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SOLVABILITY OF (K,N-K) CONJUGATE BOUNDARY-VALUE PROBLEMS AT RESONANCE

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ABSTRACT. Using the coincidence degree theory due to Mawhin and constructing suitable operators, we prove the existence of solutions for $(k, n-k)$ conjugate boundary-value problems at resonance.

1. Introduction

The existence of solutions for $(k, n-k)$ conjugate boundary-value problems at non-resonance has been studied in many papers (see [1, 2, 3, 4, 7, 8, 11, 12, 13, 14, 17, 22, 26, 27, 28, 31, 32, 33]). For example, using fixed point theorem in a cone, Jiang [13] obtained the existence of positive solutions for $(k, n-k)$ conjugate boundary-value problem

$$\begin{gathered}(-1)^{n-k}y^{(n)}(t) = f(t, y(t)), \quad 0 < t < 1, \\ y^{(i)}(0) = y^{(j)}(1) = 0, \quad 0 \le i \le k-1,\ 0 \le j \le n-k-1,\end{gathered}$$

where $f(t, y)$ may be singular at $y = 0$, $t = 0$, $t = 1$. By using fixed point index theory, Zhang and Sun [33] studied the existence of positive solutions for the problem

$$(-1)^{n-k}\varphi^{(n)}(x) = h(x)f(\varphi(x)), \quad 0 < x < 1,\ n \ge 2,\ 1 \le k \le n-1,$$

subject to the boundary conditions

$$\varphi(0) = \sum_{i=1}^{m-2} a_i\varphi(\xi_i), \quad \varphi^{(i)}(0) = \varphi^{(j)}(1) = 0, \quad 1 \le i \le k-1,\ 0 \le j \le n-k-1,$$

and

$$\varphi(1) = \sum_{i=1}^{m-2} a_i\varphi(\xi_i), \quad \varphi^{(i)}(0) = \varphi^{(j)}(1) = 0,\ 0 \le i \le k-1, \quad 1 \le j \le n-k-1,$$

respectively. Solvability of boundary-value problems at resonance has been investigated by many authors (see [5, 6, 9, 10, 15, 16, 18, 19, 20, 21, 23, 25, 29, 30, 34]). For example, in [5], using the coincidence degree theory due to Mawhin, Du, Lin

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and Ge investigated the existence of solutions for the $(n-1,1)$ boundary-value problems at resonance

$$x^{(n)}(t)=f(t,x(t),x'(t),\dots,x^{(n-1)}(t))+e(t),\quad \text{a.e. } t\in(0,1),$$
$$x(0)=\sum_{i=1}^{m-2}\alpha_i x(\xi_i),\quad x'(0)=x''(0)=\dots=x^{(n-2)}(0)=0,\quad x(1)=x(\eta).$$

Motivated by the results in [5, 13, 33], in this paper, we discuss the existence of solutions for the $(k,n-k)$ conjugate boundary-value problem at resonance

$$(-1)^{n-k}y^{(n)}(t)=f(t,y(t),y'(t),\dots,y^{(n-1)}(t))+\varepsilon(t),\quad \text{a.e. } t\in[0,1], \tag{1.1}$$

$$\begin{gathered} y^{(i)}(0)=y^{(j)}(1)=0,\quad 0\le i\le k-1,\ 0\le j\le n-k-2,\\ y^{(n-1)}(1)=\sum_{i=1}^{m}\alpha_i y^{(n-1)}(\xi_i), \end{gathered} \tag{1.2}$$

where $1\le k\le n-1$, $0<\xi_1<\xi_2<\dots<\xi_m<1$.

As far as we know, this is the first paper to study the existence of solutions for $(k,n-k)$ boundary-value problems at resonance with $1\le k\le n-1$.

In this paper, we assume the following conditions:

(H1) $0<\xi_1<\xi_2<\dots<\xi_m<1$, $\sum_{i=1}^m\alpha_i=1$, $\sum_{i=1}^m\alpha_i\xi_i\neq 1$.

(H2) $\varepsilon(t)\in L^\infty[0,1]$, $f:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ satisfies Carathéodory conditions; i.e., $f(\cdot,x)$ is measurable for each fixed $x\in\mathbb{R}^n$, $f(t,\cdot)$ is continuous for a.e. $t\in[0,1]$, and for each $r>0$, there exists $\Phi_r\in L^\infty[0,1]$ such that $|f(t,x_1,x_2,\dots,x_n)|\le\Phi_r(t)$ for all $|x_i|\le r$, $i=1,2,\dots,n$, a.e. $t\in[0,1]$.

2. Preliminaries

First, we introduce some notation and state a theorem to be used later. For more details see [24].

Let X and Y be real Banach spaces and $L:\operatorname{dom}L\subset X\to Y$ be a Fredholm operator with index zero, $P:X\to X$, $Q:Y\to Y$ be projectors such that

$$\operatorname{Im}P=\ker L,\quad \ker Q=\operatorname{Im}L,\quad X=\ker L\oplus\ker P,\quad Y=\operatorname{Im}L\oplus\operatorname{Im}Q.$$

It follows that

$$L\big|_{\operatorname{dom}L\cap\ker P}:\operatorname{dom}L\cap\ker P\to\operatorname{Im}L$$

is invertible. We denote the inverse by K_P.

Assume that Ω is an open bounded subset of X, $\operatorname{dom}L\cap\overline{\Omega}\neq\emptyset$, the map $N:X\to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N:\overline{\Omega}\to X$ is compact.

Theorem 2.1 ([24]). *Let $L:\operatorname{dom}L\subset X\to Y$ be a Fredholm operator of index zero and $N:X\to Y$ L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

1. *$Lx\neq\lambda Nx$ for every $(x,\lambda)\in[(\operatorname{dom}L\setminus\ker L)\cap\partial\Omega]\times(0,\ 1)$;*
2. *$Nx\notin\operatorname{Im}L$ for every $x\in\ker L\cap\partial\Omega$;*
3. *$\deg(QN|_{\ker L},\Omega\cap\ker L,0)\neq 0$, where $Q:Y\to Y$ is a projection such that $\operatorname{Im}L=\ker Q$.*

Then the equation $Lx=Nx$ has at least one solution in $\operatorname{dom}L\cap\overline{\Omega}$.

Take $X = C^{n-1}[0,1]$ with norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \dots, \|u^{(n-1)}\|_\infty\}$, where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$, $Y = L^1[0,1]$ with norm $\|x\|_1 = \int_0^1 |x(t)|dt$. Define the operator $Ly(t) = (-1)^{n-k} y^{(n)}(t)$ with

$$\begin{gathered}\operatorname{dom} L = \{y \in X : y^{(n)} \in Y,\ y^{(i)}(0) = y^{(j)}(1) = 0,\ 0 \le i \le k-1,\\ 0 \le j \le n-k-2,\ y^{(n-1)}(1) = \sum_{i=1}^m \alpha_i y^{(n-1)}(\xi_i)\}.\end{gathered}$$

Let $N : X \to Y$ be defined as

$$Ny(t) = f(t, y(t), y'(t), \dots, y^{(n-1)}(t)) + \varepsilon(t), \quad t \in [0,1].$$

Then problem (1.1), (1.2) becomes $Ly = Ny$.

3. Main results

By Cramer's rule, we can get the following lemmas.

Lemma 3.1. *For given $u \in Y$, the system of linear equations*

$$\begin{gathered}\frac{x_k}{k!} + \frac{x_{k+1}}{(k+1)!} + \dots + \frac{x_{n-2}}{(n-2)!} + \frac{(-1)^{n-k}}{(n-1)!}\int_0^1 (1-s)^{n-1}u(s)ds = 0\\ \frac{x_k}{(k-1)!} + \frac{x_{k+1}}{k!} + \dots + \frac{x_{n-2}}{(n-3)!} + \frac{(-1)^{n-k}}{(n-2)!}\int_0^1 (1-s)^{n-2}u(s)ds = 0\\ \dots\\ \frac{x_k}{[k-(n-k-2)]!} + \frac{x_{k+1}}{[k+1-(n-k-2)]!} + \dots + \frac{x_{n-2}}{[n-2-(n-k-2)]!}\\ + \frac{(-1)^{n-k}}{[n-1-(n-k-2)]!}\int_0^1 (1-s)^{k+1}u(s)ds = 0\end{gathered} \tag{3.1}$$

has an only one solution, $(x_k, x_{k+1}, \dots, x_{n-2})$ with

$$\begin{aligned}x_m = \int_0^1 &\frac{(-1)^{n-k-1}m!}{(m-k)!(k-1)!(n-m-2)!}\sum_{i=0}^{m-k}(-1)^{m-k-i}\frac{C_{m-k}^i}{m-i}\\ &\times \Big[\sum_{j=0}^{n-m-2}(-1)^j C_{n-m-2}^j \frac{(1-s)^{n-1-i-j}}{n-1-i-j}\Big]u(s)ds, \quad m = k, k+1, \dots, n-2.\end{aligned}$$

Lemma 3.2. *The system of linear equations*

$$\begin{gathered}\frac{x_k}{k!} + \frac{x_{k+1}}{(k+1)!} + \dots + \frac{x_{n-2}}{(n-2)!} + \frac{1}{(n-1)!} = 0\\ \frac{x_k}{(k-1)!} + \frac{x_{k+1}}{k!} + \dots + \frac{x_{n-2}}{(n-3)!} + \frac{1}{(n-2)!} = 0\\ \dots\\ \frac{x_k}{[k-(n-k-2)]!} + \frac{x_{k+1}}{[k+1-(n-k-2)]!} + \dots\\ + \frac{x_{n-2}}{[n-2-(n-k-2)]!} + \frac{1}{[n-1-(n-k-2)]!} = 0\end{gathered} \tag{3.2}$$

has an only one solution, $(x_k, x_{k+1}, \dots, x_{n-2})$ *with*

$$\begin{aligned} x_m = &-\frac{m!}{(m-k)!(k-1)!(n-m-2)!}\sum_{i=0}^{m-k}(-1)^{m-k-i}\frac{C_{m-k}^i}{m-i} \\ &\times\Big(\sum_{j=0}^{n-m-2}(-1)^j C_{n-m-2}^j\frac{1}{n-1-i-j}\Big), \quad m=k,k+1,\dots,n-2. \end{aligned}$$

Let $(B_k(u), B_{k+1}(u), \dots, B_{n-2}(u))$ denote the only solution of (3.1), and let $(A_k, A_{k+1}, \dots, A_{n-2})$ denote the only solution of (3.2), and let $A_{n-1} = 1$.

In order to obtain our main results, we firstly present and prove the following lemmas.

Lemma 3.3. *Suppose* (H1) *holds, then* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator of index zero and the linear continuous projector* $Q : Y \to Y$ *can be defined as*

$$Qu = \frac{1}{1-\sum_{i=1}^m \alpha_i\xi_i}\sum_{i=1}^m \alpha_i\int_{\xi_i}^1 u(s)ds,$$

and the linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *can be written as*

$$K_P u = \sum_{i=k}^{n-2}\frac{B_i(u)}{i!}t^i + \frac{(-1)^{n-k}}{(n-1)!}\int_0^t (t-s)^{n-1}u(s)ds.$$

Proof. By simple calculations, we obtain that

$$\ker L = \big\{y : y = c\Big(\sum_{i=k}^{n-1}\frac{A_i}{i!}t^i\Big),\ c\in\mathbb{R}\big\}.$$

Define linear operator $P : X \to X$ as follows

$$Py(t) = \Big(\sum_{i=k}^{n-1}\frac{A_i}{i!}t^i\Big)y^{(n-1)}(0).$$

Obviously, $\operatorname{Im} P = \ker L$ and $P^2 y = Py$. For any $y \in X$, it follows from $y = (y - Py) + Py$ that $X = \ker P + \ker L$. By simple calculation, we can get that $\ker L \cap \ker P = \{0\}$. So, we have

$$X = \ker L \oplus \ker P. \tag{3.3}$$

We will show that

$$\operatorname{Im} L = \big\{u\in Y : \sum_{i=1}^m \alpha_i\int_{\xi_i}^1 u(s)ds = 0\big\}.$$

In fact, if $u \in \operatorname{Im} L$, there exists $y \in \operatorname{dom} L$ such that $u = Ly \in Y$. So, we have

$$y = \sum_{i=k}^{n-1}\frac{c_i}{i!}t^i + \frac{(-1)^{n-k}}{(n-1)!}\int_0^t (t-s)^{n-1}u(s)ds.$$

Since $\sum_{i=1}^m \alpha_i = 1$ and $y^{(n-1)}(1) = \sum_{i=1}^m \alpha_i y^{(n-1)}(\xi_i)$, we have

$$\int_0^1 u(s)ds = \sum_{i=1}^m \alpha_i\int_0^{\xi_i} u(s)ds;$$

i.e., $\sum_{i=1}^m \alpha_i\int_{\xi_i}^1 u(s)ds = 0$.

On the other hand, if $u \in Y$ satisfies $\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 u(s)ds = 0$, we take

$$y = \sum_{i=k}^{n-2} \frac{B_i(u)}{i!} t^i + \frac{(-1)^{n-k}}{(n-1)!} \int_0^t (t-s)^{n-1} u(s)ds.$$

Obviously, $Ly = u$ and $y^{(n-1)}(1) = \sum_{i=1}^m \alpha_i y^{(n-1)}(\xi_i)$. By Lemma 3.1, we obtain that $y \in \operatorname{dom} L$; i.e., $u \in \operatorname{Im} L$.

Define operator $Q : Y \to Y$ as follows

$$Qu = \frac{1}{1 - \sum_{i=1}^m \alpha_i \xi_i} \Big(\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 u(s)ds \Big).$$

Obviously, $Q^2 y = Qy$ and $\operatorname{Im} L = \ker Q$. For $y \in Y$, set $y = (y - Qy) + Qy$. Then $y - Qy \in \ker Q = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$. It follows from $\ker Q = \operatorname{Im} L$ and $Q^2 y = Qy$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. So we have

$$Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

This, together with (3.3), means that L is a Fredholm operator of index zero.

Define operator $K_P : Y \to X$ as follows

$$K_P u = \sum_{i=k}^{n-2} \frac{B_i(u)}{i!} t^i + \frac{(-1)^{n-k}}{(n-1)!} \int_0^t (t-s)^{n-1} u(s)ds.$$

Now we show that $K_P(\operatorname{Im} L) \subset \operatorname{dom} L \cap \ker P$. Take $u \in \operatorname{Im} L$. Obviously, $(K_P(u))^{(n-1)}(0) = 0$. This implies that $K_P(u) \in \ker P$. It is easy to see that $(K_P(u))^{(i)}(0) = 0$, $0 \le i \le k-1$. It follows from Lemma 3.1 that $(K_P(u))^{(j)}(1) = 0$, $0 \le j \le n-k-2$. From $u \in \operatorname{Im} L$, we obtain

$$(K_P(u))^{(n-1)}(1) = \sum_{i=1}^m \alpha_i (K_P(u))^{(n-1)}(\xi_i).$$

So, $K_P(u) \in \operatorname{dom} L$.

Now we prove that K_P is the inverse of $L|_{\operatorname{dom} L \cap \ker P}$. Obviously, $LK_P u = u$, for $u \in \operatorname{Im} L$. On the other hand, for $y \in \operatorname{dom} L \cap \ker P$, we have

$$\begin{aligned} K_P Ly(t) &= \sum_{i=k}^{n-2} \frac{B_i(Ly)}{i!} t^i + \frac{(-1)^{n-k}}{(n-1)!} \int_0^t (t-s)^{n-1} (-1)^{n-k} y^{(n)}(s)ds \\ &= \sum_{i=k}^{n-2} \big(\frac{B_i(Ly) - y^{(i)}(0)}{i!} \big) t^i + y(t). \end{aligned}$$

Since y and $K_P Ly \in \operatorname{dom} L$, we have $(K_P Ly)^{(j)}(1) = y^{(j)}(1) = 0$, $0 \le j \le n-k-2$. This means that $(B_k(Ly) - y^{(k)}(0), B_{k+1}(Ly) - y^{(k+1)}(0), \dots, B_{n-2}(Ly) - y^{(n-2)}(0))$ is the only zero solution of the system of linear equations

$$\begin{gathered} \frac{x_k}{k!} + \frac{x_{k+1}}{(k+1)!} + \dots + \frac{x_{n-2}}{(n-2)!} = 0 \\ \frac{x_k}{(k-1)!} + \frac{x_{k+1}}{k!} + \dots + \frac{x_{n-2}}{(n-3)!} = 0 \\ \dots \\ \frac{x_k}{[k-(n-k-2)]!} + \frac{x_{k+1}}{[k+1-(n-k-2)]!} + \dots + \frac{x_{n-2}}{[n-2-(n-k-2)]!} = 0. \end{gathered}$$

So, we have $K_P L y = y$, for $y \in \operatorname{dom} L \cap \ker P$. Thus, $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. The proof is complete. $\square$

Lemma 3.4. *Assume* $\Omega \subset X$ *is an open bounded subset and* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$*, then* N *is* L*-compact on* $\overline{\Omega}$.

Proof. Obviously, $QN(\overline{\Omega})$ is bounded. Now we will show that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

It follows from (H2) that there exists constant $M_0 > 0$ such that $|(I-Q)Ny| \leq M_0$; a.e., $t \in [0,1]$, $y \in \overline{\Omega}$. Thus, $K_P(I-Q)N(\overline{\Omega})$ is bounded. By (H2) and Lebesgue Dominated Convergence theorem, we get that $K_P(I-Q)N : \overline{\Omega} \to X$ is continuous. Since $\big\{ \int_0^t (t-s)^j (I-Q)Ny(s)ds,\ y \in \overline{\Omega} \big\}$, $j = 0, 1 \dots, n-1$ are equi-continuous, and t^j, $j = 0, 1 \dots, n-1$ are uniformly continuous on $[0,1]$, using Ascoli-Arzela theorem, we obtain that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. The proof is complete. $\square$

To obtain our main results, we need the following conditions.

(H3) There exists a constant $M > 0$ such that if $|y^{(n-1)}(t)| > M$, $t \in [\xi_m, 1]$ then
$$\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \big[f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) + \varepsilon(s) \big] ds \neq 0.$$

(H4) There exist functions $g, h, \psi_i \in L^1[0,1]$, $i = 1, 2, \dots, n$, with $\sum_{i=1}^n \|\psi_i\|_1 < 1/2$, $\theta \in [0,1)$, some $1 \leq j \leq n$ such that
$$|f(t, x_1, x_2, \dots, x_n)| \leq g(t) + \sum_{i=1}^n \psi_i(t)|x_i| + h(t)|x_j|^\theta.$$

(H5) There exists a constant $c_0 > 0$ such that, if $|c| > c_0$, one of the following two conditions holds
$$c \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \Big[f\Big(s, c\Big(\sum_{i=k}^{n-1} \frac{A_i}{i!} s^i\Big), c\Big(\sum_{i=k}^{n-1} \frac{A_i}{(i-1)!} s^{i-1}\Big), \dots, c\Big) + \varepsilon(s) \Big] ds > 0, \quad (3.4)$$
$$c \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \Big[f\Big(s, c\Big(\sum_{i=k}^{n-1} \frac{A_i}{i!} s^i\Big), c\Big(\sum_{i=k}^{n-1} \frac{A_i}{(i-1)!} s^{i-1}\Big), \dots, c\Big) + \varepsilon(s) \Big] ds < 0. \quad (3.5)$$

Lemma 3.5. *Assume* (H1)–(H4)*. Then the set*
$$\Omega_1 = \big\{ y \in \operatorname{dom} L \setminus \ker L : Ly = \lambda N y,\ \lambda \in (0,1) \big\}$$
is bounded.

Proof. Take $y \in \Omega_1$. Since $Ny \in \operatorname{Im} L$, we have
$$\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \Big[f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) + \varepsilon(s) \Big] ds = 0. \quad (3.6)$$
Since $Ly = \lambda N y$ and $y \in \operatorname{dom} L$, it follows that
$$y(t) = \sum_{i=k}^{n-1} \frac{c_i}{i!} t^i + \frac{(-1)^{n-k}}{(n-1)!} \lambda \int_0^t (t-s)^{n-1} \Big[f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) + \varepsilon(s) \Big] ds, \quad (3.7)$$

where c_i, $i = k, k+1, \dots, n-1$ satisfy

$$\sum_{i=k}^{n-1} \frac{c_i}{i!} = -\frac{(-1)^{n-k}}{(n-1)!}\lambda \int_0^1 (1-s)^{n-1}\big[f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) + \varepsilon(s)\big]ds$$

$$\sum_{i=k}^{n-1} \frac{c_i}{(i-1)!} = -\frac{(-1)^{n-k}}{(n-2)!}\lambda \int_0^1 (1-s)^{n-2}\big[f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) + \varepsilon(s)\big]ds$$

$$\dots$$

$$\begin{aligned}\sum_{i=k}^{n-1} \frac{c_i}{[i-(n-k-2)]!} &= -\frac{(-1)^{n-k}}{[n-1-(n-k-2)]!}\lambda \int_0^1 (1-s)^{k+1} \\ &\quad \times \big[f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) + \varepsilon(s)\big]ds.\end{aligned}$$

It follows from $y^{(i)}(0) = y^{(j)}(1) = 0$, $0 \le i \le k-1$, $0 \le j \le n-k-2$ that there exists at least one point $\delta_i \in [0,1]$ such that $y^{(i)}(\delta_i) = 0$, $i = 0, 1, \dots, n-2$. So, we have

$$y^{(i)}(t) = \int_{\delta_i}^t y^{(i+1)}(s)ds, \quad i = 0, 1, \dots, n-2.$$

Therefore,

$$\|y^{(i)}\|_\infty \le \|y^{(i+1)}\|_1 \le \|y^{(i+1)}\|_\infty, \quad i = 0, 1, \dots, n-2. \tag{3.8}$$

By (3.6) and (H3), there exists $t_0 \in [\xi_m, 1]$ such that $|y^{(n-1)}(t_0)| \le M$. This, together with (3.7), implies

$$|c_{n-1}| \le M + \int_0^1 \Big|f(s, y(s), y'(s), \dots, y^{(n-1)}(s))\Big| ds + \|\varepsilon\|_1. \tag{3.9}$$

It follows from (3.7)-(3.9) and (H4) that

$$\begin{aligned}\|y^{(n-1)}\|_\infty &\le M + 2\int_0^1 \big|f(s, y(s), y'(s), \dots, y^{(n-1)}(s))\big| ds + 2\|\varepsilon\|_1 \\ &\le M + 2[\|g\|_1 + \sum_{i=1}^n \|\psi_i\|_1 \|y^{(i-1)}\|_\infty + \|h\|_1 \|y^{(j-1)}\|_\infty^\theta] + 2\|\varepsilon\|_1 \\ &\le M + 2\|g\|_1 + 2\sum_{i=1}^n \|\psi_i\|_1 \|y^{(n-1)}\|_\infty + 2\|h\|_1 \|y^{(n-1)}\|_\infty^\theta + 2\|\varepsilon\|_1.\end{aligned}$$

So, we obtain

$$\|y^{(n-1)}\|_\infty \le \frac{M + 2\|g\|_1 + 2\|\varepsilon\|_1}{1 - 2\sum_{i=1}^n \|\psi_i\|_1} + \frac{2\|h\|_1}{1 - 2\sum_{i=1}^n \|\psi_i\|_1} \|y^{(n-1)}\|_\infty^\theta.$$

Then $\theta \in [0,1)$ implies that $\{\|y^{(n-1)}\|_\infty| : y \in \Omega_1\}$ is bounded. Considering of (3.8), we obtain that Ω_1 is bounded. $\square$

Lemma 3.6. *Assume* (H1), (H2), (H5). *Then the set*

$$\Omega_2 = \{y : y \in \ker L,\ Ny \in \operatorname{Im} L\}$$

is bounded.

Proof. Take $y \in \Omega_2$, then $y(t) = c\big(\sum_{i=k}^{n-1} \frac{A_i}{i!} t^i\big)$, $c \in \mathbb{R}$ and $Ny \in \operatorname{Im} L$. So, we have

$$c\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{1} \Big[f\Big(s, c\Big(\sum_{i=k}^{n-1} \frac{A_i}{i!} s^i\Big), c\Big(\sum_{i=k}^{n-1} \frac{A_i}{(i-1)!} s^{i-1}\Big), \dots, c\Big) + \varepsilon(s)\Big] ds = 0.$$

By (H5), we obtain that $|c| \le c_0$. So, Ω_2 is bounded. ☐

Lemma 3.7. *Assume* (H1), (H2), (H5). *Then the set*

$$\Omega_3 = \{y \in \ker L : \lambda Jy + (1-\lambda)\theta QNy = 0,\ \lambda \in [0,1]\}$$

is bounded, where $J : \ker L \to \operatorname{Im} Q$ *is a linear isomorphism given by*

$$J\Big(c\sum_{i=k}^{n-1} \frac{A_i}{i!} t^i\Big) = \frac{c}{1 - \sum_{i=1}^{m} \alpha_i \xi_i}, \quad c \in \mathbb{R}$$

and $\theta = \begin{cases} 1 & \textit{if } (3.4) \textit{ holds}, \\ -1, & \textit{if } (3.5) \textit{ holds}. \end{cases}$.

Proof. For $y \in \Omega_3$, we get $y = c\big(\sum_{i=k}^{n-1} \frac{A_i}{i!} t^i\big)$ with

$$\lambda c + (1-\lambda)\theta \sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{1} \Big[f\Big(s, c\Big(\sum_{i=k}^{n-1} \frac{A_i}{i!} s^i\Big), c\Big(\sum_{i=k}^{n-1} \frac{A_i}{(i-1)!} s^{i-1}\Big), \dots, c\Big) + \varepsilon(s)\Big] ds = 0.$$

If $\lambda = 0$, by (H5), we get $|c| \le c_0$. If $\lambda = 1$, $c = 0$. For $\lambda \in (0,1)$, if $|c| \ge c_0$, then

$$\begin{aligned} \lambda c^2 = -(1-\lambda)\theta c \sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{1} \Big[f\Big(s, c\Big(\sum_{i=k}^{n-1} \frac{A_i}{i!} s^i\Big), c\Big(\sum_{i=k}^{n-1} \frac{A_i}{(i-1)!} s^{i-1}\Big), \dots, c\Big) \\ + \varepsilon(s)\Big] ds < 0. \end{aligned}$$

This is a contradiction. So, Ω_3 is bounded. ☐

Theorem 3.8. *Assume* (H1)–(H5) *Then problem* (1.1)–(1.2) *has at least one solution in* X.

Proof. Let $\Omega \supset \cup_{i=1}^{3} \overline{\Omega_i} \cup \{0\}$ be a bounded open subset of X. It follows from Lemma 3.4 that N is $L-$compact on $\overline{\Omega}$. By Lemmas 3.5 and 3.6, we obtain: (1) $Ly \ne \lambda Ny$ for every $(y,\lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,\ 1)$; and (2) $Ny \notin \operatorname{Im} L$ for every $y \in \ker L \cap \partial\Omega$. We need to prove only (3) $\deg(QN|_{\ker L},\ \Omega \cap \ker L,\ 0) \ne 0$. To do this, we take

$$H(y,\lambda) = \lambda Jy + \theta(1-\lambda)QNy.$$

According to Lemma 3.7, we know $H(y,\lambda) \ne 0$ for $y \in \partial\Omega \cap \ker L$. By the homotopy of degree, we obtain

$$\begin{aligned} \deg(QN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(\theta H(\cdot,0), \Omega \cap \ker L, 0) \\ &= \deg(\theta H(\cdot,1), \Omega \cap \ker L, 0) \\ &= \deg(\theta J, \Omega \cap \ker L, 0) \ne 0. \end{aligned}$$

By Theorem 2.1, we obtain that $Ly = Ny$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$; i.e., (1.1)-(1.2) has at least one solution in X. The prove is complete. ☐

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