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BEHAVIOR OF THE MAXIMAL SOLUTION OF THE CAUCHY PROBLEM FOR SOME NONLINEAR PSEUDOPARABOLIC EQUATION AS $|x| \to \infty$

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Abstract. We prove a comparison principle for solutions of the Cauchy problem of the nonlinear pseudoparabolic equation $u_t = \Delta u_t + \Delta\varphi(u) + h(t,u)$ with nonnegative bounded initial data. We show stabilization of a maximal solution to a maximal solution of the Cauchy problem for the corresponding ordinary differential equation $\vartheta'(t) = h(t,\vartheta)$ as $|x| \to \infty$ under certain conditions on an initial datum.

1. Introduction

In this article we consider the Cauchy problem for the pseudoparabolic equation

$$u_t = \Delta u_t + \Delta\varphi(u) + h(t,u), \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n. \tag{1.2}$$

Put $R_+ = (0,+\infty)$ and $\Pi_T = \mathbb{R}^n \times [0,T]$, $n \geq 1$, $T > 0$. Throughout this paper we suppose that the functions φ and h satisfy the following conditions:

$$\begin{gathered}\varphi(p) \text{ is defined for } p \geq 0,\ h(t,p) \text{ is defined for } t \geq 0 \text{ and } p \geq 0,\\ \varphi(p) \in C^2(\overline{R}_+) \cap C^3(R_+),\ h(t,p) \in C^{0,\alpha}_{\rm loc}(\overline{R}_+ \times \overline{R}_+) \cap C^{0,1+\alpha}_{\rm loc}(\overline{R}_+ \times R_+),\ 0 < \alpha < 1,\ h(t,0) = 0,\ t \in \overline{R}_+,\ \varphi(p) + h(t,p) \text{ does not decrease in } p \text{ for all } t \in \overline{R}_+.\end{gathered} \tag{1.3}$$

Assume that one of the following conditions is satisfied:

$$h(t,p) \geq 0, \quad t \in \overline{R}_+,\ p \in \overline{R}_+, \tag{1.4}$$

or

$$h(t,p) \text{ does not increase in } p \text{ for all } t \in \overline{R}_+. \tag{1.5}$$

Let the initial data have the following properties:

$$u_0(x) \in C^2(\mathbb{R}^n), \quad 0 \leq u_0(x) \leq M\ (M \geq 0),\ x \in \mathbb{R}^n, \tag{1.6}$$

$$\lim_{|x|\to\infty} u_0(x) = M. \tag{1.7}$$

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Equations $u_t = \Delta u_t + \Delta u^p + u^q$ and $u_t = \Delta u_t + \Delta(u^l + u^p) - u^p$, where $p, l \geq 2,\ q > 0$, are typical examples of equation (1.1) satisfying (1.3) under conditions (1.4) and (1.5) respectively.

If we suppose $u_0(x) \equiv M$ in (1.2) then a solution of the Cauchy problem for the corresponding ordinary differential equation

$$\vartheta'(t) = h(t, \vartheta), \quad \vartheta(0) = M \tag{1.8}$$

will be a solution of (1.1), (1.2).

Remark 1.1. We note that problem (1.8) may have more than one solution. Indeed, we put $h(t, \vartheta) = \vartheta^p$, $0 < p < 1$, and $M \equiv 0$ then problem (1.8) has the solutions $\vartheta_1(t) \equiv 0$ and $\vartheta_2(t) = (1-p)^{\frac{1}{1-p}} t^{\frac{1}{1-p}}$.

Definition 1.2. A nonnegative solution $\vartheta(t)$ of (1.8) is called maximal on $[0, T)$ if for any other nonnegative solution $f(t)$ of (1.8) the inequality $f(t) \leq \vartheta(t)$ is satisfied for $0 \leq t < T$.

We suppose that the maximal nonnegative solution $\vartheta(t)$ of (1.8) exists on $[0, T_0)$, $T_0 \leq +\infty$. Similarly we define the maximal solution of (1.1), (1.2).

Assume that (1.3) and (1.6) hold. Then there exists a nonnegative solution $u(x,t) \in C^{2,1}(\Pi_T)$ of (1.1), (1.2) (see [9]) satisfying for any $T < T_0$ the inequality

$$0 \leq u(x,t) \leq \vartheta(t), \quad (x,t) \in \Pi_T.$$

The main result of this article is the following statement.

Theorem 1.3. *Let* (1.3), (1.6), (1.7) *hold and* $u(x,t)$, $\vartheta(t)$ *are maximal solutions of problems* (1.1), (1.2) *and* (1.8) *respectively. Suppose that either* (1.4) *or* (1.5) *is satisfied in addition. Then we have*

$$u(x,t) \to \vartheta(t) \quad \text{as } |x| \to \infty$$

uniformly in $[0, T]\,(T < T_0)$.

Results similar to Theorem 1.3 were obtained in [5, 7] and [2, 3, 8, 11, 12, 13] respectively in studying of an asymptotic behavior of solutions of parabolic equations, systems and blow-up solutions of nonlinear heat equations and reaction-diffusion systems at infinity. Pseudoparabolic equations has been analyzed by many authors (see [14] and the references therein).

Our main research tool is a comparison principle.

Theorem 1.4. *Let* (1.3) *hold and* $u_1(x,t)$, $u_2(x,t)$ *be nonnegative bounded solutions of* (1.1) *in* Π_T *and one of them is not less some positive constant. Suppose that the corresponding initial data* $u_{01}(x)$ *and* $u_{02}(x)$ *satisfying* (1.6) *and the inequality*

$$u_{01}(x) \leq u_{02}(x), \quad x \in \mathbb{R}^n.$$

Then

$$u_1(x,t) \leq u_2(x,t), \quad (x,t) \in \Pi_T.$$

For problem (1.1), (1.2) with $\varphi(u) = u^2$ and $h(t,u) = 0$ the comparison principle was established in [1]. For an initial–boundary value problem for equation (1.1) with $h(t,u) = h(u)$ it was proved in [10].

This paper is organized as follows. In the next section we prove Theorem 1.4. Some auxiliary statements used for description a behavior of the maximal solution of (1.1), (1.2) at infinity are established in Section 3. Theorem 1.3 is proved in Section 4.

2. Proof of Theorem 1.4

Without loss of generality we may assume that $u_2(x,t) \geq \varepsilon,\ \varepsilon > 0,\ (x,t) \in \Pi_T$. Obviously, the function $w(x,t) = u_2(x,t) - u_1(x,t)$ satisfies the problem

$$w_t = \Delta w_t + \Delta(aw) + bw, \quad (x,t) \in \mathbb{R}^n \times (0,T), \tag{2.1}$$

$$w(x,0) = u_{02}(x) - u_{01}(x), \quad x \in \mathbb{R}^n. \tag{2.2}$$

Here

$$a(x,t) = \int_0^1 \varphi'(z(\theta))\,d\theta, \quad b(x,t) = \int_0^1 h_{z(\theta)}(t, z(\theta))\,d\theta,$$

where $z(\theta) = \theta u_2(x,t) + (1-\theta)u_1(x,t)$. By (1.3) the functions $a(x,t)$ and $b(x,t)$ have the following properties:

$$\begin{gathered} a(x,t) \in C^{2,0}(\Pi_T), \quad b(x,t) \in C^{\alpha,0}_{\rm loc}(\Pi_T), \\ a(x,t) + b(x,t) \geq 0, \quad |a(x,t)| + |b(x,t)| \leq m, \quad (x,t) \in \Pi_T, \end{gathered} \tag{2.3}$$

where m is some positive constant.

Lemma 2.1. *Let $a(x,t)$ and $b(x,t)$ be functions such that conditions* (2.3) *are satisfied. Then a solution of* (2.1), (2.2) *is unique.*

The proof of the above lemma is analogous to the proof the same statement for problem (1.1), (1.2) with $h(t,p) = 0$ in [6].

Let Q be a bounded domain in $\mathbb{R}^n$ for $n \geq 1$ with a smooth boundary ∂Q. We denote $Q_T = Q \times (0,T)$ and $S_T = \partial Q \times (0,T)$. Let us consider the equation

$$u_t = \Phi(x,t,u) + F(u(\cdot,t)), \quad (x,t) \in Q_T, \tag{2.4}$$

subject to the initial data

$$u(x,0) = u_0(x), \quad x \in Q, \tag{2.5}$$

where the function $\Phi(x,t,\xi)$ is defined on the set $\overline{Q} \times [0,T] \times R$ and $F(u(\cdot,t))$ is a nonlinear integral operator.

Definition 2.2. We shall say that a function $\sigma^+(x,t) \in C^{0,1}(Q_T)$, $-\infty < m_T \leq \sigma^+(x,t) \leq M_T < +\infty$, $(x,t) \in Q_T$, is a supersolution of (2.4), (2.5) in Q_T if

$$\begin{gathered} \sigma^+_t(x,t) \geq \Phi(x,t,\sigma^+) + F(\sigma^+(\cdot,t)), \quad (x,t) \in Q_T, \\ \sigma^+(x,0) \geq u_0(x),\ x \in \overline{Q}, \end{gathered} \tag{2.6}$$

where m_T, M_T are constants depending on T.

Analogously we say that $\sigma^-(x,t) \in C^{0,1}(Q_T)$, $m_T \leq \sigma^-(x,t) \leq M_T$, $(x,t) \in Q_T$, is a subsolution of (2.4), (2.5) in Q_T if it satisfies inequalities (2.6) in the reverse order. Under the assumption $\sigma^-(x,t) \leq \sigma^+(x,t),\ (x,t) \in Q_T$, we introduce the set $O(\sigma^-,\sigma^+) = \{u \in C(\overline{Q}_T) | \, \sigma^- \leq u \leq \sigma^+,\ (x,t) \in Q_T\}$ and make the following assumptions on data of (2.4), (2.5):

There exist a supersolution $\sigma^+(x,t)$ and a subsolution $\sigma^-(x,t)$ of (2.4), (2.5) in Q_T such that $\sigma^-(x,t) \leq \sigma^+(x,t),\ (x,t) \in Q_T$. (2.7)

$\Phi(x,t,\xi)$ and $\Phi_\xi(x,t,\xi)$ are continuous functions on the set $\overline{Q} \times [0,T] \times R$. (2.8)

The operator $F(u(\cdot,t))$, on $C(\overline{Q}_T)$ into $C(\overline{Q}_T)$, is completely continuous and monotone on $O(\sigma^-,\sigma^+)$. (2.9)

$$u_0(x) \in C(\overline{Q}). \tag{2.10}$$

The following existence theorem has been proved in [1].

Theorem 2.3. *Assume that* (2.7)-(2.10) *hold. Then there exists a solution of problem* (2.4), (2.5) *in* Q_T *such that*

$$\sigma^-(x,t) \le u(x,t) \le \sigma^+(x,t), \quad (x,t) \in Q_T.$$

Let $G_n(x,\xi)$ be the Green function of the boundary value problem for the operator $L = I - \Delta$ in Q. It is known that

$$G_n(x,\xi) = \mathcal{E}_n(x-\xi) + g_n(x,\xi), \quad (x,\xi) \in Q \times Q,$$

where $\mathcal{E}_n(x)$ is the fundamental solution of the operator L of $\mathbb{R}^n$ tending to zero as $|x| \to \infty$ and for any fixed $\xi \in Q$ the function $g_n \in C^2(Q) \cap C(\overline{Q})$ satisfies the equation

$$L_x g_n = 0, \quad x \in Q,$$

and the boundary condition

$$g_n(x,\xi)|_{x\in\partial Q} = -\mathcal{E}_n(x-\xi)|_{x\in\partial Q}, \quad \xi \in Q.$$

It is well known that

$$\mathcal{E}_n(x) = c_n|x|^{(2-n)/2}K_{(2-n)/2}(|x|), \tag{2.11}$$

where $K_\mu(|x|)$ is the μth order Macdonald function and c_n is the normalizing multiplier such that $\int_{\mathbb{R}^n} \mathcal{E}_n(x)d\,x = 1$.

We note some properties of the Green function (see [4]):

$$\begin{gathered}
0 < G_n(x,\xi) < \mathcal{E}_n(x-\xi), \quad (x,\xi) \in Q \times Q,\\
\frac{\partial G_n(x,\xi)}{\partial \nu_\xi} \le 0, \quad \xi \in \partial Q,\ x \in Q,\\
\int_Q G_n(x,\xi)\,d\xi = 1 + \int_{\partial Q} \frac{\partial G_n(x,\xi)}{\partial \nu_\xi}\,dS, \quad x \in Q,\\
\min_{y\in\partial Q}(-\mathcal{E}_n(x-y)) < g_n(x,\xi) < 0, \quad (x,\xi) \in Q \times Q,
\end{gathered} \tag{2.12}$$

where ν_ξ is the outward normal derivative on ∂Q in variables of ξ.

We consider the integro–differential equation, in Q_T,

$$w_t(x,t) = -a(x,t)w(x,t) + \int_Q G_n(x,\xi)[a(\xi,t)+b(\xi,t)]w(\xi,t)\,d\xi \tag{2.13}$$

subject to the initial condition, in Q,

$$w(x,0) = u_{02}(x) - u_{01}(x). \tag{2.14}$$

Let $u_{02}(x) - u_{01}(x) \le M_1$, $x \in \mathbb{R}^n$, $M_1 \in \overline{R}_+$.

Lemma 2.4. *Let conditions* (2.3) *hold. Then there exists a solution of* (2.13), (2.14) *in* Q_T *such that*

$$0 \le w(x,t) \le M_1 e^{2mt}, \quad (x,t) \in Q_T. \tag{2.15}$$

Proof. We use the following functions

$$\Phi(x,t,w) = -a(x,t)w(x,t), \quad F(w(\cdot,t)) = \int_Q G_n(x,\xi)[a(\xi,t)+b(\xi,t)]w(\xi,t)\,d\xi$$

and show that the conditions of Theorem 2.3 are valid. It is obvious, that $\sigma^-(x,t) \equiv 0$ is the subsolution of (2.13), (2.14). We shall show that $\sigma^+(x,t) = M_1 e^{2mt}$ is the supersolution of (2.13), (2.14). Indeed,

$$\begin{aligned}
\Phi(x,t,\sigma^+) + F(\sigma^+) &= -a(x,t)M_1e^{2mt} + \int_Q G_n(x,\xi)[a(\xi,t)+b(\xi,t)]M_1e^{2mt}\,d\xi \\
&\le mM_1e^{2mt} + mM_1e^{2mt} \le 2mM_1e^{2mt} \\
&= \sigma_t^+(x,t),\ (x,t)\in Q_T, \\
&\quad \sigma^+(x,0) = M_1 \ge w_0(x), \quad x\in\overline{Q}.
\end{aligned}$$

Condition (2.8) of Theorem 2.3 is satisfied by virtue of (2.3). As $a(x,t)+b(x,t)\ge 0$ then the operator F is monotone on $O(\sigma^-,\sigma^+)$. We shall prove that the operator F is completely continuous on $O(\sigma^-,\sigma^+)$. Let $w\in O(\sigma^-,\sigma^+)$ then

$$|F(w(\cdot,t))| = \Big|\int_Q G_n(x,\xi)[a(\xi,t)+b(\xi,t)]w(\xi,t)\,d\xi\Big| \le mM_1e^{2mT}.$$

Hence, the operator F is bounded. Suppose $x,y\in Q$ and $w\in O(\sigma^-,\sigma^+)$. Then

$$\begin{aligned}
|F(w(x,t)) - F(w(y,t))| &= \Big|\int_Q [G_n(x,\xi)-G_n(y,\xi)](a(\xi,t)+b(\xi,t))w(\xi,t)\,d\xi\Big| \\
&\le mM_1e^{2mT}\int_Q |G_n(x,\xi)-G_n(y,\xi)|\,d\xi,
\end{aligned}$$

that implies the validity of (2.9). Relations (1.6) for the initial data $u_{01}(x)$ and $u_{02}(x)$ are valid then all conditions of Theorem 2.3 are satisfied. Hence, there exists a solution $w(x,t)$ of (2.13), (2.14) in Q_T for which inequality (2.15) holds. □

Lemma 2.5. *If conditions* (2.3) *are satisfied then there exists a nonnegative solution of* (2.1)*,* (2.2) *in* Π_T*.*

Proof. Let $G_n(x,\xi,l)$ be the Green function of the boundary value problem for the operator $L = I - \Delta$ in $Q_l = \{x\in\mathbb{R}^n : |x| < l\},\ l>0$. Let the functions of the sequence $w_l(x,t)$ $(l=1,2,\dots)$ satisfy equation (2.13) in $Q_{l,T} = Q_l\times(0,T)$ and initial data (2.14) in Q_l. According to Lemma 2.4 there exists a solution $w_l(x,t)$ of (2.13), (2.14) in $Q_{l,T}$ such that

$$0 \le w_l(x,t) \le M_1e^{2mt}, \quad (x,t)\in Q_{l,T}. \tag{2.16}$$

Differentiating (2.13) with respect to x_i $(i=1,\dots,n)$ we obtain

$$\begin{aligned}
w_{ltx_i}(x,t) &= -a_{x_i}(x,t)w_l(x,t) - a(x,t)w_{lx_i}(x,t) \\
&\quad + \int_{Q_l} G_{nx_i}(x,\xi,l)[a(\xi,t)+b(\xi,t)]w_l(\xi,t)\,d\xi, \quad (x,t)\in Q_{l,T},
\end{aligned}$$

from which we find that

$$w_{lx_i}(x,t) = e^{-\int_0^t a(x,\tau)\,d\tau}\Big[u_{02}(x) - u_{01}(x) + \int_0^t p_l(x,\tau)e^{\int_0^\tau a(x,\tau_1)\,d\tau_1}\,d\tau\Big], \tag{2.17}$$

where

$$p_l(x,t) = -a_{x_i}(x,t)w_l(x,t) + \int_{Q_l} G_{nx_i}(x,\xi,l)[a(\xi,t)+b(\xi,t)]w_l(\xi,t)\,d\xi.$$

It follows from (2.12), (2.13), (2.16) and (2.17) that absolute values of functions $w_l,\ w_{lt},\ w_{lx_i}$ $(i=1,2,\dots,n)$ are uniformly bounded with respect to l on each set $\overline{Q}_{k,T}$, where k is an arbitrary fixed natural number, $k < l$. According to the Arzela–Ascoli theorem the sequence $w_l(x,t)$ is compact in $\overline{Q}_{k,T}$. By applying diagonal process we can extract from the sequence $w_l(x,t)$ a subsequence $w_{l_s}(x,t)$ such that

$$w_{l_s}(x,t) \to w(x,t) \quad \text{uniformly in } \overline{Q}_{k,T}. \tag{2.18}$$

Without loss of generality we assume that (2.18) is valid for the sequence $w_l(x,t)$. Integrating equation (2.13) with respect to t we obtain

$$\begin{aligned} w_l(x,t) &= u_{02}(x) - u_{01}(x) - \int_0^t a(x,\tau) w_l(x,\tau)\,d\tau \\ &\quad + \int_0^t \int_{Q_l} G_n(x,\xi,l)[a(\xi,\tau)+b(\xi,\tau)] w_l(\xi,\tau)\,d\xi\,d\tau, \quad (x,t) \in Q_{l,T}. \end{aligned} \tag{2.19}$$

Let (x,t) be an arbitrary point of Π_T and let k be such that $(x,t) \in \overline{Q}_{k,T}$, $k < l$. By virtue of (2.12), (2.16) and (2.18) we obtain

$$\begin{aligned} &\lim_{l\to\infty} \int_0^t \int_{Q_l} G_n(x,\xi,l)[a(\xi,\tau)+b(\xi,\tau)] w_l(\xi,\tau)\,d\xi\,d\tau \\ &= \int_0^t \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)[a(\xi,\tau)+b(\xi,\tau)] w(\xi,\tau)\,d\xi\,d\tau. \end{aligned} \tag{2.20}$$

Letting $l \to \infty$ in (2.19) and using (2.18) and (2.20) we conclude that

$$\begin{aligned} w(x,t) &= u_{02}(x) - u_{01}(x) - \int_0^t a(x,\tau) w(x,\tau)\,d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)[a(\xi,\tau)+b(\xi,\tau)] w(\xi,\tau)\,d\xi\,d\tau, \quad (x,t) \in \Pi_T. \end{aligned} \tag{2.21}$$

By (2.3) the solution $w(x,t)$ of (2.21) belongs to the class $C^{2,1}(\Pi_T)$ and

$$\begin{aligned} &\Delta\left(w_t(x,t) + a(x,t)w(x,t)\right) \\ &= \Delta \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)[a(\xi,t)+b(\xi,t)] w(\xi,t)\,d\xi \\ &= -[a(x,t)+b(x,t)]w(x,t) + \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)[a(\xi,t)+b(\xi,t)] w(\xi,t)\,d\xi \\ &= w_t(x,t) - b(x,t)w(x,t), \ (x,t) \in \mathbb{R}^n \times (0,T), \\ &\qquad w(x,0) = u_{02}(x) - u_{01}(x), \quad x \in \mathbb{R}^n. \end{aligned}$$

□

According to Lemmas 2.1 and 2.5 we have

$$u_2(x,t) \ge u_1(x,t), \quad (x,t) \in \Pi_T.$$

Remark 2.6. The comparison principle is valid without the condition that one of the solution is not less some positive constant if we assume that $h(t,p) \in C^{0,1+\alpha}_{\rm loc}(\overline{R}_+ \times \overline{R}_+)$, $0 < \alpha < 1$.

Remark 2.7. If the inequality $u_0(x) \ge m > 0$ and (1.4) hold then problem (1.1), (1.2) has an unique solution.

Indeed, in the same way as it was done in [9] we can show the existence of the solution $u(x,t)$ of problem (1.1), (1.2) such that $u(x,t) \ge m > 0$.

3. Auxiliary statements

Let condition (1.4) hold. We consider the Cauchy problem for equation (1.1) subject to the initial condition

$$u(x,0)=u_0(x)+\varepsilon, \quad x\in\mathbb{R}^n. \tag{3.1}$$

If we suppose $u_0(x)\equiv M$ in (3.1) then a solution of the Cauchy problem for the corresponding ordinary differential equation

$$\vartheta'(t)=h(t,\vartheta), \quad \vartheta(0)=M+\varepsilon \tag{3.2}$$

will be a solution of (1.1), (3.1).

Suppose that the solution $\vartheta_\varepsilon(t)$ of (3.2) exists on $[0,T_{0,\varepsilon})$, $T_{0,\varepsilon}\le+\infty$. It is easy to show (see [9]) that a solution $u_\varepsilon(x,t)$ of the integral equation

$$\begin{aligned} u_\varepsilon(x,t)&=u_0(x)+\varepsilon-\int_0^t\varphi(u_\varepsilon(x,\tau))\,d\tau\\ &\quad+\int_0^t\int_{\mathbb{R}^n}\mathcal{E}_n(x-\xi)[\varphi(u_\varepsilon(\xi,\tau))+h(\tau,u_\varepsilon(\xi,\tau))]\,d\xi\,d\tau \end{aligned} \tag{3.3}$$

for any $T_\varepsilon<T_{0,\varepsilon}$ solves in Π_{T_ε} problem (1.1), (3.1) and satisfies the inequality

$$\varepsilon\le u_\varepsilon(x,t)\le\vartheta_\varepsilon(t), \quad (x,t)\in\Pi_{T_\varepsilon}. \tag{3.4}$$

We note that problem (3.2) is equivalent to the integral equation

$$\vartheta_\varepsilon(t)=M+\varepsilon+\int_0^t h(\tau,\vartheta_\varepsilon(\tau))\,d\tau, \quad t\in[0,T_{0,\varepsilon}). \tag{3.5}$$

Lemma 3.1. *Let* (1.3)*,* (1.4)*,* (1.6) *and* (1.7) *hold. Then for some* $T_{*,\varepsilon}<T_{0,\varepsilon}$ *we have*

$$u_\varepsilon(x,t)\to\vartheta_\varepsilon(t) \quad \text{as } |x|\to\infty$$

uniformly in $[0,T_{*,\varepsilon}]$.

Proof. Put $u_{0,\varepsilon}(x,t)\equiv\vartheta_\varepsilon(t)$. We define a sequence of functions $u_{k,\varepsilon}(x,t)$ ($k=1,2,\dots$) in the following way

$$\begin{aligned} u_{k,\varepsilon}(x,t)&=u_0(x)+\varepsilon-\int_0^t\varphi(u_{k-1,\varepsilon}(x,\tau))\,d\tau\\ &\quad+\int_0^t\int_{\mathbb{R}^n}\mathcal{E}_n(x-\xi)[\varphi(u_{k-1,\varepsilon}(\xi,\tau))+h(\tau,u_{k-1,\varepsilon}(\xi,\tau))]\,d\xi\,d\tau. \end{aligned} \tag{3.6}$$

Fix any T_ε such that $T_\varepsilon<T_{0,\varepsilon}$ and show that the sequence $u_{k,\varepsilon}(x,t)$ converges to the solution $u_\varepsilon(x,t)$ of (1.1), (3.1) as $k\to\infty$ uniformly in some layer $\Pi_{T_{*,\varepsilon}}$ ($T_{*,\varepsilon}\le T_\varepsilon$).

At first we show that the sequence $u_{k,\varepsilon}(x,t)$ is uniformly bounded in some layer $\Pi_{T_{*,\varepsilon}}$. Using the method of mathematical induction we prove the inequality

$$\frac{\varepsilon}{2}\le u_{k,\varepsilon}(x,t)\le M+\frac{3\varepsilon}{2}+\vartheta_\varepsilon(T_\varepsilon), \quad (x,t)\in\Pi_{T_{*,\varepsilon}},\ k=0,1,\dots. \tag{3.7}$$

It is obviously that (3.7) is true for $k=0$. We assume that (3.7) holds for $k=k_0$ and we shall prove the inequality for $k=k_0+1$. Using the property of function

$\varphi + h$ and the mean value theorem we obtain

$$\begin{aligned} u_{k_0+1,\varepsilon}(x,t) &= u_0(x) + \varepsilon - \int_0^t \varphi(u_{k_0,\varepsilon}(x,\tau))\,d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)[\varphi(u_{k_0,\varepsilon}(\xi,\tau)) + h(\tau, u_{k_0,\varepsilon}(\xi,\tau))]\,d\xi\,d\tau \\ &\le M + \varepsilon + \int_0^t \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)\Big[\varphi(M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon)) \\ &\quad + h(\tau, M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon))\Big]\,d\xi\,d\tau - \int_0^t \varphi(u_{k_0,\varepsilon}(x,\tau))\,d\tau \\ &\le M + \varepsilon + \int_0^t \Big\{\varphi(M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon)) - \varphi(u_{k_0,\varepsilon}(x,\tau)) \\ &\quad + h(\tau, M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon))\Big\}d\tau \\ &\le M + \varepsilon + T_{*,\varepsilon}(M + \varepsilon + \vartheta_\varepsilon(T_\varepsilon)) \max_{\frac{\varepsilon}{2} \le \theta \le M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon)} |\varphi'(\theta)| \\ &\quad + T_{*,\varepsilon} \max_{0 \le t \le T_\varepsilon} h(t, M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon)) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &u_{k_0+1,\varepsilon}(x,t) \\ &\ge \varepsilon + \int_0^t \Big\{\varphi(\frac{\varepsilon}{2}) - \varphi(u_{k_0,\varepsilon}(x,\tau)) + h(\tau, \frac{\varepsilon}{2})\Big\}\,d\tau \\ &\ge \varepsilon - T_{*,\varepsilon}\Big((M + \varepsilon + \vartheta_\varepsilon(T_\varepsilon)) \max_{\frac{\varepsilon}{2} \le \theta \le M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon)} |\varphi'(\theta)| + \max_{0 \le t \le T_\varepsilon} h(t, \frac{\varepsilon}{2})\Big). \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we conclude that inequality (3.7) is valid for $k = k_0 + 1$ provided

$$T_{*,\varepsilon} \le \min\big\{T_\varepsilon,\ \frac{\varepsilon/2}{(M + \varepsilon + \vartheta_\varepsilon(T_\varepsilon))\lambda + \mu}\big\}, \tag{3.10}$$

where

$$\lambda = \max_{\frac{\varepsilon}{2} \le \theta \le M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon)} |\varphi'(\theta)|, \quad \mu = \max_{0 \le t \le T_\varepsilon,\ \frac{\varepsilon}{2} \le \theta \le M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon)} h(t,\theta).$$

Using the method of mathematical induction it is easy to show the validity in $\Pi_{T_{*,\varepsilon}}$ the estimate

$$|u_{k,\varepsilon}(x,t) - u_{k-1,\varepsilon}(x,t)| \le M(2\lambda + \nu)^{k-1} \frac{t^{k-1}}{(k-1)!}, \tag{3.11}$$

where

$$\nu = \max_{0 \le t \le T_\varepsilon,\ \frac{\varepsilon}{2} \le \theta \le M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_\varepsilon)} |h_\theta(t,\theta)|.$$

For $k = 1$ we have

$$|u_{1,\varepsilon}(x,t) - u_{0,\varepsilon}(x,t)| = \vartheta_\varepsilon(t) - u_0(x) - \varepsilon - \int_0^t h(\tau, \vartheta_\varepsilon(\tau))\,d\tau \le M.$$

We assume that (3.11) holds for $k = k_0$ and we shall prove the inequality for $k = k_0 + 1$. By (3.11) and the mean value theorem we have

$$|u_{k_0+1,\varepsilon}(x,t) - u_{k_0,\varepsilon}(x,t)|$$

$$
\begin{aligned}
&= \Big|\int_0^t \varphi'(\theta_1(x,\tau))[u_{k_0,\varepsilon}(x,\tau) - u_{k_0-1,\varepsilon}(x,\tau)]\,d\tau\Big| \\
&\quad + \Big|\int_0^t \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)\varphi'(\theta_2(\xi,\tau))[u_{k_0,\varepsilon}(\xi,\tau) - u_{k_0-1,\varepsilon}(\xi,\tau)]\,d\xi\,d\tau\Big| \\
&\quad + \Big|\int_0^t \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)h_{\theta_3}(\tau,\theta_3(\xi,\tau))[u_{k_0,\varepsilon}(\xi,\tau) - u_{k_0-1,\varepsilon}(\xi,\tau)]\,d\xi\,d\tau\Big| \\
&\le M(2\lambda+\nu)^{k_0}\int_0^t \frac{\tau^{k_0-1}}{(k_0-1)!}\,d\tau \\
&\le M(2\lambda+\nu)^{k_0}\frac{t^{k_0}}{k_0!},
\end{aligned}
$$

where $\frac{\varepsilon}{2} \le \theta_i \le M + \frac{3\varepsilon}{2} + \vartheta_\varepsilon(T_{*,\varepsilon})$, $i = 1,2,3$.

To show that the sequence $u_{k,\varepsilon}(x,t)$ converges uniformly in $\Pi_{T_{*,\varepsilon}}$ we consider the series

$$
u_{0,\varepsilon}(x,t) + \sum_{n=1}^{\infty}(u_{n,\varepsilon}(x,t) - u_{n-1,\varepsilon}(x,t)). \tag{3.12}
$$

Then $u_{k,\varepsilon}(x,t)$ is the $(k+1)$th partial sum of (3.12). By (3.11) every term of series (3.12) for all $(x,t) \in \Pi_{T_{*,\varepsilon}}$ is not greater than the absolute value of the corresponding term of the following convergent series

$$
\vartheta_\varepsilon(t) + M\sum_{n=0}^{\infty}(2\lambda+\nu)^n\frac{T_{*,\varepsilon}^n}{n!}.
$$

Hence, series (3.12) as well as the sequence $u_{k,\varepsilon}(x,t)$ converge uniformly in $\Pi_{T_{*,\varepsilon}}$. Let

$$
u_\varepsilon(x,t) = \lim_{k\to\infty} u_{k,\varepsilon}(x,t).
$$

Passing to the limit as $k \to \infty$ in (3.6) and using the Lebesgue theorem we obtain that the function $u_\varepsilon(x,t)$ satisfies (3.3). Hence, $u_\varepsilon(x,t)$ solves problem (1.1), (3.1) in $\Pi_{T_{*,\varepsilon}}$.

Using the method of mathematical induction we shall prove that

$$
u_{k,\varepsilon}(x,t) \to \vartheta_\varepsilon(t) \quad \text{as } |x| \to \infty,\ k = 0,1,\dots \tag{3.13}
$$

uniformly in $[0, T_{*,\varepsilon}]$.

It is obviously that (3.13) is true for $k = 0$. We assume that (3.13) holds for $k = k_0$ and we shall prove (3.13) for $k = k_0 + 1$. Fix an arbitrary $\delta > 0$. By the induction assumption for any $\delta_0 > 0$ there exists a constant $A_0 = A_0(\delta_0, \varepsilon, T_{*,\varepsilon}, k_0)$ such that if $|x| > A_0$ and $0 \le t \le T_{*,\varepsilon}$ then

$$
|u_{k_0,\varepsilon}(x,t) - \vartheta_\varepsilon(t)| < \delta_0.
$$

From (3.5) and (3.6) we have

$$
\begin{aligned}
&|u_{k_0+1,\varepsilon}(x,t) - \vartheta_\varepsilon(t)| \\
&= \Big|u_0(x) + \varepsilon - \int_0^t \varphi(u_{k_0,\varepsilon}(x,\tau))\,d\tau + \int_0^t \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)\Big[\varphi(u_{k_0,\varepsilon}(\xi,\tau)) \\
&\quad + h(\tau, u_{k_0,\varepsilon}(\xi,\tau))\Big]\,d\xi\,d\tau - M - \varepsilon - \int_0^t h(\tau, \vartheta_\varepsilon(\tau))\,d\tau\Big| \\
&\le |u_0(x) - M| + \int_0^t |\varphi'(\theta_1(x,\tau))| \cdot |u_{k_0,\varepsilon}(x,\tau) - \vartheta_\varepsilon(\tau)|\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_{|\xi|\le A_0}\mathcal{E}_n(x-\xi)(|\varphi'(\theta_2(\xi,\tau))|+|h_{\theta_3}(\tau,\theta_3(\xi,\tau))|)|u_{k_0,\varepsilon}(\xi,\tau)-\vartheta_\varepsilon(\tau)|\,d\xi\,d\tau\\
&+\int_0^t\int_{|\xi|>A_0}\mathcal{E}_n(x-\xi)(|\varphi'(\theta_2(\xi,\tau))|+|h_{\theta_3}(\tau,\theta_3(\xi,\tau))|)|u_{k_0,\varepsilon}(\xi,\tau)-\vartheta_\varepsilon(\tau)|\,d\xi\,d\tau,
\end{aligned}
$$

where $\frac{\varepsilon}{2}\le\theta_i\le M+\frac{3\varepsilon}{2}+\vartheta_\varepsilon(T_{*,\varepsilon})$, $i=1,2,3$. By (1.7) for any $\delta_1>0$ there exists a constant $A_1=A_1(\delta_1)$ such that $|u_0(x)-M|<\delta_1$ if $|x|>A_1$. Using the property of the fundamental solution $\mathcal{E}_n$ and (3.7) we obtain that for any $\delta_2>0$ there exists a constant $A_2=A_2(\delta_2,\varepsilon)$ such that if $|x|>A_2$ then

$$
\begin{aligned}
&\int_0^t\int_{|\xi|\le A_0}\mathcal{E}_n(x-\xi)(|\varphi'(\theta_2(\xi,\tau))|+|h_{\theta_3}(\tau,\theta_3(\xi,\tau))|)|u_{k_0,\varepsilon}(\xi,\tau)-\vartheta_\varepsilon(\tau)|\,d\xi\,d\tau\\
&<\delta_2.
\end{aligned}
$$

Hence, we obtain

$$
|u_{k_0+1,\varepsilon}(x,t)-\vartheta_\varepsilon(t)|<\delta_1+\delta_2+T_{*,\varepsilon}(2\lambda+\nu)\delta_0,
$$

where

$$
\lambda=\max_{\frac{\varepsilon}{2}\le\theta\le M+\frac{3\varepsilon}{2}+\vartheta_\varepsilon(T_\varepsilon)}|\varphi'(\theta)|,\quad \nu=\max_{0\le t\le T_\varepsilon,\ \frac{\varepsilon}{2}\le\theta\le M+\frac{3\varepsilon}{2}+\vartheta_\varepsilon(T_\varepsilon)}|h_\theta(t,\theta)|.
$$

Let $\delta_0=\frac{\delta}{3T_{*,\varepsilon}(2\lambda+\nu)}$, $\delta_1=\frac{\delta}{3}$, $\delta_2=\frac{\delta}{3}$ and $A=\max(A_0,A_1,A_2)$ then

$$
|u_{k_0+1,\varepsilon}(x,t)-\vartheta_\varepsilon(t)|<\delta
$$

if $0\le t\le T_{*,\varepsilon}$ and $|x|>A$. It follows that for any $\delta>0$ by suitable choosing k and A we obtain

$$
\begin{aligned}
|u_\varepsilon(x,t)-\vartheta_\varepsilon(t)|&=|u_\varepsilon(x,t)-u_{k,\varepsilon}(x,t)+u_{k,\varepsilon}(x,t)-\vartheta_\varepsilon(t)|\\
&\le|u_\varepsilon(x,t)-u_{k,\varepsilon}(x,t)|+|u_{k,\varepsilon}(x,t)-\vartheta_\varepsilon(t)|<\delta
\end{aligned}
$$

for $0\le t\le T_{*,\varepsilon}$ and $|x|>A$. □

Lemma 3.2. *Let (1.3), (1.4), (1.6) and (1.7) hold. Then for any $T_\varepsilon<T_{0,\varepsilon}$ we have*

$$
u_\varepsilon(x,t)\to\vartheta_\varepsilon(t)\quad\text{as } |x|\to\infty
$$

uniformly in $[0,T_\varepsilon]$.

Proof. Fix any T_ε such that $T_\varepsilon<T_{0,\varepsilon}$. We recall that for any $T_\varepsilon<T_{0,\varepsilon}$ the solution $u_\varepsilon(x,t)$ exists in Π_{T_ε} and satisfies inequality (3.4). Note that the solution $u_\varepsilon(x,t)$ of (1.1), (3.1) is unique by Remark 2.7.

By Lemma 3.1 there exists $T_{*,\varepsilon}\le T_\varepsilon$ such that $u_\varepsilon(x,t)\to\vartheta_\varepsilon(t)$ as $|x|\to\infty$ uniformly in $[0,T_{*,\varepsilon}]$. If $T_{*,\varepsilon}<T_\varepsilon$ then we construct for $t\ge T_{*,\varepsilon}$ new sequence $u_{k,\varepsilon}(x,t)$ in the following way:

$$
\begin{aligned}
u_{0,\varepsilon}(x,t)&\equiv\vartheta_\varepsilon(t),\\
u_{k,\varepsilon}(x,t)&=u_\varepsilon(x,T_{*,\varepsilon})-\int_{T_{*,\varepsilon}}^t\varphi(u_{k_0-1,\varepsilon}(x,\tau))\,d\tau\\
&\quad+\int_{T_{*,\varepsilon}}^t\int_{\mathbb{R}^n}\mathcal{E}_n(x-\xi)[\varphi(u_{k_0-1,\varepsilon}(\xi,\tau))+h(\tau,u_{k_0-1,\varepsilon}(\xi,\tau))]\,d\xi\,d\tau,
\end{aligned}
$$

for $k=1,2,\dots$. By the similar arguments to Lemma 3.1 we can prove that the sequence $u_{k,\varepsilon}(x,t)$ converges to the solution $u_\varepsilon(x,t)$ of (1.1), (3.1) as $k\to\infty$

uniformly in the layer $\mathbb{R}^n \times [T_{*,\varepsilon}, T_{*,\varepsilon} + \Delta T_\varepsilon]$ provided ΔT_ε satisfies condition (3.10) with $T_{*,\varepsilon} = \Delta T_\varepsilon$ and the inequality $T_{*,\varepsilon} + \Delta T_\varepsilon \le T_\varepsilon$. It follows that

$$u_\varepsilon(x,t) \to \vartheta_\varepsilon(t) \quad \text{as } |x| \to \infty$$

uniformly in $[T_{*,\varepsilon}, T_{*,\varepsilon} + \Delta T_\varepsilon]$. Repeating this procedure we obtain the conclusion of the theorem. □

4. Behavior of maximal solution at infinity

Proof of Theorem 1.3. Let (1.4) hold and $u_\varepsilon(x,t)$, $\vartheta_\varepsilon(t)$ be solutions of problems (1.1), (3.1) and (3.2) respectively. Using Theorem 1.4 for $\varepsilon_1 \ge \varepsilon_2$ we obtain:

$$\begin{gathered} u(x,t) \le u_{\varepsilon_2}(x,t) \le u_{\varepsilon_1}(x,t), \quad (x,t) \in \Pi_{T_{\varepsilon_1}}, \\ \vartheta(t) \le \vartheta_{\varepsilon_2}(t) \le \vartheta_{\varepsilon_1}(t), \quad t \in [0, T_{\varepsilon_1}]. \end{gathered}$$

According to Dini's theorem the sequences $u_\varepsilon(x,t)$ and $\vartheta_\varepsilon(t)$ convergence to some solutions $u(x,t)$ and $\vartheta(t)$ of problems (1.1), (1.2) and (1.8) as $\varepsilon \to 0$ uniformly respectively in Π_T and $[0,T]$, where $T < T_0$. It is easy to see that $u(x,t)$ and $\vartheta(t)$ are maximal solutions of problems (1.1), (1.2) and (1.8) respectively.

We fix an arbitrary $\delta > 0$ and $0 < T < T_0$. Choose $\varepsilon_1 > 0$ such that for any $\varepsilon < \varepsilon_1$ the inequality $T < T_{0,\varepsilon}$ holds. By the uniform convergence functions $u_\varepsilon(x,t)$ to $u(x,t)$ in Π_T and $\vartheta_\varepsilon(t)$ to $\vartheta(t)$ in $[0,T]$, $(T < T_0)$ as $\varepsilon \to 0$ we can take $\varepsilon_2 > 0$ such that for any $\varepsilon < \varepsilon_2$,

$$|u_\varepsilon(x,t) - u(x,t)| < \frac{\delta}{3}, \quad (x,t) \in \Pi_T, \tag{4.1}$$

$$|\vartheta_\varepsilon(t) - \vartheta(t)| < \frac{\delta}{3}, \quad t \in [0,T]. \tag{4.2}$$

Put $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. From Lemma 3.2 there exists the constant $A_0 = A_0(\delta, \varepsilon_0, T)$ such that for any $|x| > A_0$ we obtain

$$|u_{\varepsilon_0}(x,t) - \vartheta_{\varepsilon_0}(t)| < \frac{\delta}{3}, \quad (x,t) \in \Pi_T. \tag{4.3}$$

By (4.1)–(4.3) we conclude that by suitable choosing $\varepsilon = \varepsilon_0$ and $A = A_0$,

$$\begin{aligned} |u(x,t) - \vartheta(t)| &= |u(x,t) - u_\varepsilon(x,t) + u_\varepsilon(x,t) + \vartheta_\varepsilon(t) - \vartheta_\varepsilon(t) - \vartheta(t)| \\ &\le |u_\varepsilon(x,t) - u(x,t)| + |u_\varepsilon(x,t) - \vartheta_\varepsilon(t)| + |\vartheta_\varepsilon(t) - \vartheta(t)| < \delta \end{aligned}$$

for $0 \le t \le T$ and $|x| > A$.

Let (1.5) hold. Consider the Cauchy problems

$$\begin{gathered} \omega_t = \Delta\omega_t + \Delta\varphi(\omega) + h(t,\omega) - h(t,\varepsilon), \quad x \in \mathbb{R}^n,\ t > 0, \\ \omega(x,0) = u_0(x) + \varepsilon, \quad x \in \mathbb{R}^n, \end{gathered} \tag{4.4}$$

and

$$g'(t) = h(t,g) - h(t,\varepsilon), \quad g(0) = M + \varepsilon. \tag{4.5}$$

We suppose that the maximal nonnegative solution $g_\varepsilon(t)$ of (4.5) exists on $[0, T_{0,\varepsilon})$, $T_{0,\varepsilon} \le +\infty$. It is easy to show (see [9]) that for any $T_\varepsilon < T_{0,\varepsilon}$ there exists in Π_{T_ε} a solution $\omega_\varepsilon(x,t)$ of (4.4) satisfying the inequality

$$\varepsilon \le \omega_\varepsilon(x,t) \le g_\varepsilon(t), \quad (x,t) \in \Pi_{T_\varepsilon}.$$

Applying Theorem 1.4 we conclude that the solution $\omega_\varepsilon(x,t)$ of (4.4) is unique. Let $\varepsilon_1 \geq \varepsilon_2$ and $\omega_{\varepsilon_1}(x,t)$, $\omega_{\varepsilon_2}(x,t)$ are nonnegative bounded solutions of (4.4) with $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$ respectively. Then

$$\omega_{\varepsilon_1}(x,t) \geq \omega_{\varepsilon_2}(x,t), \quad (x,t) \in \Pi_{T_{\varepsilon_1}}.$$

The proof of this statement is analogous to the proof of Theorem 1.4. Then we consider the sequence $\omega_{k,\varepsilon}(x,t)$ $(k = 0,1,\dots)$:

$$\omega_{0,\varepsilon}(x,t) \equiv g_\varepsilon(t),$$

$$\begin{aligned}\omega_{k,\varepsilon}(x,t) = u_0(x) + \varepsilon - \int_0^t \varphi(\omega_{k-1,\varepsilon}(x,\tau))\,d\tau + \int_0^t \int_{\mathbb{R}^n} \mathcal{E}_n(x-\xi)\Big[\varphi(\omega_{k-1,\varepsilon}(\xi,\tau)) \\ + h(\tau,\omega_{k-1,\varepsilon}(\xi,\tau)) - h(\tau,\varepsilon)\Big]\,d\xi\,d\tau, \quad k = 1,2,\dots.\end{aligned}$$

Analogous to the arguments in Section 3 can be shown that for any $T_\varepsilon < T_{0,\varepsilon}$

$$\omega_\varepsilon(x,t) \to g_\varepsilon(t) \quad \text{as } |x| \to \infty$$

uniformly in $[0, T_\varepsilon]$. Further arguments are similar to reasoning in the proof of this theorem with condition (1.4). □

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