n)$ be the anticlockwise rotation of angle $2\pi/5$ and let Ω be a regular five-pointed star (i.e., a star pentagon) centered at the origin. In this case, the setting of Theorem 2.1 holds with $S(x) := Rx$, if H is convex and $G \geq 0$.

This gives that minimal solutions of (2.9) in the five-pointed star domain are symmetric under $2\pi/5$ rotations.

We remark that the five-pointed star domain is not Steiner-symmetric, so it is not known in general whether all the solutions endow the same symmetry (see [6]).

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