Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 116, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

RESOLUTIONS OF PARABOLIC EQUATIONS IN NON-SYMMETRIC CONICAL DOMAINS

AREZKI KHELOUFI

Abstract. This article is devoted to the analysis of a two-space dimensional linear parabolic equation, subject to Cauchy-Dirichlet boundary conditions. The problem is set in a conical type domain and the right hand side term of the equation is taken in a Lebesgue space. One of the main issues of this work is that the domain can possibly be non regular. This work is an extension of the symmetric case studied in Sadallah [13].

1. Introduction

Let Q be an open set of $\mathbb{R}^3$ defined by

$$Q = \{(t, x_1, x_2) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_t, 0 < t < T\}$$

where T is a finite positive number and for a fixed t in the interval $]0, T[$, Ω_t is a bounded domain of $\mathbb{R}^2$ defined by

$$\Omega_t = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le \frac{x_1^2}{\varphi^2(t)} + \frac{x_2^2}{h^2(t)\varphi^2(t)} < 1\}.$$

Here, φ is a continuous real-valued function defined on $[0, T]$, Lipschitz continuous on $[0, T]$ and such that

$$\varphi(0) = 0, \quad \varphi(t) > 0$$

for every $t \in\ ]0, T]$. h is a Lipschitz continuous real-valued function defined on $[0, T]$, such that

$$0 < \delta \le h(t) \le \beta \tag{1.1}$$

for every $t \in [0, T]$, where δ and β are positive constants.

In Q, we consider the boundary-value problem

$$\begin{gathered} \partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = f \in L^2(Q), \\ u\big|_{\partial Q - \Gamma_T} = 0, \end{gathered} \tag{1.2}$$

where $L^2(Q)$ is the usual Lebesgue space on Q, ∂Q is the boundary of Q and Γ_T is the part of the boundary of Q where $t = T$.

The difficulty related to this kind of problems comes from this singular situation for evolution problems; i.e., φ is allowed to vanish for $t = 0$, which prevents the domain Q from being transformed into a regular domain without the appearance of

2000 *Mathematics Subject Classification.* 35K05, 35K20.
Key words and phrases. Parabolic equation; conical domain; anisotropic Sobolev space.

Submitted May 31, 2012. Published July 9, 2012.

some degenerate terms in the parabolic equation, see for example Sadallah [12]. In order to overcome this difficulty, we impose a sufficient condition on the function φ; that is,

$$\varphi'(t)\varphi(t) \to 0 \quad \text{as } t \to 0, \tag{1.3}$$

and we obtain existence and regularity results for Problem (1.2) by using the domain decomposition method. More precisely, we will prove that Problem (1.2) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_0^{1,2}(Q) := \{u \in H^{1,2}(Q) : u\big|_{\partial Q - \Gamma_T} = 0\},$$

with

$$H^{1,2}(Q) = \{u \in L^2(Q) : \partial_t u, \partial_{x_1}^j u, \partial_{x_2}^j u, \partial_{x_1}\partial_{x_2} u \in L^2(Q), j = 1, 2\}.$$

In Sadallah [13] the same problem has been studied in the case of a symmetric conical domain; i.e., in the case where $h = 1$. Further references on the analysis of parabolic problems in non-cylindrical domains are: Alkhutov [1, 2], Degtyarev [4], Labbas, Medeghri and Sadallah [8, 9], Sadallah [12]. There are many other works concerning boundary-value problems in non-smooth domains (see, for example, Grisvard [6] and the references therein).

The organization of this article is as follows. In Section 2, first we prove an uniqueness result for Problem (1.2), then we derive some technical lemmas which will allow us to prove an uniform estimate (in a sense to be defined later). In Section 3, there are two main steps. First, we prove that Problem (1.2) admits a (unique) solution in the case of a domain which can be transformed into a cylinder. Secondly, for T small enough, we prove that the result holds true in the case of a conical domain under the above mentioned assumptions on functions φ and h. The method used here is based on the approximation of the conical domain by a sequence of subdomains $(Q_n)_n$ which can be transformed into regular domains (cylinders). We establish an uniform estimate of the type

$$\|u_n\|_{H^{1,2}(Q_n)} \le K\|f\|_{L^2(Q_n)},$$

where u_n is the solution of Problem (1.2) in Q_n and K is a constant independent of n. This allows us to pass to the limit. Finally, in Section 4 we complete the proof of our main result (Theorem 4.4).

2. Preliminaries

Proposition 2.1. *Problem* (1.2) *is uniquely solvable.*

Proof. Let us consider $u \in H_0^{1,2}(\Omega)$ a solution of Problem (1.2) with a null right-hand side term. So,

$$\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = 0 \quad \text{in } Q.$$

In addition u fulfils the boundary conditions

$$u\big|_{\partial Q - \Gamma_T} = 0.$$

Using Green formula, we have

$$\begin{aligned}\int_Q (\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u) u \, dt \, dx_1 \, dx_2 &= \int_{\partial Q} (\frac{1}{2}|u|^2 \nu_t - \partial_{x_1} u.u\nu_{x_1} - \partial_{x_2} u.u\nu_{x_2}) d\sigma \\ &\quad + \int_Q (|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2) dt \, dx_1 \, dx_2\end{aligned}$$

where ν_t, ν_{x_1}, ν_{x_2} are the components of the unit outward normal vector at ∂Q. Taking into account the boundary conditions, all the boundary integrals vanish except $\int_{\partial Q} |u|^2 \nu_t \, d\sigma$. We have

$$\int_{\partial Q} |u|^2 \nu_t d\sigma = \int_{\Gamma_T} |u|^2 \, dx_1 \, dx_2.$$

Then

$$\begin{aligned}
&\int_Q (\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u) u \, dt \, dx_1 \, dx_2 \\
&= \int_{\Gamma_T} \frac{1}{2} |u|^2 \, dx_1 \, dx_2 + \int_Q (|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2) dt \, dx_1 \, dx_2.
\end{aligned}$$

Consequently,

$$\int_Q (\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u) u \, dt \, dx_1 \, dx_2 = 0$$

yields

$$\int_Q (|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2) dt \, dx_1 \, dx_2 = 0,$$

because

$$\frac{1}{2} \int_{\Gamma_T} |u|^2 \, dx_1 \, dx_2 \geq 0.$$

This implies $|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2 = 0$ and consequently $\partial_{x_1}^2 u = \partial_{x_2}^2 u = 0$. Then, the hypothesis $\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = 0$ gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions imply that $u = 0$ in Q. This proves the uniqueness of the solution of Problem (1.2). □

Remark 2.2. In the sequel, we will be interested only by the question of the existence of the solution of Problem (1.2).

The following result is well known (see, for example, [11])

Lemma 2.3. *Let $D(0,1)$ be the unit disc of $\mathbb{R}^2$. Then, the Laplace operator $\Delta : H^2(D(0,1)) \cap H_0^1(D(0,1)) \to L^2(D(0,1))$ is an isomorphism. Moreover, there exists a constant $C > 0$ such that*

$$\|v\|_{H^2(D(0,1))} \leq C \|\Delta v\|_{L^2(D(0,1))}, \, \forall v \in H^2(D(0,1)).$$

In the above lemma, H^2 and H_0^1 are the usual Sobolev spaces defined, for instance, in Lions-Magenes [11]. In section 3, we will need the following result.

Lemma 2.4. *Let $t \in]\alpha_n, T[$, where $(\alpha_n)_n$ is a decreasing sequence to zero. Then, there exists a constant $C > 0$ independent of n such that for each $u_n \in H^2(\Omega_t)$, we have*

(a) $\|\partial_{x_1} u_n\|_{L^2(\Omega_t)}^2 \leq C \varphi^2(t) \|\Delta u_n\|_{L^2(\Omega_t)}^2$,
(b) $\|\partial_{x_2} u_n\|_{L^2(\Omega_t)}^2 \leq C \varphi^2(t) \|\Delta u_n\|_{L^2(\Omega_t)}^2$.

Proof. It is a direct consequence of Lemma 2.3. Indeed, let $t \in]\alpha_n, T[$ and define the following change of variables

$$\begin{aligned}
D(0,1) &\to \Omega_t \\
(x_1, x_2) &\mapsto (\varphi(t) x_1, h(t) \varphi(t) x_2) = (x_1', x_2').
\end{aligned}$$

Set

$$v(x_1,x_2)=u_n(\varphi(t)x_1,h(t)\varphi(t)x_2),$$

then if $v\in H^2(D(0,1))$, u_n belongs to $H^2(\Omega_t)$.

(a) We have

$$\begin{aligned}
\|\partial_{x_1}v\|^2_{L^2(D(0,1))}&=\int_{D(0,1)}(\partial_{x_1}v)^2(x_1,x_2)\,dx_1\,dx_2\\
&=\int_{\Omega_t}(\partial_{x_1'}u_n)^2(x_1',x_2')\varphi^2(t)\frac{1}{h(t)\varphi^2(t)}dx_1'dx_2'\\
&=\frac{1}{h(t)}\int_{\Omega_t}(\partial_{x_1'}u_n)^2(x_1',x_2')\,dx_1'\,dx_2'\\
&=\frac{1}{h(t)}\|\partial_{x_1'}u_n\|^2_{L^2(\Omega_t)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\Delta v\|^2_{L^2(D(0,1))}&=\int_{D(0,1)}[(\partial^2_{x_1}v+\partial^2_{x_2}v)(x_1,x_2)]^2\,dx_1\,dx_2\\
&=\int_{\Omega_t}(\varphi^2(t)\partial^2_{x_1'}u_n+(h\varphi)^2(t)\partial^2_{x_2'}u_n)^2(x_1',x_2')\ \frac{dx_1'dx_2'}{(h\varphi^2)(t)}\\
&=\frac{\varphi^2(t)}{h(t)}\int_{\Omega_t}(\partial^2_{x_1'}u_n+h^2(t)\partial^2_{x_2'}u_n)^2(x_1',x_2')\,dx_1'dx_2'\\
&\le\frac{1}{\delta}\varphi^2(t)\|\Delta u_n\|^2_{L^2(\Omega_t)},
\end{aligned}$$

where δ is the constant which appears in (1.1). Using Lemma 2.3 and the condition (1.1), we obtain the desired inequality.

(b) We have

$$\begin{aligned}
\|\partial_{x_2}v\|^2_{L^2(D(0,1))}&=\int_{D(0,1)}(\partial_{x_2}v)^2(x_1,x_2)\,dx_1\,dx_2\\
&=\int_{\Omega_t}(\partial_{x_2'}u_n)^2(x_1',x_2')h^2(t)\varphi^2(t)\frac{1}{h(t)\varphi^2(t)}dx_1'dx_2'\\
&=h(t)\int_{\Omega_t}(\partial_{x_2'}u_n)^2(x_1',x_2')\,dx_1'\,dx_2'\\
&=h(t)\|\partial_{x_2'}u_n\|^2_{L^2(\Omega_t)}.
\end{aligned}$$

On the other hand,

$$\|\Delta v\|^2_{L^2(D(0,1))}\le\frac{1}{\delta}\varphi^2(t)\|\Delta u_n\|^2_{L^2(\Omega_t)}.$$

Using the inequality

$$\|\partial_{x_2}v\|^2_{L^2(D(0,1))}\le C\|\Delta v\|^2_{L^2(D(0,1))}$$

of Lemma 2.3 and condition (1.1), we obtain the desired inequality

$$\|\partial_{x_2'}u_n\|^2_{L^2(\Omega_t)}\le C\varphi^2(t)\|\Delta u_n\|^2_{L^2(\Omega_t)}.$$

$\square$

3. Local in time result

3.1. Case of a truncated domain Q_α. In this subsection, we replace Q by Q_α

$$Q_\alpha = \{(t, x_1, x_2) \in \mathbb{R}^3 : \frac{1}{\alpha} < t < T, 0 \leq \frac{x_1^2}{\varphi^2(t)} + \frac{x_2^2}{h^2(t)\varphi^2(t)} < 1\}$$

with $\alpha > 0$.

Theorem 3.1. *The problem*

$$\begin{gathered} \partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = f \in L^2(Q_\alpha), \\ u\big|_{\partial Q_\alpha - \Gamma_T} = 0, \end{gathered} \tag{3.1}$$

admits a unique solution $u \in H^{1,2}(Q_\alpha)$.

Proof. The change of variables

$$(t, x_1, x_2) \mapsto (t, y_1, y_2) = (t, \frac{x_1}{\varphi(t)}, \frac{x_2}{h(t)\varphi(t)})$$

transforms Q_α into the cylinder $P_\alpha =]\frac{1}{\alpha}, T[\times D(\frac{1}{\alpha}, 1)$, where $D(\frac{1}{\alpha}, 1)$ is the unit disk centered on $(\frac{1}{\alpha}, 0, 0)$. Putting $u(t, x_1, x_2) = v(t, y_1, y_2)$ and $f(t, x_1, x_2) = g(t, y_1, y_2)$, then Problem (3.1) is transformed, in P_α into the variable-coefficient parabolic problem

$$\begin{gathered} \partial_t v - \frac{1}{\varphi^2(t)}\partial_{y_1}^2 v - \frac{1}{h^2(t)\varphi^2(t)}\partial_{y_2}^2 v - \frac{\varphi'(t)y_1}{\varphi(t)}\partial_{y_1} v - \frac{(h\varphi)'(t)y_2}{h(t)\varphi(t)}\partial_{y_2} v = g \\ v\big|_{\partial P_\alpha - \Gamma_T} = 0. \end{gathered}$$

This change of variables conserves the spaces $H^{1,2}$ and L^2. In other words

$$\begin{gathered} f \in L^2(Q_\alpha) \Rightarrow g \in L^2(P_\alpha) \\ u \in H^{1,2}(Q_\alpha) \Rightarrow v \in H^{1,2}(P_\alpha). \end{gathered}$$

□

Proposition 3.2. *The operator*

$$-[\frac{\varphi'(t)y_1}{\varphi(t)}\partial_{y_1} + \frac{(h\varphi)'(t)y_2}{h(t)\varphi(t)}\partial_{y_2}] : H_0^{1,2}(P_\alpha) \to L^2(P_\alpha)$$

is compact.

Proof. P_α has the horn property of Besov (see [3]). So, for $j = 1, 2$

$$\begin{array}{cccc} \partial_{y_j} & H_0^{1,2}(P_\alpha) & \to & H^{\frac{1}{2},1}(P_\alpha) \\ & v & \mapsto & \partial_{y_j} v, \end{array}$$

is continuous. Since P_α is bounded, the canonical injection is compact from $H^{\frac{1}{2},1}(P_\alpha)$ into $L^2(P_\alpha)$ (see for instance [3]), where

$$H^{1/2,1}(P_\alpha) = L^2\Big(\frac{1}{\alpha}, T; H^1\big(D(\frac{1}{\alpha}, 1)\big)\Big) \cap H^{1/2}\Big(\frac{1}{\alpha}, T; L^2\big(D(\frac{1}{\alpha}, 1)\big)\Big).$$

For the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces see for instance [11].

Consider the composition

$$\begin{array}{cccccc} \partial_{y_j} : & H_0^{1,2}(P_\alpha) & \to & H^{\frac{1}{2},1}(P_\alpha) & \to & L^2(P_\alpha) \\ & v & \mapsto & \partial_{y_j} v & \mapsto & \partial_{y_j} v, \end{array}$$

then ∂_{y_j} is a compact operator from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. Since $-\frac{\varphi'(t)}{\varphi(t)}$, $-\frac{(h\varphi)'(t)}{h(t)\varphi(t)}$ are bounded functions, the operators $-\frac{\varphi'(t)y_1}{\varphi(t)}\partial_{y_1}$, $-\frac{(h\varphi)'(t)y_2}{h(t)\varphi(t)}\partial_{y_2}$ are also compact from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. Consequently,

$$-\Big[\frac{\varphi'(t)y_1}{\varphi(t)}\partial_{y_1}+\frac{(h\varphi)'(t)y_2}{h(t)\varphi(t)}\partial_{y_2}\Big]$$

is compact from $H_0^{1,2}(P_\alpha)$ to $L^2(P_\alpha)$. □

So, to complete the proof of Theorem 3.1, it is sufficient to show that the operator

$$\partial_t-\frac{1}{\varphi^2(t)}\partial_{y_1}^2-\frac{1}{h^2(t)\varphi^2(t)}\partial_{y_2}^2$$

is an isomorphism from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$.

Lemma 3.3. *The operator*

$$\partial_t-\frac{1}{\varphi^2(t)}\partial_{y_1}^2-\frac{1}{h^2(t)\varphi^2(t)}\partial_{y_2}^2$$

is an isomorphism from $H_0^{1,2}(P_\alpha)$ to $L^2(P_\alpha)$.

Proof. Since the coefficients $\frac{1}{\varphi^2(t)}$ and $\frac{1}{h^2(t)\varphi^2(t)}$ are bounded in $\overline{P_\alpha}$, the optimal regularity is given by Ladyzhenskaya-Solonnikov-Ural'tseva [10]. □

We shall need the following result to justify the calculus of this section.

Lemma 3.4. *The space*

$$\{u\in H^4(P_\alpha):u\big|_{\partial_pP_\alpha}=0\}$$

is dense in the space

$$\{u\in H^{1,2}(P_\alpha):u\big|_{\partial_pP_\alpha}=0\}.$$

Here, ∂_pP_α is the parabolic boundary of P_α and H^4 stands for the usual Sobolev space defined, for instance, in Lions-Magenes [11].

The proof of the above lemma can be found in [7].

Remark 3.5. In Lemma 3.4, we can replace P_α by Q_α with the help of the change of variables defined above.

3.2. **Case of a conical type domain.** In this case, we define Q by

$$Q=\{(t,x_1,x_2)\in\mathbb{R}^3:0<t<T,0\le\frac{x_1^2}{\varphi^2(t)}+\frac{x_2^2}{h^2(t)\varphi^2(t)}<1\}$$

with

$$\varphi(0)=0,\quad \varphi(t)>0,\quad t\in]0,T]. \tag{3.2}$$

We assume that the functions h and φ satisfy conditions (1.1) and (1.3). For each $n\in\mathbb{N}^*$, we define Q_n by

$$Q_n=\{(t,x_1,x_2)\in\mathbb{R}^3:\frac{1}{n}<t<T,0\le\frac{x_1^2}{\varphi^2(t)}+\frac{x_2^2}{h^2(t)\varphi^2(t)}<1\}$$

and we denote $f_n=f_{/Q_n}$ and $u_n\in H^{1,2}(Q_n)$ the solution of Problem (1.2) in Q_n. Such a solution exists by Theorem 3.1.

Proposition 3.6. *There exists a constant K_1 independent of n such that*

$$\|u_n\|_{H^{1,2}(Q_n)} \le K_1 \|f_n\|_{L^2(Q_n)} \le K_1 \|f\|_{L^2(Q)},$$

where $\|u_n\|_{H^{1,2}(Q_n)} = \big(\|u_n\|^2_{H^1(Q_n)} + \sum_{i,j=1}^{2} \|\partial_{x_i}\partial_{x_j} u_n\|^2_{L^2(Q_n)}\big)^{1/2}$.

To prove Proposition 3.6, we need the following result which is a consequence of Lemma 2.4 and Grisvard-Looss [5] (see Theorem 2.2).

Lemma 3.7. *There exists a constant $C > 0$ independent of n such that*

$$\|\partial^2_{x_1} u_n\|^2_{L^2(Q_n)} + \|\partial^2_{x_2} u_n\|^2_{L^2(Q_n)} + \|\partial^2_{x_1 x_2} u_n\|^2_{L^2(Q_n)} \le C\|\Delta u_n\|^2_{L^2(Q_n)}.$$

Proof of Proposition 3.6. Let us denote the inner product in $L^2(Q_n)$ by $\langle \cdot, \cdot \rangle$, then we have

$$\begin{aligned}
\|f_n\|^2_{L^2(Q_n)} &= \langle \partial_t u_n - \Delta u_n, \partial_t u_n - \Delta u_n \rangle \\
&= \|\partial_t u_n\|^2_{L^2(Q_n)} + \|\Delta u_n\|^2_{L^2(Q_n)} - 2\langle \partial_t u_n, \Delta u_n \rangle
\end{aligned}$$

Estimation of $-2\langle \partial_t u_n, \Delta u_n \rangle$: We have

$$\partial_t u_n.\Delta u_n = \partial_{x_1}(\partial_t u_n \partial_{x_1} u_n) + \partial_{x_2}(\partial_t u_n \partial_{x_2} u_n) - \frac{1}{2}\partial_t[(\partial_{x_1} u_n)^2 + (\partial_{x_2} u_n)^2].$$

Then

$$\begin{aligned}
-2\langle \partial_t u_n, \Delta u_n \rangle &= -2\int_{Q_n} \partial_t u_n.\Delta u_n dt\, dx_1\, dx_2 \\
&= -2\int_{Q_n} [\partial_{x_1}(\partial_t u_n \partial_{x_1} u_n) + \partial_{x_2}(\partial_t u_n \partial_{x_2} u_n)] dt\, dx_1\, dx_2 \\
&\quad + \int_{Q_n} \partial_t[(\partial_{x_1} u_n)^2 + (\partial_{x_2} u_n)^2] dt\, dx_1\, dx_2 \\
&= \int_{\partial Q_n} [|\nabla u_n|^2 \nu_t - 2\partial_t u_n(\partial_{x_1} u_n \nu_{x_1} + \partial_{x_2} u_n \nu_{x_2})] d\sigma
\end{aligned}$$

where $\nu_t, \nu_{x_1}, \nu_{x_2}$ are the components of the unit outward normal vector at ∂Q_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Q_n where $t = \frac{1}{n}$, we have $u_n = 0$ and consequently $\partial_{x_1} u_n = \partial_{x_2} u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_1} = 0$, $\nu_{x_2} = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$A = \int_{\Gamma_T} |\nabla u_n|^2 \, dx_1 \, dx_2$$

is nonnegative. On the part of the boundary where $\frac{x_1^2}{\varphi^2(t)} + \frac{x_2^2}{h^2(t)\varphi^2(t)} = 1$, we have

$$\begin{gathered}
\nu_{x_1} = \frac{h(t)\cos\theta}{\sqrt{(\varphi'(t)h(t)\cos^2\theta + (h\varphi)'(t)\sin^2\theta)^2 + (h(t)\cos\theta)^2 + \sin^2\theta}}, \\
\nu_{x_2} = \frac{\sin\theta}{\sqrt{(\varphi'(t)h(t)\cos^2\theta + (h\varphi)'(t)\sin^2\theta)^2 + (h(t)\cos\theta)^2 + \sin^2\theta}}, \\
\nu_t = \frac{-(\varphi'(t)h(t)\cos^2\theta + (h\varphi)'(t)\sin^2\theta)}{\sqrt{(\varphi'(t)h(t)\cos^2\theta + (h\varphi)'(t)\sin^2\theta)^2 + (h(t)\cos\theta)^2 + \sin^2\theta}}
\end{gathered}$$

and $u_n(t,\varphi(t)\cos\theta, h(t)\varphi(t)\sin\theta)=0$. Differentiating with respect to t then with respect to θ we obtain

$$\begin{gathered}
\partial_t u_n = -\varphi'(t)\cos\theta.\partial_{x_1}u_n - (h\varphi)'(t)\sin\theta.\partial_{x_2}u_n,\\
\sin\theta.\partial_{x_1}u_n = h(t)\cos\theta.\partial_{x_2}u_n.
\end{gathered}$$

Consequently the corresponding boundary integral is

$$\begin{aligned}
J_n &= -2\int_0^{2\pi}\int_{1/n}^T \partial_t u_n.(h\varphi\cos\theta.\partial_{x_1}u_n + h\varphi\sin\theta.\partial_{x_2}u_n)\,dt\,d\theta\\
&\quad -\int_0^{2\pi}\int_{1/n}^T |\nabla u_n|^2((h\varphi)'\varphi\sin^2\theta + \varphi'(h\varphi)\cos^2\theta)\,dt\,d\theta\\
&= 2\int_0^{2\pi}\int_{1/n}^T \{(\varphi'\cos\theta.\partial_{x_1}u_n + (h\varphi)'\sin\theta.\partial_{x_2}u_n)\\
&\quad\times(h\varphi\cos\theta.\partial_{x_1}u_n + h\varphi\sin\theta.\partial_{x_2}u_n)\}\,dt\,d\theta\\
&\quad -\int_0^{2\pi}\int_{1/n}^T |\nabla u_n|^2((h\varphi)'\varphi\sin^2\theta + \varphi' h\varphi\cos^2\theta)\,dt\,d\theta\\
&= 2\int_0^{2\pi}\int_{1/n}^T |\nabla u_n|^2((h\varphi)'\varphi\sin^2\theta + \varphi' h\varphi\cos^2\theta)\,dt\,d\theta\\
&\quad -\int_0^{2\pi}\int_{1/n}^T |\nabla u_n|^2((h\varphi)'\varphi\sin^2\theta + \varphi' h\varphi\cos^2\theta)\,dt\,d\theta\\
&= \int_0^{2\pi}\int_{1/n}^T |\nabla u_n|^2((h\varphi)'\varphi\sin^2\theta + \varphi' h\varphi\cos^2\theta)\,dt\,d\theta.
\end{aligned}$$

Finally,

$$\begin{aligned}
-2\langle\partial_t u_n,\Delta u_n\rangle &= \int_0^{2\pi}\int_{1/n}^T |\nabla u_n|^2((h\varphi)'\varphi\sin^2\theta + \varphi' h\varphi\cos^2\theta)\,dt\,d\theta\\
&\quad + \int_{\Gamma_T}|\nabla u_n|^2(T,x_1,x_2)\,dx_1\,dx_2.
\end{aligned}\tag{3.3}$$

Lemma 3.8. *One has*

$$\begin{aligned}
-2\langle\partial_t u_n,\Delta u_n\rangle &= 2\int_{Q_n}(\frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n + \frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n)\Delta u_n dt\,dx_1\,dx_2\\
&\quad + \int_{\Gamma_T}|\nabla u_n|^2(T,x_1,x_2)\,dx_1\,dx_2.
\end{aligned}$$

Proof. For $\frac{1}{n}<t<T$, consider the following parametrization of the domain Ω_t

$$\begin{array}{lll}
(0,2\pi) & \to & \Omega_t\\
\theta & \to & (\varphi(t)\cos\theta, h(t)\varphi(t)\sin\theta) = (x_1,x_2).
\end{array}$$

Let us denote the inner product in $L^2(\Omega_t)$ by $\langle\cdot,\cdot\rangle$, and set

$$I_n = \big\langle \Delta u_n, \frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n + \frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n\big\rangle$$

then we have

$$I_n = \int_{\Omega_t}(\partial_{x_1}^2 u_n + \partial_{x_2}^2 u_n)(\frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n + \frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n)\,dx_1\,dx_2$$

$$\begin{aligned}
&= \int_{\Omega_t} (\frac{\varphi'}{\varphi} x_1 \partial_{x_1}^2 u_n \partial_{x_1} u_n + \frac{(h\varphi)'}{h\varphi} x_2 \partial_{x_2}^2 u_n \partial_{x_2} u_n)\, dx_1\, dx_2 \\
&\quad + \int_{\Omega_t} (\frac{\varphi'}{\varphi} x_1 \partial_{x_2}^2 u_n \partial_{x_1} u_n + \frac{(h\varphi)'}{h\varphi} x_2 \partial_{x_1}^2 u_n \partial_{x_2} u_n)\, dx_1\, dx_2.
\end{aligned}$$

Using Green formula, we obtain

$$\begin{aligned}
I_n &= \frac{1}{2} \int_{\Omega_t} (\frac{\varphi'}{\varphi} x_1 \partial_{x_1} (\partial_{x_1} u_n)^2 + \frac{(h\varphi)'}{h\varphi} x_2 \partial_{x_2} (\partial_{x_2} u_n)^2)\, dx_1\, dx_2 \\
&\quad + \int_{\Omega_t} (\frac{\varphi'}{\varphi} x_1 \partial_{x_2} (\partial_{x_2} u_n) \partial_{x_1} u_n + \frac{(h\varphi)'}{h\varphi} x_2 \partial_{x_1} (\partial_{x_1} u_n) \partial_{x_2} u_n)\, dx_1\, dx_2 \\
&= \frac{1}{2} \int_{\partial\Omega_t} (\frac{\varphi'}{\varphi} x_1 \nu_{x_1} (\partial_{x_1} u_n)^2 + \frac{(h\varphi)'}{h\varphi} x_2 \nu_{x_2} (\partial_{x_2} u_n)^2) d\sigma \\
&\quad - \frac{1}{2} \int_{\Omega_t} (\frac{\varphi'}{\varphi} (\partial_{x_1} u_n)^2 + \frac{(h\varphi)'}{h\varphi} (\partial_{x_2} u_n)^2)\, dx_1\, dx_2 \\
&\quad + \int_{\partial\Omega_t} (\frac{\varphi'}{\varphi} x_1 \nu_{x_2} + \frac{(h\varphi)'}{h\varphi} x_2 \nu_{x_1}) \partial_{x_1} u_n \partial_{x_2} u_n d\sigma \\
&\quad - \int_{\Omega_t} (\frac{\varphi'}{\varphi} x_1 \partial_{x_2} u_n \partial_{x_1 x_2}^2 u_n + \frac{(h\varphi)'}{h\varphi} x_2 \partial_{x_1} u_n \partial_{x_1 x_2}^2 u_n)\, dx_1\, dx_2
\end{aligned}$$

where ν_{x_1}, ν_{x_2} are the components of the unit outward normal vector at $\partial\Omega_t$. Then

$$\begin{aligned}
I_n &= \frac{1}{2} \int_{\partial\Omega_t} (\frac{\varphi'}{\varphi} x_1 \nu_{x_1} (\partial_{x_1} u_n)^2 + \frac{(h\varphi)'}{h\varphi} x_2 \nu_{x_2} (\partial_{x_2} u_n)^2) d\sigma \\
&\quad - \frac{1}{2} \int_{\Omega_t} (\frac{\varphi'}{\varphi} (\partial_{x_1} u_n)^2 + \frac{(h\varphi)'}{h\varphi} (\partial_{x_2} u_n)^2)\, dx_1\, dx_2 \\
&\quad + \int_{\partial\Omega_t} (\frac{\varphi'}{\varphi} x_1 \nu_{x_2} + \frac{(h\varphi)'}{h\varphi} x_2 \nu_{x_1}) \partial_{x_1} u_n \partial_{x_2} u_n d\sigma \\
&\quad - \frac{1}{2} \int_{\Omega_t} (\frac{\varphi'}{\varphi} x_1 \partial_{x_1} (\partial_{x_2} u_n)^2 + \frac{(h\varphi)'}{h\varphi} x_2 \partial_{x_2} (\partial_{x_1} u_n)^2)\, dx_1\, dx_2.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_n &= \frac{1}{2} \int_{\partial\Omega_t} \Big(\frac{\varphi'}{\varphi} x_1 \nu_{x_1} (\partial_{x_1} u_n)^2 + \frac{(h\varphi)'}{h\varphi} x_2 \nu_{x_2} (\partial_{x_2} u_n)^2\Big) d\sigma \\
&\quad - \frac{1}{2} \int_{\Omega_t} \Big(\frac{\varphi'}{\varphi} (\partial_{x_1} u_n)^2 + \frac{(h\varphi)'}{h\varphi} (\partial_{x_2} u_n)^2\Big)\, dx_1\, dx_2 \\
&\quad + \int_{\partial\Omega_t} \Big(\frac{\varphi'}{\varphi} x_1 \nu_{x_2} + \frac{(h\varphi)'}{h\varphi} x_2 \nu_{x_1}\Big) \partial_{x_1} u_n \partial_{x_2} u_n d\sigma \\
&\quad - \frac{1}{2} \int_{\partial\Omega_t} (\frac{\varphi'}{\varphi} x_1 \nu_{x_1} (\partial_{x_2} u_n)^2 + \frac{(h\varphi)'}{h\varphi} x_2 \nu_{x_2} (\partial_{x_1} u_n)^2)\, dx_1\, dx_2 \\
&\quad + \frac{1}{2} \int_{\Omega_t} \Big(\frac{\varphi'}{\varphi} (\partial_{x_1} u_n)^2 + \frac{(h\varphi)'}{h\varphi} (\partial_{x_2} u_n)^2\Big)\, dx_1\, dx_2
\end{aligned}$$

and then

$$I_n = \frac{1}{2} \int_{\partial\Omega_t} \Big(\frac{\varphi'}{\varphi} x_1 \nu_{x_1} (\partial_{x_1} u_n)^2 + \frac{(h\varphi)'}{h\varphi} x_2 \nu_{x_2} (\partial_{x_2} u_n)^2\Big) d\sigma$$

$$
\begin{aligned}
&+\int_{\partial\Omega_t}(\frac{\varphi'}{\varphi}x_1\nu_{x_2}+\frac{(h\varphi)'}{h\varphi}x_2\nu_{x_1})\partial_{x_1}u_n\partial_{x_2}u_n d\sigma\\
&-\frac{1}{2}\int_{\partial\Omega_t}\Big(\frac{\varphi'}{\varphi}x_1\nu_{x_1}(\partial_{x_2}u_n)^2+\frac{(h\varphi)'}{h\varphi}x_2\nu_{x_2}(\partial_{x_1}u_n)^2\Big)\,dx_1\,dx_2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
I_n&=\frac{1}{2}\int_0^{2\pi}\Big(\frac{\varphi'}{\varphi}\varphi h\varphi(\cos\theta.\partial_{x_1}u_n)^2+\frac{(h\varphi)'}{h\varphi}\varphi h\varphi(\sin\theta.\partial_{x_2}u_n)^2\Big)d\theta\\
&\quad+\int_0^{2\pi}(\frac{\varphi'}{\varphi}\varphi^2+\frac{(h\varphi)'}{h\varphi}(h\varphi)^2)\sin\theta\cos\theta.\partial_{x_1}u_n\partial_{x_2}u_n d\theta\\
&\quad-\frac{1}{2}\int_0^{2\pi}\Big(\frac{\varphi'}{\varphi}\varphi h\varphi(\cos\theta.\partial_{x_2}u_n)^2+\frac{(h\varphi)'}{h\varphi}\varphi h\varphi(\sin\theta.\partial_{x_1}u_n)^2\Big)d\theta\\
&=\frac{1}{2}\int_0^{2\pi}\Big(\varphi'h\varphi\Big(\cos\theta.\partial_{x_1}u_n\Big)^2+\varphi(h\varphi)'(\sin\theta.\partial_{x_2}u_n)^2\Big)d\theta\\
&\quad+\int_0^{2\pi}\Big(\varphi'\varphi+(h\varphi)'h\varphi\Big)\sin\theta\cos\theta.\partial_{x_1}u_n\partial_{x_2}u_n d\theta\\
&\quad-\frac{1}{2}\int_0^{2\pi}\Big(\varphi'h\varphi(\cos\theta.\partial_{x_2}u_n)^2+\varphi(h\varphi)'(\sin\theta.\partial_{x_1}u_n)^2\Big)d\theta.
\end{aligned}
$$

The boundary condition $u_n(t,\varphi(t)\cos\theta,h(t)\varphi(t)\sin\theta)=0$ leads to

$$
\sin\theta.\partial_{x_1}u_n=h(t)\cos\theta.\partial_{x_2}u_n;
$$

then

$$
\sin\theta\cos\theta.\partial_{x_1}u_n\partial_{x_2}u_n=h(t)(\cos\theta.\partial_{x_2}u_n)^2
$$

and

$$
h(t)\sin\theta\cos\theta.\partial_{x_1}u_n\partial_{x_2}u_n=(\sin\theta.\partial_{x_1}u_n)^2.
$$

Consequently,

$$
\begin{aligned}
I_n&=\frac{1}{2}\int_0^{2\pi}\Big(\varphi'h\varphi(\cos\theta.\partial_{x_1}u_n)^2+\varphi(h\varphi)'(\sin\theta.\partial_{x_2}u_n)^2\Big)d\theta\\
&\quad+\int_0^{2\pi}\Big(\varphi'h\varphi(\cos\theta.\partial_{x_2}u_n)^2+\varphi(h\varphi)'(\sin\theta.\partial_{x_1}u_n)^2\Big)d\theta\\
&\quad-\frac{1}{2}\int_0^{2\pi}(\varphi'h\varphi(\cos\theta.\partial_{x_2}u_n)^2+\varphi(h\varphi)'(\sin\theta.\partial_{x_1}u_n)^2)d\theta\\
&=\frac{1}{2}\int_0^{2\pi}\Big(\varphi'h\varphi(\cos\theta.\partial_{x_1}u_n)^2+\varphi(h\varphi)'(\sin\theta.\partial_{x_2}u_n)^2\Big)d\theta\\
&\quad+\frac{1}{2}\int_0^{2\pi}\Big(\varphi'h\varphi(\cos\theta.\partial_{x_2}u_n)^2+\varphi(h\varphi)'(\sin\theta.\partial_{x_1}u_n)^2\Big)d\theta\\
&=\frac{1}{2}\int_0^{2\pi}\Big\{\varphi'h\varphi(\cos\theta.\partial_{x_1}u_n)^2+\varphi(h\varphi)'(\sin\theta.\partial_{x_2}u_n)^2\\
&\quad+\varphi'h\varphi(\cos\theta.\partial_{x_2}u_n)^2+\varphi(h\varphi)'(\sin\theta.\partial_{x_1}u_n)^2\Big\}d\theta\\
&=\frac{1}{2}\int_0^{2\pi}[(\partial_{x_1}u_n)^2+(\partial_{x_2}u_n)^2](\varphi(h\varphi)'\sin^2\theta+\varphi'h\varphi\cos^2\theta)d\theta.
\end{aligned}
$$

So

$$I_n = \frac{1}{2}\int_0^{2\pi} |\nabla u_n|^2(\varphi(h\varphi)'\sin^2\theta + \varphi' h\varphi\cos^2\theta)d\theta$$

and

$$\begin{aligned}&\int_{1/n}^{T}\int_0^{2\pi} |\nabla u_n|^2(\varphi(h\varphi)'\sin^2\theta + \varphi' h\varphi\cos^2\theta)\,dt\,d\theta \\ &= 2\int_{Q_n}(\frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n + \frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n)\Delta u_n dt\,dx_1\,dx_2.\end{aligned}$$

Finally, by (3.3), it follows that

$$\begin{aligned}-2\langle\partial_t u_n, \Delta u_n\rangle &= 2\int_{Q_n}\Big(\frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n + \frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n\Big)\Delta u_n dt\,dx_1\,dx_2 \\ &\quad + \int_{\Gamma_T}|\nabla u_n|^2(T,x_1,x_2)\,dx_1\,dx_2.\end{aligned}$$

□

Now, we continue the proof of Proposition 3.6. We have

$$\begin{aligned}&|\int_{Q_n}\Big(\frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n + \frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n\Big)\Delta u_n dt\,dx_1\,dx_2| \\ &\le \|\Delta u_n\|_{L^2(Q_n)}\|\frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n\|_{L^2(Q_n)} + \|\Delta u_n\|_{L^2(Q_n)}\|\frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n\|_{L^2(Q_n)},\end{aligned}$$

but Lemma 2.4 yields

$$\begin{aligned}\|\frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n\|^2_{L^2(Q_n)} &= \int_{1/n}^T \varphi'^2(t)\int_{\Omega_t}(\frac{x_1}{\varphi(t)})^2(\partial_{x_1}u_n)^2 dt\,dx_1\,dx_2 \\ &\le \int_{1/n}^T \varphi'^2(t)\int_{\Omega_t}(\partial_{x_1}u_n)^2 dt\,dx_1\,dx_2 \\ &\le C^2\int_{1/n}^T(\varphi(t)\varphi'(t))^2\int_{\Omega_t}(\Delta u_n)^2 dt\,dx_1\,dx_2 \\ &\le C^2\epsilon^2\|\Delta u_n\|^2_{L^2(Q_n)},\end{aligned}$$

since $(\varphi(t)\varphi'(t)) \le \epsilon$. Similarly, we have

$$\|\frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n\|^2_{L^2(Q_n)} \le C^2\epsilon^2\|\Delta u_n\|^2_{L^2(Q_n)}.$$

Then

$$|\int_{Q_n}\Big(\frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n + \frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n\Big)\Delta u_n dt\,dx_1\,dx_2| \le 2C\epsilon\|\Delta u_n\|^2_{L^2(Q_n)}.$$

Therefore, Lemma 3.8 shows that

$$\begin{aligned}|2\langle\partial_t u_n, \Delta u_n\rangle| &\ge -2\Big|\int_{Q_n}\Big(\frac{\varphi'}{\varphi}x_1\partial_{x_1}u_n + \frac{(h\varphi)'}{h\varphi}x_2\partial_{x_2}u_n\Big)\Delta u_n dt\,dx_1\,dx_2\Big| \\ &\quad + \int_{\Gamma_T}|\nabla u_n|^2(T,x_1,x_2)\,dx_1\,dx_2 \\ &\ge -4C\epsilon\|\Delta u_n\|^2_{L^2(Q_n)}.\end{aligned}$$

Hence

$$\begin{aligned}\|f_n\|^2_{L^2(Q_n)} &= \|\partial_t u_n\|^2_{L^2(Q_n)} + \|\Delta u_n\|^2_{L^2(Q_n)} - 2\langle \partial_t u_n, \Delta u_n\rangle \\ &\geq \|\partial_t u_n\|^2_{L^2(Q_n)} + (1-4C\epsilon)\|\Delta u_n\|^2_{L^2(Q_n)}.\end{aligned}$$

Then, it is sufficient to choose ϵ such that $1-4C\epsilon > 0$ to get a constant $K_0 > 0$ independent of n such that

$$\|f_n\|_{L^2(Q_n)} \geq K_0\|u_n\|_{H^{1,2}(Q_n)},$$

and since

$$\|f_n\|_{L^2(Q_n)} \leq \|f\|_{L^2(Q_n)},$$

there exists a constant $K_1 > 0$, independent of n satisfying

$$\|u_n\|_{H^{1,2}(Q_n)} \leq K_1\|f_n\|_{L^2(Q_n)} \leq K_1\|f\|_{L^2(Q)}.$$

This completes the proof of Proposition 3.6.

Passage to the limit. We are now in position to prove the main result of this work.

Theorem 3.9. *Assume that the functions h and φ verify the conditions* (1.1), (1.3) *and* (3.2). *Then, for T small enough, Problem* (1.2) *admits a unique solution $u \in H^{1,2}(Q)$.*

Proof. Choose a sequence Q_n $n = 1, 2, \dots,$ of truncated conical domains (see subsection 3.2) such that $Q_n \subseteq Q$. Then we have $Q_n \to Q$, as $n \to \infty$.

Consider the solution $u_n \in H^{1,2}(Q_n)$ of the Cauchy-Dirichlet problem

$$\begin{gathered}\partial_t u_n - \partial^2_{x_1} u_n - \partial^2_{x_2} u_n = f \quad \text{in } Q_n \\ u_n\big|_{\partial Q_n - \Gamma_T} = 0,\end{gathered}$$

where Γ_T is the part of the boundary of Q_n where $t = T$. Such a solution u_n exists by Theorem 3.1. Let $\widetilde{u}_n$ the 0-extension of u_n to Q. By Proposition 3.6, we know that there exists a constant C such that

$$\|\widetilde{u}_n\|_{L^2(Q)} + \|\partial_t \widetilde{u}_n\|_{L^2(Q)} + \sum_{i,j=0,\, 1\leq i+j\leq 2}^{2} \|\partial^j_{x_1}\partial^j_{x_2}\widetilde{u}_n\|_{L^2(Q)} \leq C\|f\|_{L^2(Q)}.$$

This means that $\widetilde{u}_n$, $\partial_t\widetilde{u}_n$, $\partial^j_{x_1}\partial^j_{x_2}\widetilde{u}_n$ for $1 \leq i+j \leq 2$ are bounded functions in $L^2(Q)$. So for a suitable increasing sequence of integers n_k, $k = 1, 2, \dots,$ there exist functions $u, v, v_{i,j}$, $1 \leq i+j \leq 2$ in $L^2(Q)$ such that

$$\begin{gathered}\widetilde{u}_{n_k} \rightharpoonup u \quad \text{weakly in } L^2(Q) \text{ as } k \to \infty \\ \partial_t\widetilde{u}_{n_k} \rightharpoonup v \quad \text{weakly in } L^2(Q) \text{ as } k \to \infty \\ \partial^j_{x_1}\partial^j_{x_2}\widetilde{u}_{n_k} \rightharpoonup v_{i,j} \quad \text{weakly in } L^2(Q) \text{ as } k \to \infty,\ 1 \leq i+j \leq 2.\end{gathered}$$

Clearly,

$$v = \partial_t u, \quad v_{i,j} = \partial^i_{x_1}\partial^j_{x_2} u, \quad 1 \leq i+j \leq 2$$

in the sense of distributions in Q and so in $L^2(Q)$. So, $u \in H^{1,2}(Q)$ and

$$\partial_t u - \partial^2_{x_1} u - \partial^2_{x_2} u = f \quad \text{in } Q.$$

On the other hand, the solution u satisfies the boundary conditions $u\big|_{\partial Q - \Gamma_T} = 0$ since $u\big|_{Q_n} = u_n$ for all $n \in \mathbb{N}^*$. This proves the existence of a solution to Problem (1.2). $\square$

4. Global in time result

Assume that Q satisfies (3.2). In the case where T is not in the neighborhood of zero, we set $Q = D_1 \cup D_2 \cup \Gamma_{T_1}$ where

$$D_1 = \{(t, x_1, x_2) \in \mathbb{R}^3 : 0 < t < T_1,\ 0 \le \frac{x_1^2}{\varphi^2(t)} + \frac{x_2^2}{(h\varphi)^2(t)} < 1\}$$
$$D_2 = \{(t, x_1, x_2) \in \mathbb{R}^3 : T_1 < t < T,\ 0 \le \frac{x_1^2}{\varphi^2(t)} + \frac{x_2^2}{(h\varphi)^2(t)} < 1\}$$
$$\Gamma_{T_1} = \{(T_1, x_1, x_2) \in \mathbb{R}^3 :\ 0 \le \frac{x_1^2}{\varphi^2(T_1)} + \frac{x_2^2}{(h\varphi)^2(T_1)} < 1\}$$

with T_1 small enough.

In the sequel, f stands for an arbitrary fixed element of $L^2(Q)$ and $f_i = f\big|_{D_i}$, $i = 1, 2$.

Theorem 3.9 applied to the conical domain D_1, shows that there exists a unique solution $u_1 \in H^{1,2}(D_1)$ of the problem

$$\begin{gathered} \partial_t u_1 - \partial_{x_1}^2 u_1 - \partial_{x_2}^2 u_1 = f_1, \quad f_1 \in L^2(D_1) \\ u_1\big|_{\partial D_1 - \Gamma_{T_1}} = 0. \end{gathered} \tag{4.1}$$

Hereafter, we denote the trace $u_{1/\Gamma_{T_1}}$ by ψ which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $u_1 \in H^{1,2}(D_1)$ (see [11]).

Now, consider the following problem in D_2,

$$\begin{gathered} \partial_t u_2 - \partial_{x_1}^2 u_2 - \partial_{x_2}^2 u_2 = f_2 \quad f_2 \in L^2(D_2) \\ u_{2/\Gamma_{T_1}} = \psi \\ u_2\big|_{\partial D_2 - (\Gamma_{T_1} \cup \Gamma_T)} = 0 \end{gathered} \tag{4.2}$$

We use the following result, which is a consequence of [11, Theorem 4.3, Vol. 2], to solve Problem (4.2).

Proposition 4.1. *Let Q be the cylinder $]0, T[\times D(0, 1)$, $f \in L^2(Q)$ and $\psi \in H^1(\gamma_0)$. Then, the problem*

$$\begin{gathered} \partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = f \text{ in } Q \\ u\big|_{\gamma_0} = \psi \\ u\big|_{\gamma_0 \cup \gamma_1} = 0 \end{gathered}$$

where $\gamma_0 = \{0\} \times D(0, 1)$, $\gamma_1 =]0, T[\times \partial D(0, 1)$, admits a (unique) solution $u \in H^{1,2}(Q)$.

Remark 4.2. In the application of [11, Theorem 4.3, Vol.2], we can observe that there are no compatibility conditions to satisfy because $\partial_x \psi$ is only in $L^2(\gamma_0)$.

Thanks to the transformation

$$(t, x_1, x_2) \mapsto (t, y_1, y_2) = (t, \varphi(t)x_1, (h\varphi)(t)x_2),$$

we deduce the following result.

Proposition 4.3. *Problem (4.2) admits a (unique) solution $u_2 \in H^{1,2}(D_2)$.*

So, the function u defined by

$$u = \begin{cases} u_1 & \text{in } D_1 \\ u_2 & \text{in } D_2 \end{cases}$$

is the (unique) solution of Problem (1.2) for an arbitrary T. Our second main result is as follows.

Theorem 4.4. *Assume that the functions h and φ verify conditions* (1.1), (1.3) *and* (3.2). *Then, Problem* (1.2) *admits a unique solution $u \in H^{1,2}(Q)$.*

Acknowledgments. The author is thankful to Professor B. K. Sadallah (Ecole Normale Supérieure de Kouba, Algeria) for his help during the preparation of this work, and to the anonymous referees for their careful reading of a previous version of the manuscript, which led to a substantial improvement of this manuscript.

References

[1] Yu. A. Alkhutov; *L_p-Solubility of the Dirichlet problem for the heat equation in non-cylindrical domains*, Sbornik: Mathematics 193:9 (2002), 1243–1279.

[2] Yu. A. Alkhutov; *L_p-Estimates of solutions of the Dirichlet problem for the heat equation in a ball*, Journ. Math. Sc., Vol. 142, No.3, (2007), 2021-2032.

[3] V. Besov; *The continuation of function in L_p^1 and W_p^1*, Proc. Steklov Inst. Math. 89 (1967), 5 - 17.

[4] S. P. Degtyarev; *The solvability of the first initial-boundary problem for parabolic and degenerate parabolic equations in domains with a conical point*, Sbornik Mathematics 201 (7) (2010) 999-1028.

[5] P. Grisvard, G. Looss; *Problèmes aux limites unilatéraux dans des domaines non réguliers*, Journées Equations aux Dérivées Partielles, (1976), 1-26.

[6] P. Grisvard; *Elliptic Problems in Nonsmooth Domains*, Monographs and Studies in Mathematics 24, Pitman, Boston, 1985.

[7] A. Kheloufi, R. Labbas, B .K. Sadallah; *On the resolution of a parabolic equation in a nonregular domain of $\mathbb{R}^3$*, Differ. Equat. Appl. 2 (2) (2010) 251-263.

[8] R. Labbas, A. Medeghri, B.-K. Sadallah; *On a parabolic equation in a triangular domain*, Appl. Math. Comput. 130(2002), 511-523.

[9] R. Labbas, A. Medeghri, B.-K. Sadallah; *An L^p approach for the study of degenerate parabolic equation*, Electron. J. Diff. Equ., vol 2005 (2005), No. 36, 1-20.

[10] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva; *Linear and Quasi-Linear Equations of Parabolic Type*, A.M.S., Providence, Rhode Island, 1968.

[11] J. L. Lions, E. Magenes; *Problèmes aux Limites Non Homogènes et Applications*, 1, 2, Dunod, Paris, 1968.

[12] B. K. Sadallah; *Etude d'un problème 2m-parabolique dans des domaines plan non rectangulaires*, Boll. Un. Mat. Ital., (5), 2-B (1983), 51-112.

[13] B. K. Sadallah; *Study of a parabolic problem in a conical domain*, to appear in Mathematical Journal of Okayama University.

Arezki Kheloufi
Department of Technology, Faculty of Technology, Béjaia University, 6000 Béjaia, Algeria
E-mail address: arezkinet2000@yahoo.fr